CRASH COURSE ON COMBINATORIAL GAME THEORY

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Disclaimer: These notes are a sketch of the lectures delivered at Mathcamp 2009 containing only the definitions and main results. The lectures also included motivation, exercises, additional explanations, and an opportunity for the students to discover the patterns by themselves, which is always more enjoyable than merely reading these notes.

1. The games we will analyze

We will study *combinatorial games*. These are games:

- with two players, who take turns moving,
- with complete information (for example, there are no cards hidden in your hand that only you can see),
- deterministic (for example, no dice or flipped coins),
- guaranteed to finish in a finite amount of time,
- without loops, ties, or draws,
- **where the first player who cannot move loses**.

In this course, we will specifically concentrate on *impartial combinatorial games*. Those are combinatorial games in which the legal moves for both players are the same.

2. Nim

Rules of nim:

- We play with various piles of beans.
- On their turn, a player removes one or more beans from the same pile.
- Final position: no beans left.

Notation:

- A *P-position* is a "previous player win", e.g. (0,0,0), (1,1,0).
- An *N-position* is a "next player win", e.g. (1,0,0), (2,1,0).

We win by always moving onto P-positions.

Definition. Given two natural numbers a and b, their *nim-sum*, written $a \oplus b$, is calculated as follows:

- write a and b in binary,
- "add them without carrying", i.e., for each position digit,
  $0 \oplus 0 = 1 \oplus 1 = 0, \qquad 0 \oplus 1 = 1 \oplus 0 = 1.$

Example: $5 \oplus 6 =?$

$$\begin{array}{rclcl} & & \text{in binary} & & \\ 5 & = & 101 & & \\ 6 & = & 110 & & \\ 5 \oplus 6 & = & 011 & = & 3 \end{array}$$

Theorem: A position (a, b, c) in nim is a P-position if and only if $a \oplus b \oplus c = 0$.

Proof. To prove it, we need to check three things:

(1) The final position $(0, 0, 0)$ has nim-sum zero.
(2) If we start in a position with nim-sum zero, then we cannot move into another position with nim-sum zero.
(3) If we start in a position with nim-sum non-xero, then we can always move into at least one position with nim-sum zero.

□

3. The mex rule

Definitions:

- Given two game positions G_1 and G_2, their **sum** $G_1 + G_2$ consists of playing both games together: on their turn a player chooses whether to move in G_1 or in G_2. When a player can no longer move anywhere, they lose.
  Example: n-pile nim is the sum of n games of 1-pile nim.
- Two game positions G_1 and G_2 are **equivalent** if the outcome of $G_1 + H$ is the same as the outcome of $G_2 + H$ for any other game position H. We write $G_1 = G_2$.
- A **nimber** is a game position G which is equivalent to one pile of nim with n beans; n is called the **nim value** of G.

Theorem: [MEX rule] Every position of an impartial game is a nimber!
Specifically, let G be a position of an impartial game. Write down the list of nim values of all the positions that are followers of G. The nim value of G is the smallest number not on that list.

4. Example: subtraction games

Let S be a set of natural numbers. For this example, $S = \{1, 3, 4\}$. Rules:

- We play with various piles of beans.
- On their turn, a player removes 1, 3, or 4 beans from the same pile.

This game is the sum of various subtraction games with only one pile. We only need to calculate the nim values of one pile of beans for this game. We do this using the mex rule.

Position	Moves	Nim values of moves	Nim value of position
0	-	-	0
1	0	0	1
2	1	1	0
3	2, 0	0, 0	1
4	3, 1, 0	1, 0, 0	2
5	4, 2, 1	2, 0, 1	3
6	5, 3, 2	3, 1, 0	2
7	6, 4, 3	2, 2, 1	0
8	7, 5, 4	0, 3, 2	1

If you continue the above table, you will find the nim values for this subtraction game are periodic with period 7, and with values:

Number of beans (mod 7)	0	1	2	3	4	5	6
Nim value	0	1	0	1	2	3	2

Example: What is the winning move from $(10, 20, 30)$ for this subtraction game?

	Subtraction game	Reduce mod 7	Equivalent nim
Initial position	(10, 20, 30)	(3, 6, 2)	(1, 2, 0)
			↓
Winning move	(10, 17, 30)	(3, 3, 2)	(1, 1, 0)

5. Example: chomp

Rules:

- We play with a rectangular grid (or, better, a chocolate bar).
- The SW square is poisonous and cannot be eaten.
- On their move, a player eats one non-poisonous square. When they do that, they also have to eat all the squares that lie to the N of that square, to the E of that square, and everything in between.

Things we know about chomp:

- Starting from a full rectangle, there is always a winning move (i.e., the starting position is N), but we do not know what it is!
- Given any specific finite position, we can use the mex rule to calculate its nim value and find a winning move. However, we do not know of any general formula or pattern.

Examples of open questions:

- What is the winning move from an $n \times m$ rectangle?
- What is the nim value of an $n \times m$ rectangle?

Challenge: (Candy available for the first complete answer).
Find the winning strategy from the sum of a 4×4 chomp board and a 3×3 chomp board.

Note: The distribution of nim values among chomp boards looks a bit chaotic. If you want to see some of them, I have posted a bunch at
`http://www.math.toronto.edu/alfonso/chomp.pdf`