CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2019

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9:10 Classes

Fundamental Groups. (Kayla, Tuesday–Saturday)
In this class, we develop an algebraic tool that helps us distinguish topological spaces up to homotopy. The fundamental group is a group that we associate to a homotopy class of topological spaces that helps us detect "holes" in our space. This class will be a great introduction to algebraic topology and will give you a great sense of how group theory and topology intersect.
Chilis: 🌶🌶🌶
Homework: Recommended.
Prerequisites: Group Theory and Topology
Cluster: Topology.

Graph Coloring and Containment. (Pesto, Tuesday–Saturday)
A k-coloring of a graph is a way to assign each of its vertices one of k colors such that no pair of adjacent vertices shares a color.

The nicest statement we could hope for in graph coloring is

(1) If a graph doesn't *contain* a graph with k vertices all adjacent to each other, then it can be colored with at most $k-1$ colors.

That statement is false for your first guess of the definition of "contain", but we can change the definition of "contain" in one way ("minors") to get one of the biggest open conjectures in graph theory, and we can change the definition of "contain" in another way ("induced subgraphs") and tweak the statement slightly to get another of the biggest open conjectures in graph theory.

We'll learn about both definitions of "containment," understand both of those conjectures, prove some special cases of them, and see some other nice results in graph coloring.
Chilis: 🌶🌶
Homework: Recommended.
Prerequisites: None.
Cluster: Graph Theory.

Systems of Differential Equations. (Mark, Tuesday–Saturday)
Many models have been devised to try to capture the essential features of phenomena in economics,

ecology, and other fields using systems of differential equations. One classic example is given by the Volterra-Lotka equations from the 1920s:

$$\frac{dx}{dt} = -k_1x + k_2xy, \frac{dy}{dt} = k_3y - k_4xy,$$

in which x, y are the sizes of a predator and a prey population, respectively, at time t, and k_1 through k_4 are constants. There are two obvious problems with such models. Often, the equations are too hard to solve (except, perhaps, numerically); more importantly, they are not actually correct (they can only hope to approximate what really goes on). On the other hand, if we're approximating anyway and we have a system

$$\frac{dx}{dt} = f(x, y), \frac{dy}{dt} = g(x, y),$$

why not approximate it by a linear system such as

$$\frac{dx}{dt} = px + qy, \frac{dy}{dt} = rx + sy?$$

Systems of that form can be solved using eigenvalues and eigenvectors, and usually (but not always) the general behavior of the solutions is a good indication of what actually happens for the original (nonlinear) system if you look near the right point(s). If this sounds interesting, come find out about concepts like trajectories, stationary points, nodes, saddle points, spiral points, and maybe Lyapunov functions. Expect plenty of pictures and probably an opportunity for some computer exploration using *Mathematica* or equivalent. (If you don't want to get involved with computers, that's OK too; most homework will be doable by hand.)

Chilis:

Homework: Recommended.

Prerequisites: Linear algebra (eigenvectors and eigenvalues), calculus, a little bit of multivariable calculus (equation of tangent plane).

The Weierstrass $\wp$ Function. (Assaf, Tuesday–Saturday)

In kindergarten, we learned about the trigonometric functions and their roles in parametrizing the circle. By grade 1, after mastering calculus, we also learned about the definitions of the inverses of these functions in terms of integrals, and showed that they are derivatives of each other.

In this class, we will go up a dimension and discuss the complex analogue of a trigonometric function, called the Weierstrass $\wp$ function. This doubly-periodic meromorphic function behaves like a trigonometric function in parametrizing complex tori in $\mathbb{C}P^2$.

Along the way, we'll pass through series of meromorphic functions, the complex projective space, and hopefully prove Abel's theorem, and start looking at Riemann surfaces in $\mathbb{C}P^2$.

Chilis:

Homework: Required.

Prerequisites: Complex analysis

Cluster: Complex Analysis.

Young Tableaux and Combinatorics. (Shiyue, Tuesday–Saturday)

Young tableaux are rich combinatorial objects that inspire many combinatorial techniques and theory. Counting standard Young Tableaux (SYT) is already combinatorially interesting. Moreover, given a Young diagram, this count is also the dimension of the irreducible representation of symmetric group associated with the Young diagram. To study this, we will derive the beautiful bijection between SYT and shapes/permutations, namely the celebrated Robinson–Schensted–Knuth correspondence, Hillman-Grassel combinatorial algorithm, and SYT's connections to Schur functions (another ubiquitous gadget in algebraic combinatorics and study of symmetric groups). We will eventually derive the hook length formula that counts the number of standard Young Tableaux, using all these crucial ingredients that we have picked up on the way.

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: None.

Cluster: All About Young Tableaux.

10:10 Classes

From High School Arithmetic to Group Cohomology. (Apurva, Wednesday–Saturday)
Much of mathematics is invented by taking something ordinary and reinterpreting it in a new way – a change of perspective.

We all know how to add 2 digit numbers and can do it instinctively. But what if the digits came from non-abelian groups? Does it still make sense to talk about addition and "carrying" if the addition is non-abelian?

By understanding why the carrying process works and axiomatizing it, we will be able to define the (second) group cohomology of a group and Ext groups. The notion of group cohomology then shows up in unexpected places like field theories in physics and algebraic K-theory.

This will be an IBL class. Instead of a regular lecture, we'll learn things by solving problem sheets.

Chilis: 🌶🌶

Homework: Required.

Prerequisites: Basic group theory - group homomorphism, first isomorphism theorem, cyclic groups, products of groups, kernel and image of group homomorphisms.

Functions of a Complex Variable (2/2). (Mark, Wednesday–Saturday)
This is a continuation of the week 1 class. If you would like to switch into it, talk to Mark (or to someone who took week 1) to find out what has been covered.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Week 1 of the class or the equivalent.

Cluster: Complex Analysis.

Non-Euclidean Geometries. (Véronique, Wednesday–Saturday)
This class is an introduction to non-Euclidean geometries. We will look at Euclid's postulates, and, more precisely, theorems proved with and without the fifth postulate. Then, we will study some non-Euclidean geometries such as the spherical geometry, the projective geometry, and the inversive geometry.

Chilis: 🌶

Homework: Recommended.

Prerequisites: Group theory

Probabilistic Models and (a little bit of) Machine Learning. (Mira, Wednesday–Saturday)
Machine learning can be approached in two ways. You can put your data in a black box, try a bunch of standard techniques, find one that does a decent job on your task, and consider yourself done. If all you want is to train your machine to do a certain task, these "black-box" techniques can be very

powerful. A prime example of this approach is neural networks (which we will NOT discuss in this class).

But the black-box approach doesn't give you a deeper understanding of how learning actually happens and what intelligence really means. An alternative approach is to posit that a learner begins with some model of the world – which has to be a *probabilistic* model, since the world is full of uncertainty and we only observe it incompletely, through the lens of our data. Then learning proceeds by updating this model based on observed data according to the laws of probability (also known as Bayesian inference). This is the approach that we will pursue in this class.

In addition to learning the theory, we will actually implement some learning algorithms in simple probabilistic graphical models, using the free web-based probabilistic programming language WebPPL. The homework for this class is essential, and you will be expected to devote at least an hour to it every day.

Chilis:

Homework: Required.

Prerequisites: Solid knowledge of calculus. It will help if you have done a little bit of programming before, but WebPPL is quite easy to use, so beginners are welcome too.

Cluster: Algorithms.

Units in Algebraic Number Theory. (Kevin, Wednesday–Saturday)
J-Lo began his Intro to Number Theory class by presenting an instance of Pell's equation: $x^2-2y^2 = 1$. By factoring this using $\sqrt{2}$ and then raising the resulting factors to powers, we can generate an infinite family of solutions over the integers.

Pell's equation translates directly into the fact that $x + y\sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$. The units in these sorts of "integers" have a rich algebraic structure described fully by a famous result called Dirichlet's Unit Theorem. In this class, we'll see the surprising way that beautiful geometric ideas contribute to proving something that looks so purely algebraic.

Chilis:

Homework: Recommended.

Prerequisites: It is helpful to have some familiarity with the basics of many topics (algebraic number theory, complex numbers, group theory, linear algebra, and ring theory), but none are strictly required. There will be sections on the homework to catch you up on everything you need before it's relevant in class.

Cluster: Algebraic Number Theory.

11:10 Classes

All About Quaternions (2/2). (Assaf + J-Lo, Tuesday–Saturday)
More about quaternions!

Chilis:

Homework: Recommended.

Prerequisites: Week 2 class on quaternions, linear algebra

Problem Solving: Induction. (Misha, Tuesday–Saturday)
Some of you maybe first saw induction in the context of proving a result like

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}.$$

Such a proof is fairly straightforward, and maybe your main worry was "Can my last sentence just be 'by induction, we're done' or do I need something fancier?"

In this class, we'll see how these proofs can get much more complicated. Our induction will start out strong, and on each day of class it will get stronger than all the previous days combined. You'll see examples of crazy induction in algebra, analysis, graph theory, set theory, number theory, and other theories. You'll learn how to use induction (and how *not* to use it) to solve problems of your own, olympiad and otherwise.

In class, we will solve problems together; I will focus less on answering the question "why is this claim true?" and more on answering the question "why would we think of solving a problem this way?" There will be plenty of problems left for homework, and you will not get much out of a problem-solving class unless you spend time solving those problems.

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: None.

Cluster: Problem-Solving.

Quantum Mechanics. (*Nic Ford*, Tuesday–Saturday)
Quantum mechanics is one of the great triumphs of twentieth-century physics, and a lot of people seem to think they know a lot about it. You might have heard stories about the uncertainty principle, measurements changing the results of experiments, things being in two places at once, entangled particles communicating instantaneously over large distances, and a cat that's neither alive nor dead until you look at it. In this class, we'll talk about how quantum systems actually behave; I hope to convince you that while these popular explanations sound designed to weird you out, in a sense they're usually not strange *enough* to describe how the universe actually works!

We'll start in the first half from an abstract, axiomatic perspective to understand how quantum states and observables work mathematically; the "actual physics," where we'll examine how a couple realistic systems evolve in time, will wait until the last couple of days. My goal is to show you the mathematical description of many of the quantum phenomena you might have already heard about, like quantum measurements, entanglement, the Heisenberg uncertainty principle, and interference, but no prior knowledge of any of this is necessary.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: You'll definitely need linear algebra to follow the class, including vector spaces, bases, linear independence and dimension. Some prior exposure to inner products and eigenvalues will be helpful, but we'll also be covering them in class. Calculus will also show up, but mostly only for the second half. It will also be helpful to know a bit about how momentum and energy behave in Newtonian physics, but this isn't a hard requirement.

Real Analysis (2/2): Measures. (Ben, Tuesday–Saturday)
We like integrating things, and we like taking limits. Unfortunately, the Riemann integral runs into technical problems when dealing with taking limits of things. When trying to solve problems by methods such as Fourier analysis or approximation, this bad behavior with limits makes life a lot more difficult.

One way to solve this problem is to introduce the idea of a "measure." This lets us define a more general idea of integration that is more well-behaved with respect to taking limits! Moreover, measure theory is a gateway to a lot of useful areas of math, letting us do a great deal of analysis with more rigor and care.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: Know what limit points are (of a sequence, of a set).

The Sound of Proof. (Eric, Saturday)
Can you hear what a proof sounds like? I'll present five proofs from Euclid's Elements, and then play (recordings of) five pieces of music written to capture each proof in sound. You'll get to try and work out which piece of music lines up with which proof, and then we'll dissect how a couple of the compositions "sonify" the proofs. All of the material I'm drawing on is from an art piece entitled "The Sound of Proof" by mathematician Marcus du Sautoy and composer Jamie Perera at the Royal Northern College of Music in Manchester.

Chilis: 🌶

Homework: None.

Prerequisites: None.

Thinking of Images as Mathematical Objects. (*Olivia Walch*, Tuesday–Friday)
This class will explore different ways of talking about images and drawings mathematically. Topics covered will include: how photo editors work, how to draw a nice-looking line, vector graphics, style transfer, and generative art.

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: None.

Cluster: Math and the Real World.

1:00 Classes

Breaking Bad (RSA Encryption). (Michael, Tuesday–Saturday)
This class will be a walk through various methods in primality testing and prime factorization, and how many of them relate to each other via group theory. We'll discuss these in detail:

Pollard Rho and Pollard p-1 The Quadratic Sieve Williams p+1 and Primality testing with Lucas numbers

We will also talk about how two of these are strongly related via group theory (one of them is just for fun), and use this to understand why elliptic curves are useful for factorization and primality testing. We will, however, assume some of these common traits when we take a look at elliptic groups. We will look at how the elliptic group is constructed, the vast majority of a proof (i.e. we will look at the generic case of the Cayley-Bacharach theorem, but ignore the corner cases) that the elliptic group is actually a group, and how to perform operations in it that are useful to making it jump through the hoops we want it to on a computer.

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: None!

Cluster: The Power of Groups.

Everything You Ever Wanted to Know About Finite Fields. (Eric, Tuesday–Saturday)
Finite fields! They pop up in many places across mathematics and are a great place to get experience thinking about abstract algebra. In this class you'll work through all of the basic structural properties of finite fields, see how to construct them in a variety of hands on and not hands on ways, and see some

Galois theory for infinite extensions in action by determining the full list of subfields of the algebraic closure of a finite field.

Chilis:

Homework: Recommended.

Prerequisites: You should know ring theory, to the level where you're comfortable quotienting a ring by an ideal.

Required for: Counting Points over Finite Fields (W4)

Not-Proofs of Fermat's Last Theorem. (Gabrielle, Friday–Saturday)
In this class, we're going to learn many ways that Fermat's Last Theorem was not proved. This will be a history class in part, but we are going to look at the proofs themselves–a proof when $n = 4$ (Fermat), a failed-but-fixed proof for $n = 3$ (Euler), one of the first proofs for infinitely many primes (Germain), and an idea which would have worked to prove the statement for all p, had it not been for nonunique factorization (Kummer).

Chilis:

Homework: Optional.

Prerequisites: None.

Permutation Combinatorics. (Bill, Tuesday–Saturday)
In this class, we'll study the underlying combinatorial structures of permutations. We'll find that many simple questions about permutations (e.g. how many permutations contain no decreasing subsequence of length three?) have startlingly beautiful answers (e.g., The n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$). We'll also find that other questions about permutations have quite *surprising* answers. For one example of this, consider the following riddle:

Suppose 100 players $1, 2, \ldots, 100$ each walk into identical rooms. In each room, there are 100 boxes in a row, containing the numbers $1, 2, \ldots, 100$ in a random order (though the order is the same in all 100 rooms). Each player opens 75 boxes, and wins a cheap plastic trophy if one of the boxes that they open contains their number. If, however, all 100 players win trophies, then all 100 players also win $\$1,000,000$.

At face value, the cash prize seems nearly impossible to win. Surprisingly, there's a simple strategy that guarantees a $> 70\%$ chance of winning it. (Wait, what!?) Perhaps even more surprisingly, the motivation and analysis for the strategy comes straight from studying the cycle structure of permutations.

Chilis:

Homework: Recommended.

Prerequisites: Know what a permutation is.

Polytopes (Higher Dimensional Polygons). (*Angélica Osorno*, Tuesday–Thursday)
In this series we will explore how to generalize the definition of a polygon to higher dimensions. We will learn about intricacies and subtleties of this higher dimensional geometry. We will also explore applications to optimization theory (e.g., the traveling salesman problem). If time permits, we will also talk about some open research problems in this area.

Chilis:

Homework: Recommended.

Prerequisites: None.

Quiver Representations. (Will, Tuesday–Saturday)
One of the central ideas of linear algebra is that, while we often express maps between vector spaces by matrices, they aren't exactly the same thing. Indeed, if we change the bases of the vector spaces involved, the matrix expression for our map changes as well. By choosing coordinates strategically, we can express a map by a particularly nice matrix, which depends only on the rank of the map. This can reveal structure that wasn't visible in the original matrix.

If we consider maps from a vector space to itself, thus only allowing one change of basis, the situation gets more complicated. But it turns out we can still pick a basis that puts our matrix in a nice canonical form, known as Jordan normal form.

What if we have *two* linear maps between a pair of vector spaces? Or a chain of maps $U \to V \to W$? Can we still pick the bases of the spaces involved such that all the maps are simultaneously described by nice-looking matrices? The subject of quiver representations provides a unifying framework for questions like these. In this class, we'll see how considering networks of vector spaces and maps between them can tell us when these problems have satisfying answers. And while there won't be a direct connection to cluster algebras, some actors from Véronique's week 1 class will make an unexpected reappearance.

Chilis:

Homework: Required.

Prerequisites: Familiarity with linear algebra. In particular, you should know what effect change of basis has on a matrix. Knowing what it means to diagonalize a matrix would also help.

Cluster: The Cluster Algebra Cluster.

Required for: Representation Theory of Associative Algebras (W4)

COLLOQUIA

The Most Depressing Theorem I Know. (Mira, Tuesday)
Say there is a human trait that (e.g. probability of being in a car accident) on which two groups (e.g. men and women) differ.

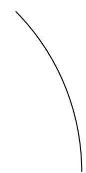
Say also that there is a cost or benefit associated with this trait. In our example, if your probability of being in an accident is high, then you insurance goes up – that's the cost.

The theorem says that any algorithm for predicting such a trait will either be either biased or unfair. I'll tell you precisely what I mean by "biased" and "unfair" in the talk, but the upshot is: there *are* no fair algorithms for prediction. And it's not because the people who write these algorithms are racist or sexist or careless (though that may be true too), but because, in many cases, perfect fairness is just mathematically impossible.

Why Should We Care About Category Theory? (*Angélica Osorno*, Wednesday)
One of the first mathematical concepts we learn as children is counting, and when we do so, we think of counting the number of elements in a specific set. Soon after, we forget about sets and we just consider the abstract numbers themselves. This abstraction simplifies many things, but it also makes us forget about some structure that we had when we were thinking about sets. That structure can be encoded by a category. In this talk, we will describe certain concepts in category theory, and you will realize that in most of your mathematics classes, you have been working with categories – you just didn't know about it. There will be plenty of examples that will show that category theory provides a unifying language for mathematics, and that many constructions are more naturally understood when they are seen through the categorical lens.

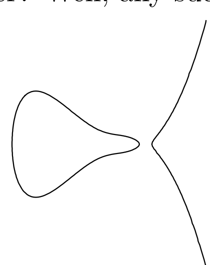
Your Clock on Your Phone: Tracking Your Body's Internal Time with Math. (*Olivia Walch*, Thursday)
The body's circadian clock regulates timing for almost all major physiological functions, with the clock itself primarily regulated by light exposure. Measuring circadian time experimentally can be time-consuming and expensive, but mathematical modeling can help overcome this challenge. By providing a model of the circadian clock with a person's recent light exposure, the model can generate a prediction of that person's internal time. In this talk, I'll discuss one such model of the circadian clock, as well as our experiences putting that model into mobile apps used around the world. I'll also talk about connections between this work and sleep, jet lag, and optimal control.

Dictionary Shapes. (J-Lo, Friday)
Have you ever wondered what it would be like to see every right triangle at once? Well here they are!



And here's every possible grid in the plane!

Is it possible to find a pair of triangles – one right, one isosceles – with integer side lengths, which have the same area and the same perimeter? Well, any such pair can be found somewhere on here:



Visitor Bios

Nic Ford. Nic has been teaching at Mathcamp since 2010. He likes the intersection of algebraic geometry and combinatorics, especially when he gets to draw lots of little pictures that each correspond to some complicated geometric object. He's also recently become more interested in physics and is hoping to share that with many of you this summer!

Yulia Gorlina. Camper '99-'01, JC '04. Love West Coast Swing and Salsa dancing.

Angélica Osorno. I'm an algebraic topologist who is also interested in higher category theory. At Mathcamp, I'll probably teach something in the intersection of discrete geometry (polytopes) and combinatorics.