CLASS DESCRIPTIONS—WEEK 4, MATHCAMP 2019

Contents

9:10 Classes

Long Live Determinants! (Will, Wednesday–Saturday)
The determinant of a square matrix is a polynomial in its entries which contains a lot of important information. It's sometimes fiddly to work with: when Sheldon Axler wrote his highly-regarded textbook *Linear Algebra Done Right*, he adopted the motto of "Down with determinants!", arguing that they are computationally impractical and pedagogically unhelpful.

On the other hand, determinants are really cool by themselves! In this class, we'll look at the several different definitions of determinant, and see that, in spite of its arbitrary-seeming formulas, it is an important and natural feature of a matrix. Then in the rest of the class, we'll look at some remarkable identities involving the determinant: as time allows, we'll count things, prove magic formulas, and/or index lines in projective space.

Chilis:

Homework: Recommended.

Prerequisites: Basic familiarity with linear algebra (linear maps, manipulating matrices). Abstract vector spaces and eigenstuff would help. You do not need to know what a determinant is.

Musical Lattices. (J-Lo, Wednesday–Saturday)
Ignoring the difference between black and white keys, a piano is essentially just a set of equally spaced points on a line: press a point, get a note. Each point plays a note $2^{1/12}$ times the frequency of the previous one. But where did this particular spacing come from? Why 12? More importantly, why should all the music we want to create be built out of a single interval in the first place?

What if, instead, our piano went in *two directions*: say, steps along the x-axis go up perfect fifths, and steps along the y-axis go up octaves? And if we wanted to play major thirds too, we could make a *three-dimensional piano*![1]

But why should we even stop there — we can make n-*dimensional pianos*.

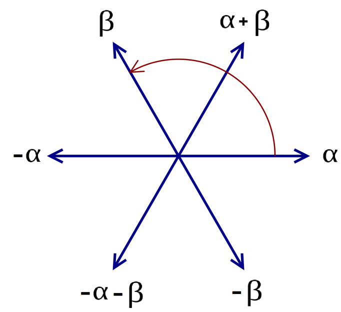
This class will explore the properties of n-dimensional pianos (more commonly known as "lattices"). Through a combination of lecture and guided exercises, we will study Diophantine approximation, Minkowski's Theorem, simultaneous approximation of real numbers, and more, in order to discover facts about musical scales.

Chilis:

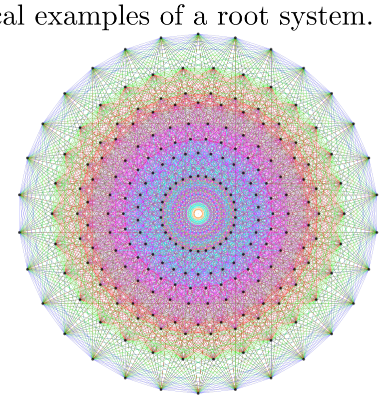
[1] `http://stanford.edu/~jonlove/3dpiano.cdf`

Homework: Recommended.
Prerequisites: None. (The concepts may seem more familiar to you if you've taken linear algebra or group theory before, but everything we need will be developed from scratch).
Cluster: The Power of Groups.

Root Systems. (Kevin, Wednesday–Saturday)
Root systems are arrangements of vectors in some Euclidean space that satisfy certain basic reflection and projection properties. Here's an example:



This is A_2, one of the prototypical examples of a root system. Here's another example:



This is (a two dimensional projection of) the infamous E_8 root system, one of the few exceptions to an otherwise elegant classification of root systems.

Root systems are an important tool to understand Lie algebras and Lie groups, but in this class, we'll focus on their combinatorial properties and their beautiful representation by Dynkin diagrams[2].

Chilis: 🌶🌶
Homework: Recommended.
Prerequisites: None.

The Mathematics of Fairness. (Mira, Wednesday–Saturday)
This class will be a hodgepodge of independent topics related in some way to fairness. Days 1 and 2 will be at least partially active learning; Days 3 and 4 will be interactive lecture. Come for as many days as you like!

Days 1 and 2: Apportionment. The US Constitution mandates that "representatives … shall be apportioned among the several states … according to their respective numbers." This is usually taken

[2]If you took Cluster Algebras, you might have seen the exact same Dynkin diagrams... even though what we do in this class has nothing to do with cluster algebras!

to mean that the number of representatives in each state should be proportional to its population. But exact proportionality is not possible: for example, California cannot have 54.37 representatives. (Europeans face the same problem when they have to figure out how many representatives each party should have in Parliament.)

This is the problem of apportionment – and it is a lot trickier and more interesting than might appear at first glance. Over the course of US history, Congress went through five different apportionment methods, always accompanied by fierce political debates and sometimes leading to paradoxes that no one expected. The method that we currently use was proposed by a Harvard mathematician named Huntington in 1921 and adopted by Congress on the recommendation of the US Academy of Sciences! The math behind it is surprisingly non-trivial.

Day 3: An ancient fair division problem. Here is a passage from the Mishnah, the 2nd century codex of Jewish law:

> *A man has three wives; he dies owing one of them 100 [silver pieces], one of them 200, and one of them 300.*
>
> *If his total estate is 100, they split it equally.*
>
> *If the estate is 200, then the first wife gets 50 and the other two get 75 each.*
>
> *If the estate is 300, then the first wife gets 50, the second one 100, and the third one 150.*
>
> *Similarly, any joint investment with three unequal initial contributions should be divided up in the same way.*

For 1800 years, this passage had baffled scholars: what could possibly be the logic behind the Mishnah's totally different ways of distributing the estate in the three cases? Then, in 1985, a pair of mathematical economists produced a beautifully simple explanation based on ideas from game theory. They showed that for any number of creditors and for any estate size, there is a unique distribution that satisfies certain criteria, and it turns out to be exactly the distribution proposed in the Mishnah. The proof is very cool, based on an analogy with a simple physical system.

Day 4: TBD. I have a bunch of ideas, and will also take student preferences into account.

Chilis: 🌶

Homework: Optional.

Prerequisites: None.

Cluster: Math and the Real World.

Tychonoff's Theorem. (Ben, Wednesday–Saturday)

The topological idea of "compactness" is, roughly speaking, a finiteness condition. However, while the infinite product of finite sets is not finite, it turns out that the infinite product of compact spaces is still compact. This surprising result, called "Tychonoff's Theorem," has applications not only in topology but also in analysis, logic, and graph theory.

This course will prove Tychonoff's Theorem and then show some of its applications to various areas of mathematics.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: Point-set topology (know what it means for a topological space to be compact).

10:10 Classes

Counting, Involutions, and a Theorem of Fermat. (Mark, Tuesday)

Involutions are mathematical objects, especially functions, that are their own inverses. Involutions show up with some regularity in combinatorial proofs; in this class we'll see how to use counting and an involution, but no "number theory" in the usual sense, to prove a famous theorem of Fermat about primes as sums of squares. (Actually, although Fermat stated the theorem, it's uncertain whether he had a proof.) If you haven't seen why every prime $p \equiv 1 \pmod 4$ is the sum of two squares, or if you would like to see a relatively recent (Heath-Brown, 1984, Zagier 1990), highly non-standard proof of this fact, do come!

Chilis: 🌶🌶

Homework: None.

Prerequisites: None.

Game Theory. (Kayla, Tuesday–Saturday)

This class will take an economical approach to the study of game theory. We will explore ideas including optimal strategies, iterated deletion of dominated strategies, Nash equilibrium, etc. The aim of the course is to play some classical games and formalize the idea of strategizing to win a game!

Chilis: 🌶

Homework: Optional.

Prerequisites: None.

Randomized Algorithms. (Bill, Wednesday–Saturday)

The only thing I love more than algorithms is *randomized* algorithms. In this class we'll see examples where a little bit of randomization, used in just the right way, can let us solve algorithmic problems that are otherwise unapproachable (or very difficult). Examples of questions we'll answer include:

- Given a binary sequence x and another shorter binary sequence p, how quickly can I determine whether p appears as a contiguous subsequence of x?
- Given a graph $G = (V, E)$, how do I efficiently split the vertices into two parts V_1, V_2 that minimize the number of edges between V_1 and V_2?
- How do I route messages on a computer network in order to minimize network congestion?

In the last day of the class, we'll analyze a special case of the so-called constructive Lovász Local Lemma. In doing so, we'll see one of the weirdest and coolest algorithm analyses I know, in which we argue an algorithm's correctness using the fact that random sequences of zeros and ones are difficult to compress.

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: Familiarity with probability. Familiarity with Big-O notation would be useful but not necessary. No knowledge of Chernoff Bounds or advanced probability is required.

Cluster: Algorithms.

Reciprocity Laws in Algebraic Number Theory. (Eric, Tuesday–Saturday)

If I give you an integer polynomial $f(x)$, it may factor mod p for some primes p but not for others. How can we determine for which primes a given polynomial factors? What sorts of answers can we expect?

This turns out to be super hard to answer in general. One of the goals of modern number theory is to give good general answers to this and related questions. I hope to give you a tour of this landscape of algebraic numbers, zeta functions, and Galois groups.

We'll explore this landscape starting in the foothills with quadratic reciprocity, testing our mettle on the slopes of class field theory, and we'll try to gain a glimpse of the peaks of the Langlands mountains by the end.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: Algebraic number theory to the level of knowing that every prime number factors into a product of prime ideals in the ring of integers of a number field. You should be familiar with the statement of quadratic reciprocity. I will introduce a few pieces of Galois theory as necessary.

Cluster: Algebraic Number Theory.

Riemann Surfaces. (Apurva, Tuesday–Saturday)
Riemann surfaces were invented by Riemann to understand inverses of complex functions. In this class, we'll explore Riemann's ideas of gluing pieces of the complex plane together and the theory of ramified coverings. We'll prove the Riemann-Hurwitz formula and as an application prove Fermat's last conjecture for polynomials in one variable, namely that

$$(x(t))^p + (y(t))^p = (z(t))^p$$

has no non-trivial solutions if $p > 2$.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Basic knowledge of complex analysis – complex analytic functions, open mapping theorem, Laurent and Taylor series. We'll also need basic notions from topology like continuity and compactness.

Cluster: Complex Analysis.

Young Tableaux and Enumerative Geometry. (Shiyue, Tuesday–Saturday)
Enumerative geometry is a subfield of algebraic geometry where mathematicians are interested in asking "How many A are there such that B?" where A is usually algebraic varieties and B is some geometric conditions. In the 19th century, Hermann Schubert raised a series of such questions like "how many lines in $\mathbb{P}^3$ can pass through 4 generic lines?" If you know the answer to this question (the answer is two), how about proving it? Hilbert asked in his 15th problem to systemize our knowledge of such questions, and hence the subfield of algebraic geometry and combinatorics – Schubert Calculus. To get there, we will start from projective geometry, basic facts about intersections of projective varieties, basic intersection-theoretic objects such as Grassmannians, and Schubert varieties. The meat of this course will be seeing how Littlewood-Richardson rule combined with counting semistandard Young tableaux will give us a proof of the question above, and a few other examples.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Linear algebra.

Cluster: All About Young Tableaux.

11:10 Classes

Bhargava's Cube. (*Dave Savitt*, Tuesday–Thursday)
One of the fundamental problems in number theory is to determine which integers are represented by a given polynomial: which integers have, say, the form $x^n + y^n$, or $x^3 + y^3 + z^3$, or $x^2 + xy + 2y^2$, and so forth. In this class we will focus on polynomials of the latter kind: quadratic polynomials in two variables. First studied systematically by Gauss, remarkably such polynomials are still generating new mathematics of the highest caliber, for example leading recently to the Fields-medal-winning work of Manjul Bhargava. In this class we will learn enough of the theory of binary quadratic forms to gain

an appreciation for one of Bhargava's key contributions, the remarkable "Bhargava cube" – an idea that Gauss missed!

Chilis:

Homework: Recommended.

Prerequisites: None.

Building Mathematical Sculptures. (*Zach Abel*, Tuesday–Saturday)
Come transform ordinary items into extraordinary geometric sculptures! In these small yet intricate construction projects (that you'll take home when complete), we will assemble flexible paperclip cubes, precarious pencil weavings, gorgeous drinking straw jumbles, and more! Browse `http://zacharyabel.com/sculpture/` for examples of the types of projects this course may feature. Assembling these mathematical creations requires scrutiny of their elegant mathematical underpinnings from such areas as geometry, topology, and knot theory, so come prepared to learn, think, and build!

Chilis:

Homework: Recommended.

Prerequisites: None.

Magic. (*Don Laackman*, Friday–Saturday)
It turns out that you can give a quite precise meta-mathematical definition of what a magic trick is. For magic tricks involving deception or misdirection, there isn't much mathematical content, but there is a huge world of magic tricks where not knowing what you don't know about math can quite literally generate magic. In this class, we'll take a look at a couple of my favorite mathematical magic tricks, consider what they have in common, and even come up with some tricks of our own!

Chilis:

Homework: Recommended.

Prerequisites: None.

Representation Theory of Associative Algebras. (Véronique, Wednesday–Saturday)
Representation theory is a branch of mathematics that studies abstract algebraic structures (in our case, algebras) by representing their elements as linear transformations of vector spaces, and studies modules over these abstract algebraic structures. In this class, we will define modules and algebras, in particular, path algebras. Then, we will prove how modules on a path algebra are equivalent to quiver representations, studied in Wil's class. The class on quiver representation is not an absolute prerequisite to this class, but you should definitely know what a quiver representation is. If it is not the case, ask me or Will!

Chilis:

Homework: Recommended.

Prerequisites: Course on quiver representations (just need to know the definition of a quiver representation)

Cluster: The Cluster Algebra Cluster.

The Hopf–Poincaré Index Formula. (Assaf, Tuesday–Saturday)
I bet you've heard of hairy balls, and how the spheri-ness of the sphere prevents one from combing it. That's really weird when you think about it in those terms. How could the global structure of the sphere be related to combability?

This class will discuss surfaces, Euler characteristics, and vector fields, and the truly surprising relation between them. Along the way, we'll discuss curves on surfaces, homotopy groups, winding numbers, and the classification of surfaces.

Chilis:

Homework: Recommended.

Prerequisites: Some calculus and topology in $\mathbb{R}^n$ would be helpful, but is not necessary.

Cluster: Topology.

Wedderburn's Theorem. (Mark, Tuesday)

You may well have seen the quaternions, which form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$ with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j.$$

Have you seen any examples of *finite* division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Chilis:

Homework: None.

Prerequisites: Some group theory; knowing what the words "ring" and "field" mean. Familiarity with complex roots of unity would help.

Zeta Functions. (*Sachi Hashimoto*, Tuesday–Saturday)

Many problems in mathematics, like Fermat's last theorem, which asks which integer nth powers are the sums of two nth powers, are deep statements about zeta functions. The most famous zeta function is the Riemann Zeta Function, whose values at special integers can be computed; Euler showed that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Special values of zeta appear in another millennium problem, the BSD conjecture, which relates the zeta value to the number of points on that curve. These statements about zeta functions are truly mysterious. We will meet a few zeta functions and contemplate their mystery. In concrete terms, zeta functions count the number of points of a geometric object. Throughout the class, we will make use of the open source programming language Sage, a Python-based language, but no programming experience is necessary.

Chilis:

Homework: Recommended.

Prerequisites: Group theory; quadratic reciprocity.

1:00 Classes

(Dys)functional Analysis. (*Viv Kuperberg*, Tuesday–Saturday)

Let's say you want to do linear algebra, but you're working with an infinite dimensional vector space. Many things get much worse very quickly! We will talk about some of those things, including: different notions of length and how they interact with each other; bases and spanning sets, and how the span of a set depends on the order you write it in; dual spaces, and when you can and can't say that a vector space is its dual's dual; linear transformations and how they can misbehave; and more! We'll also talk about some examples of these spaces as they're used and, in particular, the joy of calling functions vectors.

Chilis:

Homework: Recommended.

Prerequisites: This doesn't need much calculus, but we will write down some infinite series and even an integral or two.

Counting Points over Finite Fields. (*Aaron Landesman*, Tuesday–Saturday)
How many invertible matrices are there? How many subspaces of a vector space are there? How many squarefree polynomials of degree d are there? How many points does a curve have? You might guess the answer to all these questions is "infinitely many". And indeed, you'd be correct over the real or complex numbers. But what if you work over a finite field, such as the integers modulo a prime p? Then there are only finitely many of these objects. In this class, we'll learn how to count them.

After answering the above questions, we'll take the occasion to learn some algebraic geometry in order to define and explore the notion of zeta functions over finite fields. Roughly speaking, zeta Functions are a tool to count points over all finite fields simultaneously, and are closely related to the Riemann zeta function from number theory.

Chilis:

Homework: Required.

Prerequisites: Linear algebra, group theory, and finite fields.

Infinite-Ness. (Susan, Tuesday–Saturday)
What happens when you take an infinite set and add an element to it? This looks like a silly question if we're thinking in terms of cardinality. The answer is: nothing happens. Adding a single element to an infinite set has no effect on its cardinality. But if we're thinking of an *ordered* infinite set, the question becomes much more interesting. The answer is: it depends on where you put the extra element.

In this class we'll be discussing two different perspectives on infinite sets—cardinality and ordinality. We'll consider the relationships between these two notions of infinite-ness, and how to find interpretations for the operations of addition, multiplication, and exponentiation in either context.

We'll finish off by discussing examples of sets that are really, really big. We'll discuss inaccessible cardinals, and prove that they are so big they prove their own independence.

Chilis:

Homework: Recommended.

Prerequisites: Some familiarity with cardinality could be helpful—knowing how to show that the rationals and the naturals are the same size, but that the real numbers are bigger should suffice. A quick conversation with Susan (or really any mentor of your choice) should bring you up to speed.

Matching Bears With Campers. (Rice, Tuesday–Saturday)
Hooray! Mathcamp just received a shipment of 128 adorable teddy bears, one for each camper! Each camper writes down a complete ranking of teddy bears, ordered by how much they like each bear.

Unbeknownst to the campers, the teddy bears are sentient creatures, and they are hungry. Each bear writes down a complete ranking of campers, ordered by how much they want to eat each camper.

The Mathcamp staff collect these rankings and match the bears up with the campers. Their goal is to create a *stable* matching: a matching M where there is no bear b and camper c such that b and c prefer each other over their match in M. After all, if the matching is unstable, b and c can just run off with each other, abandoning their matches in M, and that would be sad.

Day 1: Can the staff succeed, no matter what rankings the bears and campers submit? (And more!)

Day 2: Some stable matchings are better for campers, others for bears. Can we make a stable matching that's good for both bears and campers, or is the bears' and campers' welfare fundamentally at odds?

Day 3: Oops! The staff accidentally bought 127 bears, just one short. What happens now? (The answer is really surprising!)

Days 4 and 5: The staff realize that the bears want to eat the campers, use their staff powers to fight off the bears, and this time order 128 bears that are *actually* just teddy bears. Now that only campers have preferences, what are some fair ways for the bears to be matched with the campers?

Note: days 1 and 2 of the class will be IBL, i.e. you'll derive the results yourselves in small groups!

Chilis:
Homework: Recommended.
Prerequisites: None!

Unique and Nonunique Factorization. (Gabrielle, Tuesday–Saturday)
In this class, we are going to be studying the nice properties of unique factorization domains, ways to determine if a ring has unique factorization or not, and cool results that we can use unique factorization to prove. Then, because we are not afraid of challenges, we are going to confront some examples of domains where there is nonunique factorization and learn about the tools (elasticity and catenary degree) we can use to study them.
Chilis:
Homework: Recommended.
Prerequisites: Some comfort with rings, integral domains, ideals (i.e. know and be comfortable with the definitions)

Colloquia

Women and Gender Minorities in Math. (Staff, Tuesday)
Various staff will speak about the history of women and gender minorities in math, present anecdotes about their personal experiences, and discuss what we can all do to make math a more welcoming place. The colloquium will be followed by discussions in small groups (and pizza).

Stably Surrounding Spheres with Smallest String Size. (*Zach Abel*, Wednesday)

Once upon some tricky casework, while I pondered how much lacework,
I would need to tie some netting tightly 'round my unit orb,
While the sphere I thus tried trapping, suddenly a bound came, capping
How much string is needed wrapping, wrapping round my precious orb.
Merely 3π is required to accomplish such a chore!
Only this? Well, just eps more.

Ah, distinctly I remember, how I then wished to dismember,
Each and every strand that emb(e)raced that rare and radiant orb.
Seeking to dislodge my prize, I quickly saw, to my surprise, by
Choosing k strands to incise, my sphere from net could not be torn!
How much–how much length is needed? Tell me–tell me, I implore!
$(2k + 3)\pi$, and more.

Knowledge Puzzles. (*Don Laackman*, Thursday)
I know what this colloquium is about.

You don't know what this colloquium is about.

I know you don't know what this colloquium is about.
You know I know what this colloquium is about.
You know I know you don't know what this colloquium is about.
Now, you know what this colloquium is about.

Knowledge puzzles are a type of logical problem in which everyone involved has imperfect information about their universe; and yet, by only sharing information about how much they know, they can often discover everything. The theory of knowledge puzzles has only barely been explored, but I will talk about some of the coolest problems and theorems the field has to offer!

Future of Mathcamp. (Staff, Friday)
Do you have opinions about what would make Mathcamp better? Then come to this event for brainstorming and discussion in groups about what we can change in the future.

Visitor Bios

Zach Abel. I'm a former camper and mentor who is delighted to return for my 8th Mathcamp! You can often find me turning office supplies into geometric sculptures (join my evening activities at camp!), solving or writing puzzles (ask about my elephants!), researching computational geometry (often origami-related!), and using too many asides and exclamation marks (!).

Sachi Hashimoto. Sachi has been a camper, JC, and mentor at Mathcamp. Now she is a graduate student at Boston University where, in a weird coincidence of events, her advisor is a Mathcamp alum!

Her research interests lie in using geometry to study rational number solutions to polynomials and understanding how data, collected from massive mathematical coding projects, can advance theoretical work in arithmetic geometry. Her recent work is on computing databases of solutions to polynomial equations.

Viv Kuperberg. Viv is a graduate student at Stanford studying analytic number theory. She was once a camper at Mathcamp! She then went on to have the roles of sticker supplier, All Star meme enthusiast, sticker receiver, mentor, swing dance instructor, occasional juggler, JC, and Winnie-the-Pooh.

She also likes puzzles.

Don Laackman. Once upon a time, Don was a Mathcamper, a JC, a Mentor, an Academic Ordinator, and now he is excited to be a Mathcamp visitor! When not visiting Mathcamp, Don works full time running another math camp, BEAM, in Los Angeles, whose goal is to create a pathway for underserved middle schoolers to eventually attend programs like Mathcamp. He thinks a lot about logic puzzles, category theory, algebraic algebraic geometry, precision cooking, benign accusations, board games, horrifyingly bad puns, and Otter Pops.

Aaron Landesman. Aaron Landesman was a camper in 2010, 2011, and 2012, and a mentor in 2017 and 2018. He is currently a grad student at Stanford studying algebraic geometry and arithmetic geometry. In 2017 and 2018 he taught classes at Mathcamp on solving the cubic equation, which led to research involving understanding average ranks of elliptic curves. Among the classes he has taught at Mathcamp, his favorites were "Math Until We Die" and a class on finite fields in Twelve-Halves Tongue. He enjoys puzzles, games, swimming, and running.

David Roe. This will be my 20th Mathcamp. When I'm not at camp, I do computational number theory at MIT, go backpacking, play board games and travel a lot.