WEEK 5 CLASS PROPOSALS, MATHCAMP 2019

Andrew's Classes

The Matrix-Tree Theorem. (🌶🌶–🌶🌶🌶, Andrew, 2–3 days)
One of the slicker mathematical facts I know is Cayley's formula: given n vertices labeled from 1 to n, the number of ways to draw a tree through all n vertices is n^{n-2}. (Go to Mark's class, "The Pruefer Correspondence," for one explanation of this!) One way this can be rephrased is in terms of **spanning trees**: if we start with the complete graph K_n, there are n^{n-2} ways to remove some of the edges and end up with a tree on those n vertices.

What may be more surprising is that we can generalize this: even if we start with a graph G that isn't K_n, we can still calculate the number of spanning trees that are subgraphs of G! This class will cover the matrix-tree theorem, a way to count such trees by using matrix determinants. We'll start by reviewing various linear algebra tidbits that you may have picked up over the first four weeks (or learned from elsewhere). From there, we'll be able to formulate and prove the matrix-tree theorem, apply it to some cute examples, and hopefully gain a stronger foundation in some details of linear algebra.

Homework: Recommended.

Prerequisites: A good understanding of linear algebra concepts (primarily eigenvalues and determinants), though I'll review most of the tools being used. The class is probably 3-chili if you're seeing linear algebra for the first time at camp this year and 2-chili otherwise.

Apurva's Classes

All Things Manifoldy. (🌶🌶, Apurva, 4 days)
Who said that mathematicians are not real doctors? We perform surgeries all the time. In this class, we'll take baby steps towards understanding manifolds. We'll learn some of the uber awesome techniques invented by topologists to study manifolds, perform surgeries on them, and do origami using simplices. By the end of the class you'll be able to visualize (some) manifolds in higher dimensions.

Incidentally, when Einstein tried to combine special relativity with Newton's gravity, nothing seemed to work. It took him a decade to finally realize a beautiful solution to the conundrum: our universe is a 4-dimensional manifold, and gravity is a measure of how the manifold curves. But what is a manifold?

Homework: Recommended.

Prerequisites: None.

A Unique "Prime" Factorization Theorem for Matrices. (🌶🌶🌶, Apurva, 4 days)
The goal of this class is to prove the existence of Jordan Canonical Forms by first proving the structure theorem of finitely generated modules over a PID. This is also an excuse to learn about PIDs and modules.

Homework: Recommended.

Prerequisites: Basic linear algebra and group theory courses.

Morse Theory. (🌶🌶🌶🌶, Apurva, 4 days)
Morse theory is a technique for producing cell structures on manifolds using critical points of functions. Morse theory was used by Milnor to produce exotic differentiable structures on seven-dimensional spheres and lies at the intersection of differential geometry and algebraic topology.

We will go through the first few sections of Milnor's book on Morse theory.

This class will probably be more than 4 chilis.

Homework: Recommended.

Prerequisites: Multivariable calculus, basic manifold stuff. Topology – you should have a general idea of what a cell complex is.

Assaf's Classes

Exploring Exponents. (🌶🌶🌶, Assaf, 1–3 days)
This expository expo expounds my experiments in explicitly expanding exponents. The expanded expression expels explosive exploration of exp and explains its expansive exploits. Explicitly, we'll experience how exponentiation exports algebras to groups and exposes exploitable expressways to solving ODEs and PDEs.

Homework: Required.

Prerequisites: Linear Algebra, Calculus.

Geometric Group Theory. (🌶🌶–🌶🌶🌶, Assaf, 4 days)
Geometric group theory is the study of groups using metric spaces. The Cayley graph of an infinite group can be thought of as a metric space, and looking at the geometric properties of this graph can give information about the group. For example, one can detect whether a group has a finite index free group by looking at this graph.

Moreso, if a group has a so-called Gromov hyperbolic Cayley graph, then we can recover many more properties of the group, like the kinds of spaces it can act on, or even its abelian subgroups.

This field is huge, and we'll only scratch the surface of what's possible if we're just willing to shift our perspective from algebra to geometry.

Homework: Required.

Prerequisites: Group Theory, knowledge of graph theory, and some topology.

Hyperbolic Surfaces. (🌶🌶🌶, Assaf, 1–3 days)
If you take a piece of paper and try wrapping it around a sphere, it will crinkle up. That's because the sphere has positive curvature, and the paper has zero curvature. The same problem happens when we try to wrap a hyperbolic piece of paper around a torus. The curvature just doesn't add up. In this class, we'll see how to glue together the hyperbolic plane in order to build high genus surfaces. If those sound like words you know nothing about and would like to hear more about, come by!

Homework: Recommended.

Prerequisites: Non-Euclidean Geometries.

Möbius Transformations. (🌶🌶🌶, Assaf, 2–3 days)
A Möbius transformation is a function from $\mathbb{C}$ to $\mathbb{C}$ of the form $f(z) = \frac{az+b}{cz+d}$. These functions have some remarkable properties. In this class, we'll go through some of their properties, like their geometric interpretation as isometries of the Riemann Sphere, their group structure, their use in hyperbolic geometry, and their classification.

Homework: Recommended.

Prerequisites: Functions of a Complex Variable.

The Matrix Tree Theorem and Gerrymandering. (🌶🌶, Assaf, 1 day)
The matrix tree theorem, while amazing and interesting on its own, has also recently (i.e., summer of 2018) been applied to gerrymandering. To argue that a districting plan is gerrymandered, we need to produce many other plans to compare it to, and we do this with spanning trees. Come learn some cool applications of cool math!

Homework: Recommended.

Prerequisites: The Matrix-Tree Theorem.

There Is No Straight Line on the Tetrahedron. (🌶🌶, Assaf, 2–3 days)
If an ant starts at the corner of a regular tetrahedron, and walks in a straight line on its surface, it will never return to the corner where it started. There's a cute proof of this involving techniques from billiards. On the way, we will talk about tilings, tori, and "branched double covers."

Homework: None.

Prerequisites: None.

The Word Problem in Groups. (🌶🌶🌶, Assaf, 2 days)
The word problem asks the following question: given a finite group presentation and a word in the generators, does there exist an algorithm that tells you whether the word is trivial? The answer to this is no. To show this, we will embed the halting problem into the structure of a group, to construct a really really really ugly group.

Homework: Recommended.

Prerequisites: Group theory; Homophones, Anagrams, and Free Groups.

Traffic. (🌶, Assaf, 1 day)
As the saying goes, "you're not in traffic, you are traffic." Traffic is a game that almost[1] everyone plays. Everyone wants to be rational, but sometimes this rationality comes back and bites the collective. In this class, we'll explore scenarios where this effect happens. We'll look at Braess' Paradox, the Bus Motivation Problem, and spend some time discussing the formalism of congestion games.

Homework: None.

Prerequisites: None.

Ben's Classes

Baire Category Theorem. (🌶🌶🌶, Ben, 2–4 days)
Not about bears, not about categories, but it is a theorem. The Baire Category Theorem is a standard result in metric space topology, but one that is surprisingly applicable in many areas. It offers one proof that the real numbers are uncountable, and can be used to study the points where functions can be continuous or discontinuous. It gives us a way to show the existence of a nowhere-differentiable continuous function (albeit somewhat nonconstructively), and it has many applications in the field of functional analysis.

Homework: Recommended.

Prerequisites: Topology may be helpful but shouldn't be necessary. Linear algebra for the 4-day version (which is likely to include some Banach space topics).

[1] except for those annoying cyclists

Compactification. (🌶🌶🌶–🌶🌶🌶🌶, Ben, 2–3 days)
Compactness, roughly speaking, is a notion of a topological space being "small." Naturally, to take a space that isn't compact and make it compact, you add more points so that it becomes small.

There are a few natural questions about compactification: how small can a compactification be? How large can it be? These are the questions we will try to answer in this course. If you like category theory or want to get a better feel for it, some of the objects we'll study in this course have elegant universal properties, which we'll hopefully have time to discuss.

Homework: Recommended.

Prerequisites: Topology (knowing what compactness is).

Measure For Measure. (🌶🌶🌶🌶, Ben, 2–4 days)
Measure theory for the sake of teaching more measure theory! Topics here include more convergence theorems: hopefully Lebesgue's Dominated Convergence Theorem, and perhaps even some other topics.

Homework: Recommended.

Prerequisites: Measure Theory.

The Polish Attack on the Enigma. (🌶🌶–🌶🌶🌶, Ben, 1–2 days)
In the early 1930s, the German Enigma machine seemed to offer a great deal of security, due to the sheer number of possible setups for the machine. However, Polish intelligence realized that the code could be broken, using group theory.

We'll see how the Polish attack on the early Enigma worked, which is both a historically interesting topic and a testament to the power of group theory.

Homework: Recommended.

Prerequisites: Group theory.

Ben and Assaf's Classes

PARTIAL DIFFERENTIAL EQUATIONS. (🌶🌶🌶🌶🌶, Ben and Assaf, 4 days)
PAIN IS WEAKNESS LEAVING THE BODY! PARTIAL DIFFERENTIAL EQUATIONS? MORE LIKE PAINFULLY DIFFICULT EXERCISE! WANT FOURIER TRANSFORMS? WANT INFINITE DIMENSIONAL VECTOR SPACES? WANT INTEGRATION BY PARTS? HAVE YOU EVER TAKEN THE EXPONENT OF A DERIVATIVE? SOLVE ALL OF QUANTUM MECHANICS, SOLVE HEAT AND MOTION AND WAVES AND MAKE LAPLACE'S EQUATION LOOK LIKE ADDITION! YOU WILL TRANSCEND TIME ITSELF. IN THIS CLASS, YOU DO THE WORK, YOU GET RESULTS! CALL US AT 1888-MCSP-PDE FOR YOUR FREE TRIAL TODAY! Limited time offer, some restrictions apply, misuse of this product may cause blindness and hair loss, not available in Alaska or Hawaii.

Homework: Required.

Prerequisites: Topology, multivariable calculus, complex analysis, linear algebra, nonlinear algebra, measure theory, and an undergraduate degree in physics.

Bill's Classes

Cauchy–Davenport with Combinatorial Nullstellensatz. (🌶🌶🌶, Bill, 1 day)
The Cauchy–Davenport Theorem states that for a prime number $p > 2$, and for $A, B \subseteq \mathbb{Z}_p$,

$$|A + B| \geq \min(p, |A| + |B| - 1).$$

In this class, we'll prove the Cauchy–Davenport Theorem using a powerful technique known as Combinatorial Nullstellensatz, in which one carefully defines a polynomial whose set of zeros captures some combinatorial problem; and then one uses a (somewhat peculiar) theorem about the zeros of multivariate polynomials in order to extract properties of the original combinatorial problem.
Homework: None.
Prerequisites: None.

Elegant Applications of Linear Algebra to Combinatorics. (Bill, 2 days)
In this class we'll use ideas from linear algebra to prove beautiful (and surprising!) theorems in discrete math. For example, we'll prove:

- For any irrational number x, it's impossible to tile a $1 \times x$ rectangle with finitely many squares (*even* if the squares are permitted to have both irrational and rational side-lengths).

The theme of the class will be that basic ideas in linear algebra (e.g., linear independence, dimension, bases) can be immensely useful when analyzing seemingly unrelated problems (e.g., tilings of a rectangle, or set systems satisfying certain intersection properties).
Homework: None.
Prerequisites: Basic familiarity with linear algebra. Understanding of terms: vector space, basis, dimension, linear function, linear independence. No familiarity with eigenstuff is required.

Generating-Function Magic. (Bill, 2 days)

"A generating function is a clothesline on which we hang up a sequence of numbers for display."
– Herbert Wilf

The *generating function* of a number sequence $a_0, a_1, \ldots$ is the formal power series $\sum_{i=1}^{\infty} a_i x^i$. By encapsulating the number sequence within a single algebraic object, generating functions allow for us to derive results about the original number sequence in what are often highly unexpected ways.

In this class, we'll use generating functions to prove two of my favorite results:

- The number of integer partitions with odd-sized parts is equal to the number of integer partitions with distinct-sized parts. (We'll also see a beautiful bijective proof of this result!)
- The n^{th} Catalan number C_n, which counts the number of valid configurations of n pairs of parentheses, satisfies $C_n = \binom{2n}{n}/(n+1)$.

Homework: None.
Prerequisites: Students who have seen an example of generating functions used in the past will have an easier time, but I will make sure this is not necessary.

Randomized Cup Games (a.k.a. Bill's Research). (Bill, 1 day)
If you've taken any of my classes you probably know that I like combinatorics, algorithms, and probability. But what do I actually study in my research? (Hint: the answer is, all of the above, simultaneously.)

In the *cup game on n cups*, there are n initially empty cups that are each capable of holding arbitrarily large amounts of water. In each step of the game, a *filler* distributes one new unit of water among the cups; a *emptier* then selects a single cups, and removes (up to) one unit of water from that cup. The emptier's goal is to minimize the *backlog* of the cup system, which is defined to be the amount of water in the fullest cup.

The greedy emptying algorithm (i.e., always emptying from the fullest cup) is known to achieve backlog $O(\log n)$. Until recently, the following question has been open: can randomized emptying algorithms do better?

In this class, we'll give an overview of the design and analysis of a recent randomized algorithm that achieves backlog $O(\log\log n)$. (This is so recent that it was published *during* the first week of camp this year.)

Homework: None.

Prerequisites: Big-O notation would be useful.

The (Almost) Secret Algorithm Researchers Used to Break Thousands of RSA Keys. (🌶–🌶🌶, Bill, 1 day)
By computing pairwise GCDs between millions of RSA private keys, researchers were able to break every key that shared a common prime factor with any other key. If the prime factors of keys were selected truly randomly, then common factors would be extremely rare; but due to the widespread use of badly-seeded random-number generators, certain prime factors appeared much more prolifically than others.

Then, in an unusual move, the researchers intentionally tried to hide the key algorithmic ideas used in their research, hinting only that they had been able to run a computation that should have naively taken years in a matter of hours. This class will describe the algorithm that the researchers used.

Homework: None.

Prerequisites: Familiarity with Big-O notation would be useful (but the class can still be enjoyed without it).

Eric's Classes

Counting Curves using Linear Algebra. (🌶🌶, Eric, 2 days)
We'll work through some problems in enumerative geometry, trying to do things by counting dimensions of vector spaces of polynomials as much as possible. As an example of the sort of thing we'll prove is that 5 points in sufficiently generic position determine a plane conic, 9 determine a plane cubic, and so on; along the way we'll work on understanding exactly what "sufficiently generic" means in contexts like this. We can hit things like constructing the group law on elliptic curves and the Cayley–Bacharach theorem along the way.

Homework: Recommended.

Prerequisites: Linear algebra, at the level of knowing dimensions of intersections. Having seen projective spaces before is helpful but not necessary.

Homophones, Anagrams, and Free Groups. (🌶🌶, Eric, 2 days)
What happens if you decided that words that sound the same should BE the same? For example, the words "knight" and "night" are pronounced the same, what happens if we decide they should be the same word? If you work out all the consequences of this, it turns out you don't end up with a lot of words. We'll talk about how to interpret this (and other) game(s) with words as statements about group theory.

Homework: Optional.

Prerequisites: Group theory, at least to know what normal subgroups and quotient groups are.

Infinitely Many Proofs of Infinitely Many Primes. (🌶🌶, Eric, 1 day)
I'll fit as many proofs as I can into one class of the fact that there are infinitely many prime numbers. Some will be straightforward, others will connect us to great open problems (the Riemann zeta function will appear!), others will use topology (??), and more!

Homework: None.

Prerequisites: Some proofs may require a small amount of knowledge about topology or rings, but nothing beyond the basic definitions.

The 1-Dimensional Analogue of Wiles' Proof. (🌶🌶🌶🌶, Eric, 2–3 days)
Get a start towards understanding Wiles' proof of the modularity of (semi-stable) elliptic curves by proving a 1-dimensional analogue of his result using the same techniques. We'll prove the Kronecker–Weber theorem which states that any abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field. We'll introduce some fancy rings parametrizing group homomorphisms, take some complicated commutative algebra theorems and boil them down to straightforward topological facts, and induct on how bad our fields are. I hope to emphasize and make comparisons to the key points of Wiles' argument in this simpler (but still complicated) situation.

Homework: Recommended.

Prerequisites: This class will move fast. You should be happy thinking about rings and their quotients. Algebraic number theory to the point of knowing that prime numbers factor into prime ideals. We'll use some Galois theory for number fields, so you should be familiar with the statements of Galois theory (at the level used in my week 4 class is fine).

You Can't Solve the Quintic. (🌶🌶, Eric, 2–3 days)
We'll prove that there's no general formula in radicals that works to produce the roots of a degree 5 polynomial. This particular proof due to Arnold is a beautiful combination of combinatorial and topological arguments. The only things you'll need to know going in are what complex numbers, the quadratic formula, and permutations are; but we'll make tons of connections with classes on group theory, fundamental groups, Riemann surfaces and more from the first 4 weeks of camp.

Homework: Recommended.

Prerequisites: Be comfortable with the complex plane, know the quadratic formula, know cycle notation for permutations.

Gabrielle's Classes

Factoring in the Chicken McNugget Monoid. (🌶, Gabrielle, 1–2 days)
When they were first released, Chicken McNuggets were sold in packs of 6, 9, and 20. The Chicken McNugget Monoid is the set of numbers of Chicken McNuggets that can be purchased (Chicken McNugget numbers), i.e. numbers of the form $6a + 9b + 20c$. Notice that this expansion is not unique: $18 = 6 \cdot 3$ and $18 = 9 \cdot 2$. This is a form of nonunique factorization!

We will analyze this problem, and related problems in numerical monoids, through the lens of nonunique factorization.

Homework: Recommended.

Prerequisites: None.

Integer-Valued Polynomials. (🌶🌶–🌶🌶🌶, Gabrielle, 1–3 days)
The ring of integer-valued polynomials, denoted $\mathrm{Int}(\mathbb{Z})$, is the ring of polynomials f with rational coefficients such that $f(\mathbb{Z}) \subseteq \mathbb{Z}$. It's... kind of weird. We will discuss some basic properties of $\mathrm{Int}(\mathbb{Z})$, factorization properties, and generalizations to other integral domains (and their properties).

Homework: Recommended.

Prerequisites: Last 2 days of Unique and Nonunique Factorization, or talk to Gabrielle about some concepts that you should know.

The Class Number. (🌶🌶–🌶🌶🌶, Gabrielle, 2–3 days)
Kummer was able to prove Fermat's Last Theorem in the case where n is a regular prime. In this class, we're going to figure out what that means and why that is useful.

Homework: Recommended.

Prerequisites: Intro to Algebraic Number Theory.

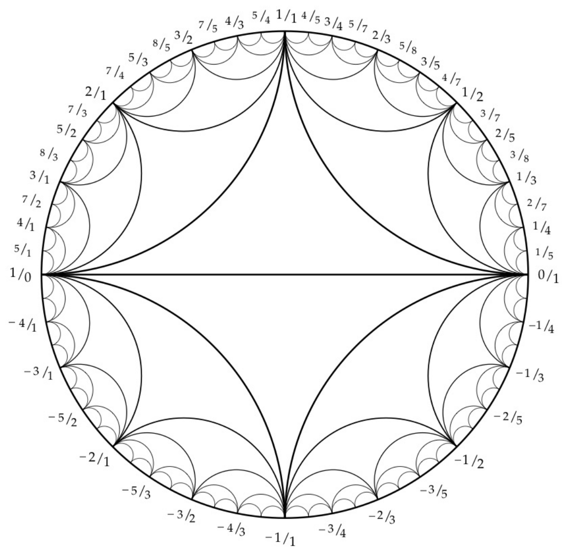
J-Lo's Classes

All the Absolute Values. (🌶🌶, J-Lo, 1 day)
The rationals have an absolute value that we all know and love. But they also have p-adic absolute values: Absolute values that measure how divisible a number is by some given prime. How many absolute values does $\mathbb{Q}$ have? In this class, we will classify all of them, a result known as Ostrowski's Theorem.

Homework: None.

Prerequisites: None.

Farey Tales. (🌶🌶–🌶🌶🌶, J-Lo, 1–5 days)
The goal of this course is to understand this picture:



Along the way we will encounter Farey series, the Euclidean algorithm, and a geometric interpretation of continued fractions. But really it's all about the picture.

Homework: Recommended.

Prerequisites: Basic theory of 2×2 matrices (multiplication, inverses, determinant, how they act on vectors).

Isogeny Graphs. (🌶🌶🌶🌶, J-Lo, 1–5 days)
The vertices are elliptic curves defined over a finite field. The edges are certain functions between them. The resulting graphs are complicated enough that they have applications to cryptography, and yet are just structured enough that we can still meaningfully study their properties.

Homework: Recommended.

Prerequisites: Finite Fields, group theory.

Lattice-Based Cryptography. (🌶🌶–🌶🌶🌶, J-Lo, 2–3 days)
Lattices are essentially just a set of regularly-spaced points in $\mathbb{R}^n$, and this makes them seem pretty simple and innocent. But be warned! In high enough dimensions, they become so complicated that people are developing cryptosystems based on them—betting their security on the hardness of certain problems about lattices. In this class we'll talk a bit about hard lattice problems, and then work together to try to design a cryptosystem based on these ideas.
Homework: Optional.
Prerequisites: Linear algebra.

When Graph Theory Solved a Geometry Problem. (🌶🌶, J-Lo, 1 day)
The *diameter* of an arbitrary shape is the largest possible distance between two points in the shape. In 1932, Karl Borsuk asked the following question:

Given a bounded subset of $\mathbb{R}^n$, can it be split into $n+1$ pieces, each of which has smaller diameter than the original set?

For example, if you try to split an equilateral triangle into two pieces, one of them will always contain two of the vertices, and so its diameter will not be any smaller than that of the original triangle. However, any figure in the plane can be decomposed into 3 pieces of smaller diameter.

Over the course of the 20^{th} century, evidence began to grow that $n+1$ pieces would always be enough. That is, until 1993, when the first counterexample was found... in 1325 dimensions. Currently the smallest known counterexample is a 64-dimensional set which requires more than 65 pieces.

And yes, the counterexamples come from graph theory. This class will explain what on earth graph theory has to say about this inherently geometric problem.
Homework: None.
Prerequisites: Intro Graph Theory.

Why the Millennium Problems? (🌶🌶, J-Lo, 1 day)
With such a wide variety of unsolved problems out there, why did seven in particular stand out enough to counted as Millennium Prize Problems? In this class I will give a brief overview of the history behind each problem, and even though I won't be able to state all the problems precisely, I hope at least to convey a sense of the mathematical significance of each, and why people care about them.
Homework: None.
Prerequisites: None.

Yes, You Can Solve the Quintic. (🌶🌶🌶🌶, J-Lo, 2–4 days)
In this class we will describe a method for solving a general quintic equation.

"But that's impossible!" you may protest. Indeed, there is no solution *in radicals*. But if you're allowed to use other tools, such as modular forms, then everything changes.
Homework: Recommended.
Prerequisites: Complex Analysis.

J-Lo, Assaf, Eric, and Apurva's Classes

Perspectives on Cohomology. (🌶🌶🌶, J-Lo, Assaf, Eric, and Apurva, 4 days)
Cohomology is a useful mathematical tool that comes in a variety of flavours, and many of your staff friends use cohomology in their research. In this class we'll have a different staff every day talk about their favourite flavour of cohomology, what is special about it, what it is useful for, and how they think about it.
Homework: Optional.

Prerequisites: Each lecture will aim to be mostly self-contained, but some familiarity with linear algebra will be useful for all of the lectures.

J-Lo and Rice's Classes

How To Keep People From Lying To You. (🌶–🌶🌶, J-Lo and Rice, 1–2 days)
Jordan the JC is planning a hiking trip and wants to know how likely it is to rain. Mark the Meteorologist knows that there is a 30% chance that it will rain, but Jordan does not know this. Jordan tells Mark: "Tell me the probability p that it will rain, and as a gift I will give you weather balloons in proportion to the probability that you assign to the event that actually occurs. If it rains I will give you $100p$ weather balloons, and if it doesn't I will give you $100(1-p)$ weather balloons." Mark responds: "Why thank you, Jordan! The probability that it will rain is 0."

Jordan has made an unfortunate mistake: the incentive structure (scoring rule) that they gave to Mark made it in Mark's interest to lie about the probability of rain. What sorts of scoring rules could Jordan have offered Mark so that he would tell the truth? Come to this class to find out!

(A historical note: in 1950, Glenn Brier (employed by the US Weather Bureau) proposed a better scoring system, for which the forecaster's optimal strategy IS to accurately report their own certainty. In his own words, the forecaster "is fooling nobody but himself if he thinks he can beat the verification system by putting down only zeros and unities when his forecasting skill does not justify such statements of extreme confidence." Since his paper, this score has become an important tool in the study of formal epistemology, a branch of philosophy in which mathematical tools are used to study how people know, believe, and reason.)

Homework: None.

Prerequisites: Probability (expected value); familiarity with derivatives.

Kayla's Classes

Introduction to Morse Theory. (🌶🌶–🌶🌶🌶, Kayla, 2–4 days)
Topologists like donuts. Suppose that we want to cover our donut in chocolate sauce when it is oriented vertically (so that the hole of the donut is perpendicular to the floor). What would the chocolate sauce function look like? Would this function have critical points? What does it mean to be a critical point of a chocolate sauce function?

Morse theory is a branch of topology that enables us to analyze the topology of a manifold by studying differentiable functions on that manifold. Moreover, some differentiable functions can tell us information about how to build a CW complex structure for our space!

Homework: Optional.

Prerequisites: Topology and calculus.

More Algebraic Topology. (🌶🌶🌶–🌶🌶🌶🌶, Kayla, 3–4 days)
Some of you have seen the limitations of the fundamental group as an algebraic structure to associate to a topological space. Are there better algebraic associations we can make to topological spaces? The answer is yes! Namely, we will discuss the idea of singular and simplicial homology and hopefully be able to develop an axiomatic framework to work under for our homology theories to coincide.

Homework: Recommended.

Prerequisites: Topology and ideally Fundamental Group.

Kevin's Classes

A Stupid Float Trick. (🌶, Kevin, 1 day)
Did you know that if x is a float, then "float"(1597463007 - "int"(x)/2) is quite close to 1/sqrt(x)? We will learn how floating point numbers work in order to understand whence this number comes.
Homework: None.
Prerequisites: A rough sense of what float means. (Knowing precisely how they work is not necessary.)

A Very Difficult Definite Integral. (🌶🌶🌶🌶, Kevin, 1 day)
In this class, we will show

$$\int_0^1 = \frac{\log(1+x^{2+\sqrt{3}})}{1+x}\,dx = \frac{\pi^2}{12}(1-\sqrt{3}) + \log(2)\log(1+\sqrt{3}).$$

We'll start by turning this Very Difficult Definite Integral into a Very Difficult Series. Then we'll sum it!

We'll need an unhealthy dose of clever tricks involving some heavy-duty algebraic number theory. It will be Very Difficult.

(This class is based on a StackExchange post by David Speyer.)
Homework: None.
Prerequisites: If you have taken all of the algebraic number theory classes at camp, you will recognize some of the ingredients in results that we will assume.

Calculus on Young's Lattice. (🌶🌶🌶, Kevin, 1 day)
Young's lattice is a beautiful poset capturing the structure of partitions, with connections to combinatorics, geometry, and representation theory. Young's lattice is almost unique among posets[2], with the remarkable property that we can do calculus on it. We'll discuss how, and we'll see how doing calculus on Young's lattice yields powerful enumerative results, including some famous identities involving Young tableaux. Plus, if you like exponentiating things, we'll even get a chance to exponentiate xD!
Homework: None.
Prerequisites: Differential calculus. Familiarity with partitions or any of Shiyue's Young Tableaux classes will be helpful.

Cluster Algebras of Invariants. (🌶🌶🌶, Kevin, 2–4 days)
Véronique's class covered cluster algebras arising from surfaces. In this class, we'll discuss another source of cluster algebras: classical rings of invariants. We'll talk about the Grassmannian and how funny objects called plabic graphs give us a way to visualize our clusters.
Homework: Optional.
Prerequisites: Cluster Algebras.

Marisa's Classes

Chromatic Number of a Surface. (🌶🌶, Marisa, 2–3 days)
The chromatic number of a graph G is the smallest k such that the vertices of G can be properly colored with k colors. Pick your favorite surface—let's say it's the sphere—and we can consider *all* graphs that embed in the sphere (i.e., are planar), and ask about *all* of their chromatic numbers. The largest number of colors we could possibly need is called the chromatic number of that surface.

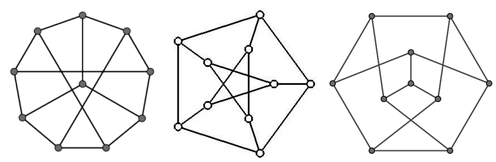
[2] Morally, there's only one other such poset, and no other lattices! Conjecturally, at least.

Spoiler: any graph that embeds on the sphere can be colored with at most 4 colors. (This is the famous 4-Color Theorem.) In this class, we'll instead ask the same question for other surfaces. What is the value of "4" for the torus? the Klein Bottle? We'll prove a beautiful upper bound and exhibit that it's sharp.

Homework: Optional.

Prerequisites: None. If you've seen some topology (e.g. the classification of surfaces) and are comfortable with basic graph theory vocabulary (e.g. average degree, plane drawings), you'll be extra prepared, but the class will be self-contained.

Crossing Numbers, from 1954 to 2012. (🌶🌶, Marisa, 2–4 days)
Suppose our goal is to draw graphs with as few edges crossing as possible. Take, for instance, the Petersen graph. We can draw it in a nice way, displaying very pretty symmetry, with 6 edge crossings. (Terrible!) Or, exhibiting different symmetry, with five crossings. (Still terrible!) Or, exhibiting yet another symmetry, with three crossings. (Closer.) The best possible is actually two crossings; can you do it?



In this class, we'll prove nice bounds (and look for exact answers) for the minimum number of edge crossings of a graph, and we'll find very quickly that many questions about crossing numbers are wide open. In fact, even the crossing number of $K_{m,n}$ on the plane is open! (An answer to this question was conjectured—and a proof claimed—in the 1950s by Zarankiewicz, and was followed by a 1969 paper entitled "The decline and fall of Zarankiewicz's theorem.")

Homework: Recommended.

Prerequisites: None.

Mixed-Up Sarongs. (🌶, Marisa, 1 day)
If we have ten Mathcampers and ten sarongs, out of all of the 10! ways of matching the campers and the sarongs, what is the probability of matching them all wrong (so that nobody gets their own sarong back? Which is more likely - getting them all wrong, or getting at least one right?

This is a one-off about the Hat-Check Problem, using a sly approach: counting the number of 1-factors in the graph $K_{n,n}$ minus a 1-factor by solving a recurrence relation.

Homework: None.

Prerequisites: None. I'll briefly use the Taylor Series for e^x, but those who haven't seen calculus can just take my word for it.

MARK'S CLASSES

A Tour of Hensel's World. (🌶🌶–🌶🌶🌶, Mark, 1 day)
In one of Euler's less celebrated papers, he started with the formula for the sum of a geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and substituted 2 for x to arrive at the apparently nonsensical formula $1 + 2 + 4 + 8 + \cdots = -1$. More than a hundred years later, Hensel described a number system in which this formula is perfectly correct. That system and its relatives (for each of which 2 is replaced by a different prime number

p), the p-adic numbers, are important in modern mathematics; we'll have a quick look around this strange "world."

Homework: None.

Prerequisites: Some understanding of the idea of convergence of a series.

Exploring the Catalan Numbers. (🌶🌶, Mark, 1 day)
What's the next number in the sequence $1, 2, 5, 14, \ldots$? If this were an "intelligence test" for eighth graders, the answer might be 41; that's the number that continues the pattern in which every number is one less than three times the previous number. If the sequence gives the answer to some combinatorial question, though, the answer is more likely to be 42. We'll look at a few questions that, in fact, give rise to this sequence (with 42), and we'll see that the sequence is given by an elegant formula, which has a lovely combinatorial proof. If time permits, we may also look at an alternate proof using generating functions.

Homework: None.

Prerequisites: None, but at the end, generating functions and some calculus may show up.

Galois Theory Crash Course. (🌶🌶🌶🌶, Mark, 4 days)
In 1832, the twenty-year-old mathematician and radical (in the political sense) Galois died tragically, as the result of a wound he sustained in a duel. The night before Galois was shot, he hurriedly scribbled a letter to a friend, sketching out mathematical ideas that he correctly suspected he might not live to work out more carefully. Among Galois' ideas (accounts differ as to just which of them were actually in that famous letter) are those that led to what is now called Galois theory, a deep connection between field extensions on the one hand and groups of automorphisms on the other (even though what we now consider the general definitions of "group" and "field" were not given until fifty years or so later). If this class happens, I expect to be rather hurriedly (but not tragically) scribbling as we try to cover as much of this material as reasonably possible. If all goes well, we might conceivably be able to get through an outline of the proof that it is impossible to solve general polynomial equations by radicals once the degree of the polynomial is greater than 4. (This depends on the simplicity of the alternating group, which we won't have time to show in this class but which *may* be shown in a separate week 5 class.) Even if we don't get that far, the so-called Galois correspondence (which we should be able to get to, and prove) is well worth seeing!

Homework: Recommended.

Prerequisites: Group theory; linear algebra; some familiarity with fields and with polynomial rings.

Multiplicative Functions. (🌶🌶–🌶🌶🌶, Mark, 2 days)
Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such "multiplicative" functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, and/or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, this class is for you.

Homework: Optional.

Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.)

Perfect Numbers. (🌶, Mark, 1 day)
Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own

divisors, not counting itself. Such integers are called *perfect*, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes – a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.
Homework: None.
Prerequisites: None

Rescuing Divergent Series. (, Mark, 1 day)
Consider the infinite series $1-1+1-1+1-1+\dots$. What is its sum? Maybe $(1-1)+(1-1)+\cdots = 0$, maybe $1-(1-1)-(1-1)\cdots = 1$. At one time mathematicians were quite perplexed by this, and one even thought the issue had theological significance. Now presumably it's nonsense to think that the "real" answer is $\frac{1}{2}$, just because the answers 0 and 1 seem equally good, right? After all, how could the sum of a series of integers be anything other than an integer?
Homework: None.
Prerequisites: A bit of experience with the idea of convergence.

Simplicity Itself: A_n and the "Other" A_n. (, Mark, 2 days)
The monster group (of order roughly $8 \cdot 10^{53}$) gets a lot of "press," but it's not the largest finite simple group; it's the largest exceptional finite simple group. (Reminder: A simple group is one which has no normal subgroups other than the two "trivial" ones; by using homomorphisms, all finite groups can be "built up" from finite simple groups. The complete classification of finite simple groups was a monumental effort that was completed successfully not far into our new millennium.) What about the unexceptional finite simple groups? They come in infinite families, and in this class we'll look at two of those families: the alternating groups A_n and one class of groups of "Lie type", related to matrices over finite fields. (If you haven't seen finite fields, think "integers mod p" for a prime p.) By the way, the simplicity of the alternating groups plays a crucial role in the proof that in general, polynomial equations of degree 5 and up cannot be solved by radicals (there is no "quintic formula").

We'll prove that A_n is indeed simple for $n \geq 5$, and we should be able to prove simplicity for the other class of groups also, at least for 2×2 matrices.
Homework: None.
Prerequisites: Basic group theory and linear algebra; familiarity with finite fields would help a bit but is not needed.

The Cayley–Hamilton Theorem. (, Mark, 1 day)
Take any square matrix A and look at its characteristic polynomial $f(\lambda) = \det(A - \lambda I)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(\lambda) = \lambda^4 - 6\lambda^3 - 5\lambda^2 + 17\lambda - 8$, then compute $f(A) = A^4 - 6A^3 - 5A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" (or "adjugate") of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.
Homework: None.
Prerequisites: Linear algebra, including a solid grasp of determinants (perhaps, but not necessarily, from Will's week 4 class).

The Pruefer Correspondence. (–, Mark, 1 day)
Suppose you have n points around a circle, with every pair of points connected by a line segment. (If

you like, you have the complete graph K_n.) Now you're going to erase some of those line segments so you end up with a *tree*, that is, so that you can still get from each point to each other point along the remaining line segments, but in only one way. (This tree will be a spanning tree for K_n.) How many different trees can you end up with? The answer is a surprisingly simple expression in n, and we'll go through a very nice combinatorial proof.

Homework: None.

Prerequisites: None.

The Riemann Zeta Function. (🌶🌶–🌶🌶🌶, Mark, 2–3 days)
Many highly qualified people believe that the most important open question in pure mathematics is the Riemann hypothesis, a conjecture about the zeros of the Riemann zeta function. Having been stated in 1859, the conjecture has outlived not only Riemann and his contemporaries, but a few generations of mathematicians beyond, and not for lack of effort! So what's the zeta function, and what's the conjecture? By the end of this class you should have a pretty good idea. You'll also have seen a variety of related cool things, such as the probability that a "random" positive integer is not divisible by a perfect square (beyond 1) and the reason that $-691/2730$ is a useful and interesting number.

Homework: None.

Prerequisites: Single-variable calculus, including infinite series. Previous experience with functions of a complex variable may help a bit, but is *not* required.

Mira's Classes

Information Theory and the Redundancy of English. (🌶🌶, Mira, 4 days)
NWSFLSH: NGLSH S RDNDNT!! (BT DN'T TLL YR NGLSH TCHR SD THT)

The redundancy of English (or any other language) is what allows you to decipher the above sentence. It's also what allows you to decipher bad handwriting or to have a conversation in a crowded room. The redundancy is a kind of error-correcting code: even if you miss part of what was said, you can recover the rest.

How redundant is English? There are two ways to interpret this question:

- How much information is conveyed by a single letter of English text, relative to how much could theoretically be conveyed? (But what is information? How do you measure it?)
- How much can we compress English text? If we encode it using a really clever encoding scheme, can we reduce the length of the message by a factor of 2? 10? 100? (But how will we ever know if our encoding is the cleverest possible one?)

Fortunately, the two interpretations are related. In this class, we will first derive a mathematical definition of information, based on our intuitive notions of what this word should mean. Then we'll prove the Noiseless Coding Theorem: the degree to which a piece of text (or any other data stream) can be compressed is governed by the actual amount of information that it contains. We'll also talk about Huffman codes: the optimal way of compressing data if you know enough about its source. (That's a big "if", but it's still a very cool method.)

Finally, we'll answer our original question — how redundant is English? — in the way that Claude Shannon, the father of information theory, originally answered it: by playing a game I call Shannon's Hangman and using it as a way of communicating with our imaginary identical clones!

The class is 4 days long, but you can skip some of the days and still come to the others. Here's how it works:

Day 1: Introduction and definition of information. (Required for the rest of the class.)

Days 2, 3: Noiseless coding and Huffman codes. (The mathematical heart of the class, where we'll prove the Noiseless Coding Theorem.)

Day 4: Shannon's Hangman and the redundancy of English. (You can come to this class even if you don't come on Days 2 and 3 — you just need the material from Day 1.)

Homework: Recommended.

Prerequisites: None. **Please note:** This class is *exactly* the same as the Week 5 class I taught last summer. I'm only offering it again because I heard that new campers wanted more information theory. Please do not vote for this class if you took it last year, even if you liked it. :)

MCMC. (🌶🌶–🌶🌶🌶, Mira, 3–4 days)
MCMC (which stands not for "Mathcamp Mathcamp," but for "Markov Chain Monte Carlo") is a technique for sampling from very complicated probability distributions. That might not sound particularly glamorous, but in fact, MCMC lies at the heart of the Bayesian revolution in statistics that began in the 1990's and is still going strong. The revolution is changing the way scientists use mathematical models in their work. There is literally no branch of the natural or social sciences where researchers aren't currently using MCMC.

In this class, we'll define exactly what MCMC does and why this is so important. We'll analyze it mathematically and see how and why it works. And we'll talk about it in the broader context of the Bayesian revolution, the ongoing replication crisis in science, and why you should not take statistics in high school if you can possibly avoid it.

Note: For people in my MCMC class at MC18, a lot in this class will be review, except that this time I actually intend to prove that Metropolis-Hastings works! For people in my probabilistic models class at MC19, this class will answer the question "so how **does** WebPPL sample from the posterior distribution?" People who were in neither of those classes should have no trouble enjoying the class on its own terms.

Homework: Optional.

Prerequisites: None.

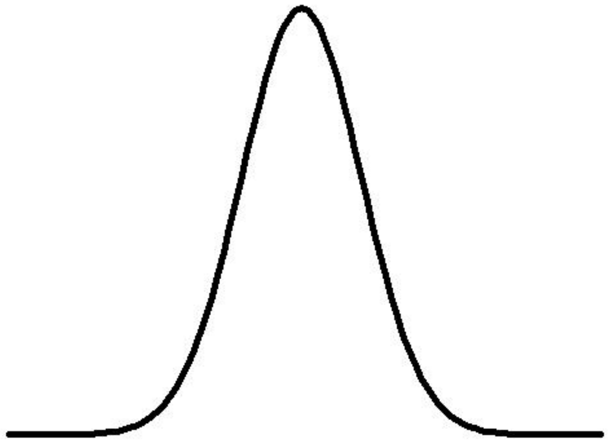
The Bell Curve. (🌶🌶🌶, Mira, 4 days)

> I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error." The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason.

Sir Francis Galton, 1889

Human heights; SAT scores; errors in scientific measurements; the number of heads you get when you toss a million coins; the number of people per year who forget to write the address on a letter they mail. . . . what do all of these (and numerous other phenomena) have in common?

Empirically, all of these phenomena turn out to be distributed according to "the bell curve":



The bell curve, known in the 19th century as the "Law of Error", is now usually called the *normal* or *Gaussian* distribution. It is the graph of the function $e^{x^2/2}/\sqrt{2\pi}$ (scaled and translated appropriately). We will see how Gauss derived this function from a completely backward argument – a brilliant leap of intuition, but pretty sketchy math. We'll see how the great probabilist Laplace explained its ubiquity through the Central Limit Theorem. (Maybe you've learned about CLT in your statistics class … but do you know the proof?) We'll talk about how the normal distribution challenged the nineteenth century concept of free will. Finally, we'll look at some other mathematical contexts in which the normal distribution arises – it really is everywhere!

Homework: Recommended.

Prerequisites: Calculus (comfort with integrals).

The Good and Bad Mathematics of Pennsylvania's Gerrymandering Lawsuit. (🌶–🌶🌶, Mira, 1–2 days)
Partisan gerrymandering has never been successfully challenged in US federal courts, and with the Supreme Court's most recent decision, pretty much all hope is gone.

On the other hand, in 2018, the Pennsylvania Supreme Court declared that Pennsylvania's gerrymandered map violates the Pennsylvania Constitution's guarantee of "free and fair elections". Since many other states have similar guarantees, the focus of anti-gerrymandering activism looks like it's going to switch to state courts.

That all sounds promising, except for one thing. What the PA Court called the "most compelling" argument in the case was based on completely bogus statistics. There was also a really good statistical argument presented to the court, but the judges didn't seem to understand it. (The good argument, made by mathematicians, was complicated; the bad one, made by political scientists, had nice pictures.)

In this class, we'll look at some of the original expert witness testimony presented before the court, the original research papers that both arguments are based on, the opposing side's attempted rebuttal, and the judges' reactions. You will get a sense of both how important and how difficult it can be to communicate mathematical concepts to non-mathematicians. You will also be in a better position to evaluate similar anti-gerrymandering arguments going forward; there are likely to be many in the next few years, and the popular press is likely to make a complete mess out of them.

Homework: Optional.

Prerequisites: None. (In particular, you can take this whether or not you took Assaf's Week 1 class.)

Two Games and a Code. (🌶🌶, Mira, 2 days)
Question: What do the following two games have in common?

(1) You think of a number between 0 and 15. I ask you seven yes/no questions about it. You are allowed to tell at most one lie (or, if you prefer, to answer truthfully throughout). At the end, I'll tell you if and when you lied, and then I'll guess your number! (We'll actually play this in class.)

(2) You and six friends are playing a cooperative game. You are each given a black or white hat at random. As usual, each person can see the color of everyone else's hats and has to guess the color of his or her own. Each of you writes your guess — either "BLACK" or "WHITE" — on a piece of paper, without showing it to anyone else. If a player doesn't want to guess, they also have the option of writing "PASS". Then everyone holds up their papers simultaneously. Your team wins if, among you, you have *at least one correct guess and no incorrect guesses.* ("Passes" don't make you lose, but they don't help you win either.) You are allowed to agree on a strategy before the hats are passed out, but no communication is allowed afterwards. What strategy will maximize your probability of winning, and what is the best probability you can achieve? (We'll play this in class if we have time.)

Answer: The same strategy, based on a perfect code!

Come and find out what that means.

Homework: None.

Prerequisites: None.

Misha's Classes

Geometric Programming. (🌶🌶, Misha, 2 days)
If you ever take a calculus class, I promise that you will spend a day on solving the following problem: what is the largest rectangular region you can fence off with 100 feet of fencing, if one side of the region is already blocked off by a long wall?

But maybe you also already know that you don't need calculus to solve this problem: the AM-GM (arithmetic mean-geometric mean) inequality is enough.

As such problems get tougher, though, your life gets harder. There are many ways to apply the AM-GM inequality, and they all give different answers. Which AM-GM is the best AM-GM? That's geometric programming.

In the end, we will take an optimization problem and turn it into a different optimization problem called its "dual", and then make the two problems fight each other until they're both easy to solve.

Homework: Optional.

Prerequisites: Some calculus (taking derivatives to find a critical point). The AM-GM inequality.

Going in Cycles. (🌶🌶–🌶🌶🌶, Misha, 2 days)
A *knight* is a chess piece that jumps from a square to any other square exactly $\sqrt{5}$ units away. Put one of these in the corner of an 8×8 chessboard. Can it visit every other square of the board exactly once, then come back to the start?

This is an instance of the Hamiltonian cycle problem. In general, it's very hard to solve. We will talk about some ways we can guarantee a solution exists—or quickly demonstrate that it doesn't.

Homework: Optional.

Prerequisites: It will help to be familiar with some graph-theoretic terminology.

Packing Permutation Patterns. (🌶🌶🌶, Misha, 2 days)
Prepare by picking a permutation π and a pattern P. Probabilistically pick $|P|$ pieces of π: perhaps putting them together produces P? Let $\rho_P(\pi)$ be the probability of producing P.

To *pack* P in π, puff up this probability, making P as plentiful as possible. We will ponder the packing problem for $P = 132$ (and plenty of its pals) using a progression of powerful problem-solving procedures.

(For returning campers: this material is a subset of my class on flag algebras last year.)

Homework: Recommended.

Prerequisites: None.

The Simplex Algorithms. (🌶🌶–🌶🌶🌶, Misha, 2–4 days)
Let's say you already know how to solve systems of equations $A\mathbf{x} = \mathbf{b}$.

Negative numbers are no fun, so let's assume $\mathbf{x} \geq \mathbf{0}$. Now we have plenty of solutions, but what's the best one? Specifically, what solution maximizes a linear objective function $\mathbf{c} \cdot \mathbf{x}$?

We'll consider the simplex method, which starts with one solution $\mathbf{x}$ and improves it until it gets to the best solution. But maybe that's boring; instead, how would you like to start with the best $\mathbf{x}$, period... and make it more and more of a solution until it is one?

What if you realize you forgot an equation, and would like to add it back in to fix things? And how can we use that to solve the problem when $\mathbf{x}$ is required to be an integer?

(I don't know how much of that we'll get through in 2 days or in 4 days; I haven't planned things out that far. I promise that there is enough cool stuff for any number of days.)

Homework: Recommended.

Prerequisites: Linear algebra: you should be able to solve a system of linear equations, and in the case of infinitely many solutions, parametrize them by the values of the free variables.

Rice's Classes

Communicating Set Equality. (🌶🌶–🌶🌶🌶, Rice, 1–2 days)
Apurva and Ben have subsets A and B, respectively, of $\{1, 2, \dots, n\}$. They want to figure out if their sets are the same. The naïve way for them to do this is for Apurva to send his entire set to Ben for comparison. However, this method has a high communication cost: it requires n bits to be communicated. Is there a way for Apurva and Ben to succeed while communicating fewer than n bits? Come find out!

Homework: None.

Prerequisites: None.

Procrastination. (🌶, Rice, 1 day)
If you're like me, you've had the experience of putting off all your homework until the last minute, despite knowing the whole time that pacing yourself would make your life way easier. Come learn about an elegant mathematical model of procrastination, and about how simply realizing that you're going to procrastinate will (usually) cause you to make much wiser decisions.

Homework: None.

Prerequisites: None.

Shiyue's Classes

A (Hopefully) Gentle Invitation to Algebraic Geometry. (🌶🌶🌶–🌶🌶🌶🌶, Shiyue, 3–4 days)
Algebraic geometry is the study of zeros of polynomials. For this invitation to be gentle, we will look at examples of objects that provoke algebraic geometers' interest and give us playgrounds for reasoning about the following questions: how do we translate a geometric shape into algebraic symbols, how do we endow a topology to a geometric shape using those algebraic structures, how are topological

properties exhibited via algebraic properties, etc. Formally, the course will touch on the spectrum of a ring, Zariski topology, irreducibility, connectedness, Noetherian-ness, open sets and closed sets of an algebraic variety, and Hilbert's Nullstellensatz (might not be able to prove it).

Homework: Recommended.

Prerequisites: Ring theory, Linear Algebra.

How to Blow Up the Plane? (🌶🌶🌶, Shiyue, 1–2 days)
Two mathematicians were boarding an airplane. One of them was talking to the other about blowing up the plane. This quickly caught the flight attendant's attention. The flight attendant asked, "Are you trying to blow up the plane?" The mathematician showed their papers of scratch work, only to confuse the flight attendant even more! The mathematician soon was brought back to security for further investigation.

Talking about blowing up the plane while boarding sure seems dangerous. But how about talking about it at Mathcamp? – How to blow up the plane? Worry not. We will not blow up any actual aircraft to practice knowledge obtained in the class; on the other hand, we will learn about blowing up a variety: a sequence of surgeries to resolve singularities of the variety. The first interesting example is the blow up of the affine plane. We will think about an algebraic structure called a blow-up algebra, given a commutative graded ring and an ideal. We will also briefly describe singularities of a variety by studying the cotangent bundle and tangent bundle defined in a commutative algebraic sense. Our favorite example will be the singular cubic curve that looks like a fish! We will see that blowing up of the plane does resolve the singularity of the fish by turning it into a beautiful curve.

Homework: Recommended.

Prerequisites: Ring theory, knowledge about localization is helpful but not required.

The Joy of Finite Sets and Injective Maps. (🌶🌶–🌶🌶🌶, Shiyue, 2–3 days)
Who don't love infinite-ness? But finite sets deserve some careful thought as well. In particular, when you only consider the injective maps between finite sets, they actually form an interesting category that serves as the skeleton of many other algebraic/topological structures that arise in nature. What's more fun is that you can now organize your favorite objects that you learned at Mathcamp, say finite dimensional vector spaces, homology, cohomology, the polynomial ring over $\mathbb{C}$ with n variables, or the dictionary (moduli) shape (space) of manifolds with n marked points, using this category that consists only of finite sets and injective maps and say some interesting things about them. This belongs to a new and exciting math field called representation stability, which all starts with finite sets and injective maps!

Homework: Recommended.

Prerequisites: Linear algebra, group theory, some knowledge about rings and topological spaces will be helpful but not crucial.

Shiyue and Andrew's Classes

Young Tableaux and Probability. (🌶–🌶🌶, Shiyue and Andrew, 1–2 days)
Guess who is super excited about Young tableaux in MC2019? In this course, we will give a different proof for the Hook Length formula, using probability. Maybe in Week 2, you have seen that the Hook Length Formula gives us a way of counting the number of standard Young tableaux, which turns out to be the dimension of irreducible representations of the symmetric group S_n. In Week 3, you might have seen a complicated proof of the Hook Length Formula which required building up several identities involving generating functions. In this course, we will use probability to think about hook walks and the recursive relations between standard Young tableaux and their sub-standard Young tableaux

(which is a very natural idea when one tries to build up a standard Young tableaux). Eventually, we will give an elegant derivation of the Hook Length Formula.

Homework: Recommended.

Prerequisites: Induction, some knowledge about probability will be helpful but not necessary.

Susan's Classes

Continued Fractions. (🌶🌶–🌶🌶🌶, Susan, 1–2 days)
Do you know what's great? Number theory! Do you know what else is great? Analysis! Still with me here? Then continued fractions might just be for you!

A continued fraction looks like this:

$$2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}}}}}}}}}}$$

This continued fraction is equal to $1+\sqrt{2}$. But what does this mean? Do all continued fraction expansions converge to real numbers? Does every real number have a continued fraction expansion? Are these expressions unique? Who are these crazy people, and why are they all wearing math capes? Come to this class to find the answers to (most of) these questions!

Homework: Optional.

Prerequisites: None.

Finite-ness. (🌶🌶🌶🌶, Susan, 1 day)
What does it mean for a set to be finite? One possible definition is that the set can be put into bijective correspondence with a natural number. But is that really the definition we want? It seems kind of inelegant to just take all the finite sets we already know and say, "That's all, folks!" Here are a couple possible definitions of finite-ness that don't reference the natural numbers:

- **Dedekind's Definition:** A set S is finite if every injection $f : S \to S$ is a surjection.
- **Tarski's Definition:** A set S is finite if every total-order on S is a well-order.
- **Kuratowski's Definition:** A set S is finite if and only if it is an element of the smallest class of sets containing $\emptyset$ and all $\{s\}$ for $s \in S$ which is closed under unions.

All of these definitions are equivalent to the natural numbers definition… as long as you believe the Axiom of Choice. Otherwise, only one of them is. And also the proofs are surprisingly subtle. Wait… is finiteness hard?

Homework: None.

Prerequisites: None.

Martin's Axiom. (🌶🌶🌶🌶, Susan, 2–3 days)
Induction is awesome! You know what else is awesome? Posets! In fact, my one problem with induction is that I only ever get to do it on well-ordered sets. First I prove my result for zero, then for one, then for two, etc. If I'm lucky, maybe I also get to prove my result for infinite ordinals. That's kinda cool. But nowhere near as cool as it could be—I wanna do induction on posets!

Enter Martin's Axiom: an extra-set-theoretic axiom that allows us to do just that. I take the poset of all partially-proved results, and then glue them haphazardly into a giant, glorious Franken-Sign of a proof that gives us an uncountable result.

Wanna see how it works? Come to this class!

Homework: Recommended.

Prerequisites: None.

Special Aronszajn Trees. (🌶🌶🌶, Susan, 1 day)
An Aronszajn tree is called "special" if it is the union of countably many antichains. In this class we will construct a special Aronszajn tree. No extra-set-theoretic axioms required!

Homework: None.

Prerequisites: Infinite Trees Week 1.

Suslin Lines and Suslin Trees. (🌶🌶🌶, Susan, 1 day)
In Infinite Trees, we proved that if a Suslin Line exists, then a Suslin Tree exists. The converse statement turns out to be trickier. Wanna see it? Come to this class!

Homework: None.

Prerequisites: Infinite Trees Weeks 1 and 2.

The Mathematical ABC's. (🌶, Susan, 1 day)
Ever noticed how no one ever uses "d" as a variable in a calculus class? Or how "i," "j," and "k" are perfectly fine natural numbers if you're doing combinatorics, but not if you're studying noncommutative rings? In this class we'll go over the alphabet from A to Z, and talk about how to use (or not to use) these letters we know and love. If we have time, maybe we'll learn Mathematical Greek!

Homework: None.

Prerequisites: None.

Tim!'s Classes

A Proof of the Sensitivity Conjecture. (🌶🌶🌶, Tim!, 2–4 days)
The Sensitivity Conjecture was a big open problem in theoretical computer science for thirty years. I spent some time thinking about it earlier in grad school. I would have offered to show you the proof earlier in camp (you know, back when I was at camp), but when we were making the four-week schedule, there was no proof yet (as far as the world knew)! In fact, Hao Huang posted his proof during Week 2 of camp. The proof is amazingly short, tantalizingly simple and very beautiful, yet also clever enough that it eluded prominent mathematicians for decades. It uses ideas from graph theory and linear algebra, by looking at eigenvalues.

More about the conjecture itself: The Sensitivity Conjecture is (or I should say, *was*) a conjecture about the complexity of boolean functions. Theoretical computer scientists care a lot about boolean functions (in part) because computers operate with boolean values. They care about complexity because they care about how complicated their programs are (in various senses of the word "complicated" — how long they take to run, how much memory they use, etc.).

There are many ways to measure complexity. Many of these complexity measures are known to be equivalent (or more precisely, *polynomially related*) to each other; these include *degree*, *(deterministic) decision tree complexity*, *bounded-error randomized decision tree complexity*, *certificate complexity*, *block sensitivity*, *approximate degree*, and *quantum decision tree complexity with bounded error*. This is both surprising (because these complexity measures have very different-sounding definitions) and a relief (because if all these complexity measures disagreed with each other, then who would you trust?).

However, there was one complexity measure — called *sensitivity* — that was not known to be in the club. The Sensitivity Conjecture asserted (sensibly) that sensitivity is polynomially related to all those other complexity measures.

It seems fortuitous that this problem, which I've spent time on and which has been open for so long, was solved during Mathcamp and has such an elegant proof. If I were still at camp, I'd definitely propose a Week 5 class on it. Right now, though, I am in the Midwest, land of cows and rolling cornfields. But you know what... someone hold my cheese; I'm on my way!

In this class, we'll follow the story of the Sensitivity Conjecture from its development up to and including its proof.

Homework: Optional.

Prerequisites: You should know what *degree* means in graph theory, and what an *eigenvalue* is in linear algebra.

Calculus without Calculus. (🌶🌶, Tim!, 1–2 days)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Lizka is 5 cubits tall and Eric is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Lizka's head to the top of Eric's head that touches the ground in the middle. What is the shortest length of string you can use?
- Assaf rides a bike around an elliptical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 20 meters way along the shoreline throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- Where in a movie theater should you sit so that the screen takes up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Homework: Recommended.

Prerequisites: Some calculus will be useful for context, but we won't actually use calculus (that's the point).

Dynamic Programming. (🌶🌶, Tim!, 1–2 days)
Dynamic programming is a reliable way to get efficient algorithms to all sorts of problems. Here are a few such problems:

- If you have k eggs and an n story building, find a strategy to determine the highest floor of the building you can drop an egg from without it breaking.
- If you have a bunch of text that you would like to typeset (for instance, in LaTeX), how should you break the text into lines so that it is the most aesthetically pleasing?
- Given a collection of items each with a given integer weight and value, find a set of items that weighs at most 100 kilograms and has the maximum possible value.

- Given a level of Super Mario Bros., determine whether it is possible to beat the level — assuming that any part of the level that scrolls off-screen gets reset (without this assumption, the problem is NP-hard. . .).

Come find some quick algorithms!

Homework: Recommended.

Prerequisites: None.

Error-Correcting Codes. (🌶🌶, Tim!, 1–2 days)
Ben and I are secretly planning to take over the camp. Shhh, don't tell anyone; it's a secret! Of course, we have a secret code. Ben sends me messages by knocking or tapping four times on my door. For instance, knock-tap-knock-tap means "I've hacked the class schedule and replaced every class with Category Theory", tap-tap-tap-knock means "Tonight is the night to steal Susan's idol of power", and so on.

One night, Ben knock-knock-knock-knocks on my door, but I mishear it as knock-tap-knock-knock. So, instead of the message "Let me in", I respond to the message "May Day! Burn down the dorms!".

This is a setback.

The problem is that if I mishear even one of the knocks, I get the wrong message-phrase. But there is a solution! There are codes that are error-detecting — if I mishear one of the knocks/taps, I'll know just from what I heard that something has gone wrong. Even more amazingly, there are codes that are error-correcting — if I mishear one of the knocks/taps, then the knocks/taps I do hear will tell me exactly what I misheard and what the correct message was supposed be. It seems too good to be true, but the simplest error-correcting codes are easy to construct, and the best ones are used everywhere. They're used in DVDs so the discs can play even if scratched. They are used on books (for the ISBN and barcode on the back). They are used in computer systems all over the world (and all over the solar system — the Cassini probe that crashed into Saturn in 2017 really wanted to make sure its photos and data get transmitted back to earth correctly before its mission ended).

We'll see the power and magic of these codes!

Homework: Optional.

Prerequisites: None.

Véronique's Classes

Auslander–Reiten Quivers. (🌶🌶🌶🌶, Véronique, 3–4 days)
A module over an algebra is a generalization of the notion of vector space over a field. To actually compute the indecomposable modules and the homomorphisms between them, the notions of irreducible morphisms and almost split sequences are particularly useful. In this class, we introduce the notions of irreducible morphisms and almost split morphisms, Finally, we define a new quiver, called the Auslander–Reiten quiver; this quiver allow us to easily visualize the irreducible the morphisms between modules over an algebra and the relations between them.

Homework: Recommended.

Prerequisites: None, but a good intuition in ring theory is an advantage.

Conway-Coxeter Frieze Patterns. (🌶–🌶🌶, Véronique, 2–3 days)
Conway and Coxeter introduced frieze patterns in 1973: a frieze pattern over a field K is a map that associates to specific points in $\mathbb{Z}^2$ elements of K following a given rule. In this class, you will study properties of frieze patterns (periodicity, construction from a given diagonal, construction from a given row). More important, you will study the connection between frieze patterns and triangulations of an

n-gon. Indeed, the number of triangles incident to each vertex appear in a frieze pattern, as well as a labeling of the vertices of the triangulated n-gon based on the triangles they share.

Homework: Optional.

Prerequisites: None.

Véronique and Kayla's Classes

Laurent Phenomenon. (🌶🌶–🌶🌶🌶, Véronique and Kayla, 2–3 days)
The Laurent phenomenon is a fundamental result in cluster algebra that states that any cluster variable can be expressed as Laurent polynomial (a polynomial divided by a monomial) in any cluster. In this class, we will discuss a proof of this theorem.

Homework: Recommended.

Prerequisites: None, but you should know what a mutation in cluster algebra is. If it is not the case, ask Véronique or Kayla (or even Kevin)!

Markov Triples and Continued Fractions. (🌶🌶–🌶🌶🌶, Véronique and Kayla, 3 days)
Markov numbers are integers that appear in the solution triples of the Diophantine equation, $x^2+y^2+z^2 = 3xyz$. A conjecture, unproven for over 100 years, states that a Markov triple $\{x, y, z\}$ is uniquely determined by its maximum. In this class, we prove a statement related to this conjecture to determine an ordering on subsets of the Markov numbers based on their corresponding rational. Indeed, there is a natural map from the rational numbers between zero and one to the Markov numbers. The proof uses the cluster algebra of the torus with one puncture and a resulting reformulation of the conjectures in terms of continued fractions.

Homework: Recommended.

Prerequisites: None, but cluster algebra is an advantage.

Will's Classes

Categories and Universal Properties. (🌶🌶🌶, Will, 2 days)
Throughout mathematics, certain constructions appear over and over again—such as products, quotients, or free things—in a variety of different contexts. The definitions differ depending on whether we're talking about groups, rings, topological spaces, or something else, but there's a uniting thread that makes each definition the "right" one: they satisfy universal properties.

A universal property characterizes an object entirely by how it maps to other objects. This is one of the main motivations for category theory: by defining things in this way, we can often streamline proofs and uncover hidden connections to other types of objects. For, example, we'll see that the direct product of two groups and the maximum of two numbers are secretly the same thing!

The first day of this class will examine examples of universal properties in many different contexts, and look at their similarities and differences. The second day will step back and look at some of the general category theory at play.

Homework: Recommended.

Prerequisites: Exposure to groups and rings. Familiarity with topological spaces, vector spaces, posets, and/or graphs is not necessary but will help (the more, the better).

Electrifying Random Trees. (🌶🌶, Will, 2 days)
Consider a graph, and a particular edge in this graph. A natural question to ask is: how important is this edge in connecting the graph together? Here are two ways we might quantify this importance:

1) The spanning trees of the graph give all the ways of minimally connecting the vertices using a subset of the edges. So look at all the spanning trees, and figure out how many of them contain our edge. Put another way, what is the probability that a random spanning tree contains the edge?

2) Build your graph out of 1-ohm resistors. Hook up a 1-volt battery to the ends of your edge, and then measure how many amps of current are passing through that edge (as opposed to going through other paths in the graph).

These two numbers are actually the same! In this class, we'll see why. On the way, we'll see how one might generate a random spanning tree in the first place.

Homework: Recommended.

Prerequisites: Familiarity with the language of graphs (i.e., what a tree is; what the degree of a vertex is).

Everything Is a Quiver Grassmannian. (🌶🌶🌶🌶, Will, 1–2 days)
The Grassmannian $\mathrm{Gr}(k, V)$ is a "dictionary space" (as in J-Lo's colloquium) in which every point represents a k-dimensional subspace of the vector space V. It can be realized as a projective variety: the solution set of a system of polynomial equations in projective space. Grassmannians have many special properties that set them apart from other varieties.

By analogy, we can define a quiver Grassmannian, whose points now correspond to families of vector subspaces which are mapped into each other by fixed linear maps. (That is, its points are subrepresentations of a fixed quiver representation.) We might ask: what special properties do these spaces have?

The somewhat anticlimactic answer is that they have no special properties—not because they're individually boring, but because *every* projective variety can be described as a quiver Grassmannian! This class will outline what this means and examine the surprisingly simple proof, which uses an important tool from algebraic geometry known as the Veronese embedding.

Homework: Optional.

Prerequisites: Linear algebra (ranks and kernels of linear maps). Prior exposure to projective space and/or homogeneous polynomials would help. You don't need to have taken Quiver Representations.

Matroids and Greed. (🌶🌶–🌶🌶🌶, Will, 1–2 days)
Kruskal's algorithm is a way of choosing, in a graph with weighted edges, a spanning tree of minimal weight. It's a very simple greedy algorithm: just keep adding the least-weight edge you can, subject to the condition that you don't form a cycle.

A matroid is a mathematical structure that captures the idea of "independent subsets of a set." Key motivating examples include linearly independent subsets of a finite set of vectors, as well as forests and trees in a graph.

And in fact, just as in the case of graphs, we can use a greedy algorithm to find minimum-weight independent subsets of any matroid. But more is true: matroids are actually precisely characterized by the fact that a greedy algorithm works in this way. In this class, we'll look at a few equivalent ways of defining matroids, and then see why the greedy algorithm property is one of them.

Homework: Recommended.

Prerequisites: Know what "linear independence" is. Know what a tree (in the context of graph theory) is.

Will and Kevin's Classes

Hyperplane Arrangements. (🌶🌶, Will and Kevin, 2 days)
Suppose we cut space with a collection of planes. How many pieces does the space get divided into?

We can answer this question by looking how the planes intersect. To arrive at this answer involves the use of some handy combinatorial tools, including the Möbius function.

Once we have an answer to this question, we'll connect it to graphs, and prove a remarkable theorem about what it means to color a graph with -1 colors.

Homework: Recommended.

Prerequisites: Know what a graph coloring is.