CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2019

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9:10 Classes

Eigenstuff and Beyond. (Mark, Tuesday–Saturday)

If after a sunny day, the next day has an 80% probability of being sunny and a 20% probability of being rainy, while after a rainy day, the next day has a 60% probability of being sunny and a 40% probability of being rainy, and if today is sunny, how can you (without taking 365 increasingly painful steps of computation) find the probability that it will be sunny exactly one year from now? If you are given the equation $8x^2 + 6xy + y^2 = 19$, how can you quickly tell whether this represents an ellipse, a hyperbola, or a parabola, and how can you then (without technology) get an accurate sketch of the curve? These are two of many problems that can be solved using "eigenstuff" – more formally, eigenvalues and eigenvectors of square matrices. After defining those and seeing how one might in principle find them (although there are complications for 5×5 and larger matrices), we'll look at orthogonal (distance-preserving) linear transformations, at applications such as the above, and if time permits, at a seemingly magical, and fundamental, fact about square matrices called the Cayley–Hamilton Theorem. By the way, we'll probably also see how one might *come up with* Binet's formula

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

for the Fibonacci numbers. (If someone gives you the formula, you can prove it by induction.)

Chilis:

Homework: Recommended

Prerequisites: Beginning linear algebra; for example, Apurva's week 1 class.

Required for: Systems of Differential Equations (W3); Long Live Determinants (W4)

Galois Correspondence of Covering Spaces. (Apurva, Tuesday–Saturday)

Galois theory first arose in the study of roots of polynomials. Its main theorem establishes a mirrored structure, called the Galois correspondence, between the algebraic extensions of fields and certain automorphism groups.

A parallel theory exists in topology with field extensions replaced by covering spaces and the absolute Galois group replaced by the fundamental group. This parallel theory existed independently for a while, and was unified by Grothendieck under a broader umbrella of algebraic geometry.

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In this class, we'll see the topological side of the story: we will define covering spaces, deck transformations, and prove the correspondence between sub-covering spaces and subgroups of deck transformations. As an application, we will prove the Nielsen–Schreier theorem which states that *every subgroup of a free group is free*.

We won't need any topology, as we'll mostly be working with graphs and surfaces. The intrepid student is very welcome to generalize to higher dimensions.

Disclaimer: We won't be talking about the Galois correspondence for fields at all. This is a purely geometry/topology class.

Chilis: ୬୬୬୬

Homework: Recommended

Prerequisites: You should be very comfortable with group theory stuff like normal subgroups, group actions, and the orbit-stabilizer theorem.

Hedetniemi's Conjecture. (Yuval Wigderson, Tuesday-Saturday)

Two months ago, the world of graph theory was shocked when Yaroslav Shitov, a relatively unknown Russian mathematician, published a three-page paper disproving Hedetniemi's conjecture, a famous open problem that's been intensively studied for more than fifty years. In this class, we'll go through Shitov's beautiful and subtle argument to disprove this conjecture.

Hedetniemi's conjecture concerns proper colorings of graphs, which are a central topic in graph theory (for instance, the famous Four-Color Theorem says that every graph that can be drawn in the plane can be properly colored with at most four colors). More specifically, Hedetniemi's conjecture deals with proper colorings of the tensor product of two graphs, which is a simple construction that generates a new graph $G \times H$ from two graphs G, H. As we'll see, the definition of the tensor product implies that if G or H can be properly colored with k colors, then $G \times H$ can also be colored with k colors. Thus, if $\chi(G)$ denotes the minimum number of colors needed to properly color G, then this implies that

$$\chi(G \times H) \le \min{\{\chi(G), \chi(H)\}}.$$

Hedetniemi conjectured that this inequality is actually always an equality, and this is what Shitov disproved, by finding graphs G, H with $\chi(G \times H) < \min\{\chi(G), \chi(H)\}$.

You might wonder why it took fifty years to disprove this conjecture, given that a counterexample amounts to writing down two graphs satisfying a simple property. One reason is that the graphs Shitov constructs are absolutely enormous—to my knowledge, the smallest known G has about 2,000,000 vertices, while the smallest known H has about $10^{4,000,000}$ vertices; for comparison, the number of particles in the universe is about 10^{80} , so H is incomprehensibly huge. So if you're interested in thinking about truly enormous graphs, or interested in seeing what real-life math research looks like, this class is probably for you.

Chilis:

Homework: Recommended

Prerequisites: None, but the class will be easier to follow if you've seen a little bit of graph theory before.

Cluster: Graph Theory

Introduction to Algebraic Number Theory. (J-Lo, Tuesday–Saturday)

Here are two helpful perspectives that can be helpful when solving problems:

- (1) Sometimes the easiest way to answer a question about numbers is to change what "number" means.
- (2) Factoring can become a whole lot nicer if you do it with infinitely many numbers at once.

In this course, we will develop these perspectives through a guided series of exercises that introduce you to algebraic numbers, algebraic integers, and ideals.

Chilis:

Homework: Recommended

Prerequisites: Dividing polynomials with remainder, adding and multiplying mod n.

Cluster: Algebraic Number Theory

Required for: Reciprocity Laws in Algebraic Number Theory (W4)

Mathcamplandia. (Luke Joyner, Tuesday–Saturday)

The powers that be (THE POWERS THAT BE!!!) have noticed that Mathcamp moves around year to year, and more generally that mathematicians often work in spaces and places that may not be well suited to their needs. They've asked you to build a city of math (CITY OF MATH!!!) that will house a permanent home for Mathcamp, any alumni and friends who might want to live in such a place, other mathematicians, and roughly 100000–400000 other people. They want you to find a real location for this city, and design the whole thing... in a week. (You're welcome to adapt an existing place, or create a new place from scratch, if you can find a site that allows it.) Along the way, we'll dive into the fascinating world of cities and urban design... there'll be a little topology and geometry involved, and you're welcome to bring math to your project in whatever creative ways you see fit, but this class mostly focuses on mapping, design and making.

Chilis: 🌶

Homework: Required

Prerequisites: Very basic understanding of Graph Theory, an open mind, willingness to develop ideas through stages of editing (Note: drawing skills are not required, but drawing by hand or on the computer will be an important part of our final project; I will lead an optional seminar on drawing techniques for anyone interested on Wednesday).

Note: homework for this class will be a mix of math, reading, group activities, and mostly progress toward designing a city in small groups. This is a one-chili class for the low level of math knowledge required, but it is not exactly an *easy* class, as it is not so easy to design a city well in a week. It's a "you get what you put in" kind of class.

Cluster: Math and the Real World

10:10 Classes

Algorithms in Number Theory. (Misha, Tuesday–Saturday)

We will discuss which number-theoretic problems can be solved efficiently, and what "efficiently" even means in this case.

We will learn how to tell if a 100-digit number is prime, and may be also how we can tell that $2^{82589933} - 1$ is prime.

We will talk about what's easy and what's hard about solving (linear, quadratic, higher-order) equations modulo n. We may also see a few applications of these ideas to cryptography.

Chilis:

Homework: Recommended

Prerequisites: Number theory: you should be comfortable with modular arithmetic (including inverses and exponents) and no worse than mildly uncomfortable with quadratic reciprocity.

Cluster: Algorithms

¹from an n-dimensional space where each coordinate is an integer mod 3

Cap Sets, SET, and ProSet. (Elizabeth Chang-Davidson, Friday-Saturday)

One formulation of the cap set problem asks: how many points¹ can you have in a set before you are guaranteed three collinear points? The question is interesting for a number of reasons, one of which is that the current best general upper bound was proved quite recently, in 2016 by Ellenberg and Gijswijt, and another is that we only know the exact answer for up to 6-D space. In this class, we will talk about how the cap set problem relates to the card game SET, prove some of the known exact answers, discuss the 2016 result briefly, and then talk about a different card game, projective set.

Chilis:

Homework: Recommended

Prerequisites: Some familiarity with arithmetic mod 3 is necessary. Linear algebra experience is helpful but not necessary.

Chaos in Voting. (Ben, Friday–Saturday)

Suppose you and your friends want to figure out how many croutons to order for your event "Having Lots of Croutons Around." One way to pick a result that you're all kinda OK with is to have a sequence of majority votes. Eventually, you'll probably end up choosing to have the amount of croutons that the person in the middle of your group wants, the "median crouton wanter."

Now, you might think that this is also a good way to decide how many croutons and how much soda to buy for your "Having Lots of Crouton Salads Around" event. What we will show in class is that this is not only a bad idea, but it is a bad idea in the worst possible way—it's possible (depending on a few things) that you'll end up ordering a number of croutons AND an amount of salad that makes everyone very sad.

Chilis: 🌶

Homework: Recommended

Prerequisites: None.

Functions of a Complex Variable (1/2). (Mark, Tuesday–Saturday)

Spectacular (and unexpected) things happen in calculus when you allow the variable (now to be called z = x + iy instead of x) to take on complex values. For example, functions that are "differentiable" on a disk in the complex plane now automatically have power series (Taylor) expansions. If you know what the values of such a function are everywhere along a closed curve, then you can deduce its value anywhere inside the curve! Not only is this quite beautiful math, it also has important applications, both inside and outside math. For example, functions of a complex variable were used by Dirichlet to prove his famous theorem about primes in arithmetic progressions, which states that if a and b are positive integers with gcd(a,b) = 1, then the sequence $a, a + b, a + 2b, a + 3b, \ldots$ contains infinitely many primes. This was probably the first major result in analytic number theory, the branch of number theory that uses complex analysis as a fundamental tool and that includes such key questions as the Riemann Hypothesis. Meanwhile, in an entirely different direction, complex variables can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow. Dirichlet's theorem is certainly beyond the scope of this class, and heat conduction probably is too, but we should see a proof of the so-called "Fundamental Theorem of Algebra", which states that any nonconstant polynomial (with real or even complex coefficients) has a root in the complex numbers. We should also see how to compute some impossible-looking improper integrals by leaving the real axis that we're supposed to integrate over and boldly venturing forth into the complex plane! This class runs for two weeks, but it should be worth it. (If you can take only the first week, you'll still get to see a good bit of interesting material, including one or two of the things mentioned above.)

Chilis:

Homework: Recommended

Prerequisites: Multivariable calculus, including Green's Theorem; if the week 1 crash course doesn't get to Green's Theorem, it will be covered near the beginning of this class.

Cluster: Complex Analysis

Required for: The Weierstrass \wp Function (W3); Functions of a Complex Variable (2/2) (W3); Rie-

mann Surfaces (W4)

Introduction to Ring Theory. (Will, Tuesday–Saturday)

Rings are abstract algebraic structures (like groups and vector spaces) which capture the idea of a system with addition, subtraction, and multiplication, but not necessarily division. They show up in any mathematical subject that involves algebra.

In this class, we'll develop the axioms and basic properties of rings: homomorphisms, ideals, quotients, and localization: the practice of strategically adding multiplicative inverses of certain elements. We'll pay particular attention to the rings of integers and polynomials, which share many intriguing parallels.

Chilis:

Homework: Required

Prerequisites: Some examples may reference basic facts about groups, matrices, and vector spaces, but you won't need to be familiar with these things to take the course.

Required for: Everything You Ever Wanted to Know About Finite Fields (W3); Unique and Nonunique

Factorization (W4)

Sperner, Monsky, and Brouwer. (Laura Pierson, Tuesday–Wednesday)

Is it possible to divide a square into an odd number of triangles of equal area? If you stir your coffee, must there always be a point that doesn't move? If you have a big triangle divided into small triangles, and you color all the vertices so that the vertices of the big triangle are all different colors and each edge of the big triangle has only two different colors, must there be a small triangle all of whose vertices are different colors? In this class, we'll answer all these questions and see how they're related to each other. Along the way, we'll also encounter the game of hex and the 2-adic numbers!

Chilis:

Homework: Recommended

Prerequisites: None.

THE Intuitive Proof of the Hairy Ball Theorem. (Assaf, Thursday)

The usual proof of the Hairy Ball Theorem is some magic wand nonsense involving homology, degree maps, and wishy-washy hand waving. In this class, we'll get rid of everything except for the wishy-washy hand waving to prove the Hairy Ball Theorem using only an intuitive definition of continuity.

Chilis:

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Homework: None

Prerequisites: A good grasp on the intuitive meaning behind continuity

Take it to the Limit. (Ben, Tuesday–Thursday)

The normal idea of a limit has a lot of nice and wonderful properties. We'll start this class by stating some of these properties, as well as another not-so-wonderful property: it is not always defined.

In this course, we will give up some of the other wonderful properties of the limit in order to define limits of more sequences. This will include some normal methods—such as the Cesàro limits—and some less normal ones—such as ultrafilters.

Chilis: 🍎🍎🍎

Homework: Recommended

Prerequisites: None.

11:10 Classes

All About Quaternions (1/2). (Assaf + J-Lo, Wednesday-Saturday)

On October 16, 1843, William Rowan Hamilton was crossing the Brougham Bridge in Dublin, when he had a flash of insight and carved the following into the stone:

$$i^2 = j^2 = k^2 = ijk = -1.$$

This two-week course will take you through a guided series of exercises that explore the many implications of this invention, and how they can be used to describe everything from the rotations of 3-D space to which integers can be expressed as a sum of four squares.

Chilis: 🍎🍎

Homework: Recommended

Prerequisites: None.

Required for: All About Quaternions (2/2) (W3)

Discrete Derivatives. (Tim!, Wednesday–Saturday)

Usually, we define the derivative of f to be the limit of $\frac{f(x+h)-f(x)}{h}$ as h goes to 0. But suppose we're feeling lazy, and instead of taking a limit we just plug in h=1 and call it a day. The thing we get is kind of a janky derivative: it's definitely not a derivative, but it acts sort of like one. It has its own version of the power rule, the product rule, and integration by parts, and it even prefers a different value of e. We'll take an expedition into this bizarre parallel universe. Then we'll apply what we find to problems in our own universe: we'll talk about Stirling numbers, and we'll solve difference equations and other problems involving sequences.

Chilis:

Homework: Recommended

Prerequisites: Calculus (derivatives)

Group Theory & Rubik's Cubes. (Gabrielle, Wednesday–Saturday)

Group theory has many practical applications! We will not study any of those in this class. Instead, we will describe the set of moves one can perform on a Rubik's cube as a group, and we will use group theory (some of which you may learn in this class, such as group actions) to analyze this group, in order to characterize all valid configurations of the Rubik's cube and determine some solution methods. This class is an active learning class, which means that you will play with Rubik's cubes and discover the answer to these questions yourself (with some guidance).

Chilis:

Homework: Recommended

Prerequisites: Group theory (definition of groups, subgroups, cosets, homomorphisms; know what a symmetric group is; know what is meant by "even" and "odd" permutations); talk to me if you're interested but lack some of the prerequisites!

Cluster: The Power of Groups

Multi-Coefficient Solving of Problems. (Pesto, Wednesday–Saturday)

Polynomials are a frequent topic of olympiad-style competitions, since there are many and interesting problems using them. For instance,

(1) If a_1, \ldots, a_n are distinct real numbers, find a closed-form expression for

$$\sum_{1 \le i \le n} \prod_{1 \le j \le n, j \ne i} \frac{a_i + a_j}{a_i - a_j}.$$

(2) Find the product of the lengths of all the sides and diagonals of a regular n-gon of diameter 2. Don't see the polynomials? Come to class and find them.

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to olympiad-style problems you will have solved as homework the previous day.

Chilis:

Homework: Required

Prerequisites: Linear algebra: understand the statement "Two-variable polynomials of degree at most

2 are a vector space of dimension 6."

Cluster: Problem-Solving

Real Analysis (1/2): Limits. (Véronique, Wednesday–Saturday)

How can we prove that $\lim_{n\to\infty} \frac{n}{n+1} = 1$? Because, for large n, $\frac{n}{n+1}$ is almost 1? Sure, but how can we prove this formally? And what is a continuous function, other than a function that you can draw without lifting your pen? And finally, how do we compute $\lim_{x\to 0} \frac{\sin(x)}{x}$? We answer these questions in this class, among others.. We will learn about a little bit of topology and a lot of analysis: we'll discuss numerical sequences and functions (limits and continuity).

Chilis:

Homework: Recommended Prerequisites: A bit of calculus

Required for: Real Analysis (2/2): Measures (W3)

1:10 Classes

Analysis with Prime Numbers. (Eric, Tuesday–Saturday)

I'm a number theorist, so I just want to do modular arithmetic all day long. But I had to take all these calculus and analysis classes instead of doing things I actually wanted to do! Wouldn't it be nice if I could do something that looks like calculus, but is secretly just number theory?

In this class, we will learn about p-adic numbers and analytic functions. We'll talk about the strange world of non-Archimedean spaces, where all triangles are isosceles and the distance you travel on a hike is the same as the biggest step you took. With the help of some new versions of our old friends e^x and $\log(x)$, we'll prove a cool theorem about the structure of linear recurrence sequences (e.g. things like Fibonacci numbers) that doesn't look like it should have anything to do with calculus.

Chilis: 🌶🌶

Homework: Recommended

Prerequisites: None! If you've seen functions defined by power series and the exponential and logarithmic functions before that's great, but it's not at all necessary. Come talk to me if you have questions!

Infinite Trees (2/2). (Susan, Tuesday–Saturday)

Infinite Trees Week 2: Electric Boogaloo

Chilis:

Homework: Recommended

Prerequisites: Infinite Trees Week 1

The Probabilistic Method. (Bill, Tuesday–Saturday)

The probabilistic method is the art of proving combinatorial results using elegant probabilistic arguments. Often, the result being proven will seem to have *nothing to do with probability*. For example, this theorem from combinatorial number theory:

Theorem: Call a set $A \subseteq \mathbb{N}$ sum-free if for all distinct $x, y \in S$, the quantity $x + y \notin S$. Then every set $B = \{b_1, \ldots, b_n\}$ of n nonzero integers contains a sum-free subset A of size $|A| > \frac{1}{3}n$.

In this class we'll see some of the most beautiful (and mind-blowing!) applications of the probabilistic method. Along the way we'll prove famous results from Ramsey theory, combinatorial number theory, graph theory, and even real analysis. Often the proofs that we will encounter will be an order of magnitude shorter than any other known proofs (and in some cases, it will turn out that there are no other known proofs!).

Chilis:

Homework: Recommended

Prerequisites: Familiarity with probability. Familiarity with linearity of expectation would be helpful but will also be reviewed during the class.

Topology. (Kayla, Tuesday–Saturday)

Want to understand why mathematicians joke that coffee mugs and donuts are actually the same? Come learn about this in topology! In the class, we will be introducing many basic concepts in point-set topology. We will be discussing when we think of two topological spaces as the same, maps between topological spaces, basic properties of topological spaces and how they play with maps between the spaces, the product topology and the quotient topology (how to formalize the idea of "gluing" things together)!

Chilis:

Homework: Required Prerequisites: None. Cluster: Topology

Required for: Fundamental Groups (W3); Tychonoff's Theorem (W4); Riemann Surfaces (W4)

Young Tableaux and Representation Theory. (Shiyue, Tuesday–Saturday)

If you run into any representation theorist and ask them what rep theory is, they will probably tell you that it is a way of studying your algebraic structures as maps into vector spaces (think: understanding group theory (hard) using linear algebra (easy)). This can still be quite scary. But believe it or not, representation theory of symmetric groups can be highly combinatorial and not scary at all!

In this course, we will see a fascinating and ubiquitous interplay between algebra and combinatorics. We will play around with a fun combinatorial object called Young tableaux (think: stacking boxes and throwing balls in the boxes), learn representation theory, and see how Young tableaux give us simple and elegant ways of understanding symmetric groups and their representations. More particularly, we will construct irreducible representations of symmetric groups using Young tableaux, and prove that the dimensions of these irreducible representations can be understood combinatorially.

Chilis: එඑඑ

Homework: Recommended

Prerequisites: Linear algebra, group theory

Cluster: All About Young Tableaux

Colloquia

Project Selection. (Staff, Tuesday)

This is not a colloquium.

Many Mathcampers enjoy working on *some* kind of long-term project throughout camp: on their own, or in groups, and possibly with guidance from a staff member.

These projects range from reading math papers to folding origami to doing original resesarch to baking. They can take lots of time every day or just some planning once or twice a week.

If this sounds appealing to you, and you have a project you'd like to work on, just talk to any of the Mathcamp staff about it! We'd be happy to help out.

If this sounds appealing to you, but you don't have a project in mind yet, then come to this event: the project selection fair! Staff will have their own project ideas for you to sign up for.

Degrees of Unsolvability. (Steve Schweber, Wednesday)

There are some mathematical problems – even some *interesting* ones – which, in a precise sense, cannot be solved by a computer (even in principle). For example, there is no way to write a computer program which will determine whether a given computer program will ever stuck in an endless loop. Perhaps surprisingly, in a precise sense some unsolvable problems are *more unsolvable* than others! There is a rich structure of "degrees of unsolvability" – also called "Turing degrees" – and after saying a bit about why unsolvable problems exist in the first place, I'll discuss some of its features. In particular, we'll see why there are "incomparably unsolvable" problems (that is, being able to solve one wouldn't help you solve the other).

Good Practice: Teaching, Learning & Beauty in Math and Beyond. (Luke Joyner, Thursday)

"In mathematics, the art of asking questions is more valuable than solving problems." – Georg Cantor Mathematicians often speak of "elegance" as a desired quality in a proof, and more generally, of "pure math" as something to hold dear... but as soon as math gets applied – which can mean a lot of things – notions of beauty and truth get a lot more complicated, to the extent they matter at all. Meanwhile, here at Mathcamp, a premium is put on math that is fun and satisfying, which often means either creative problem-solving, a series of proofs that lead to important results, or math wearing unexpected clothes. In this colloquium, we'll consider a range of attitudes toward teaching and learning, both in math, and in other fields that pursue beauty. In particular, I'll share examples from my own work in mapmaking, architecture & urban design, fields that must negotiate the messy social tangle of people and the world, without the luxuries of pure abstraction and validating proof.

Superpermutations, Traveling Salesmen, and The Melancholy of Haruhi Suzumiya. (Ari Nieh, Friday)

Suppose that, not content with linear time, you want to watch your favorite TV series with the episodes in every possible order. That is, you want every possible arrangement of all n episodes to appear as a contiguous block. How many total episodes would you have to watch? Obviously it will require some repetition, but how much? For example, if your favorite show has two episodes, you can watch them in the order 1-2-1, which will give you all 2! possible orderings.

Until 2014, it was conjectured that this problem's answer was given by a very straightforward formula. In this talk, we'll see why that formula is wrong, and what this has to do with the traveling salesman problem, 4chan, and an anime called *The Melancholy of Haruhi Suzumiya*.

VISITOR BIOS

Elizabeth Chang-Davidson. Elizabeth was a camper in '13, '14, and '15, a JC in '16 and '17, and a visitor for '18. She just graduated from MIT with a double major in mechanical engineering and math (she blames Mathcamp for the math half), and will be starting in the fall as a PhD student at Carnegie Mellon, studying metal additive manufacturing. Her mathematical interests include math with fun pictures, discrete math, and differential equations for engineers. (The strange nature of this list can be attributed to the fact that math is her secondary major.) Talk to her about mechanical engineering, math, or the intersection of the two, and she will try her best to produce interesting responses!

Luke Joyner. Luke walked from Chicago to Pittsburgh one time. He designs cities, and makes maps and things. He loves to do math, but isn't a mathematician.

Ari Nieh. Ari was first a student at Mathcamp in 1996, and has spent the last few decades as a JC, mentor, faculty, co-pear, and visitor. His mathematical interests include knot theory and gerrymandering. When not teaching math, he is a professional bass-baritone singer and works as a game designer on Magic: the Gathering.

Laura Pierson. Laura was a camper at Mathcamp in '13, '14, and '17 and a JC in '18. She is currently studying applied math at Harvard with a minor in educational studies. She does lots of teaching and volunteer work, and also enjoys painting, social dance, theater, baking, playing with her dog, and trying to play piano and ukulele.

Yuval Wigderson. Yuval has been a camper, a JC, and a mentor at various times in the past. Now he's a grad student at Stanford, trying to understand things like very big graphs and stupidly large numbers. He also enjoys crossword puzzles, ancient Greek literature, and the color purple.