WEEK 5 CLASS PROPOSALS, MATHCAMP 2023

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About the weird symbols appearing instead of chilis

We are trying an experiment in week 5 this summer! Instead of labeling every class with 🌶, 🌶🌶, 🌶🌶🌶, or 🌶🌶🌶🌶, we are using a two-symbol code to represent two aspects of the class. A one-word summary of the two axes could be "abstraction" and "pacing", but they are described in more detail below.

As always, feel free to talk to us to learn more about what our classes will be like! Also, no matter which symbols a class has next to it, we are always happy to spend more time outside of class to help you understand the material, resolve lingering doubts, or indulge your curiosity.

The first dimension: ⚑, 🏁, and ⚐

- ⚑: Just like the ⚑ flag is fully colored in, the class is "fully filled in" with examples. The instructor makes an effort to make the ideas easier to grasp immediately, even at the cost of presenting the material less generally.
- 🏁: A class with 🏁 next to it is halfway between ⚑ and ⚐. For example, you might see how familiar objects are special cases of general, less intuitive ideas.
- ⚐: In a class with ⚐ next to it, you will often have to grapple with ideas that are hard to reduce to concrete examples. You might have to wait to see how the new abstract ideas relate to things you already know.

The second dimension: 🚲, 🚗, and ✈

- 🚲: A class with 🚲 next to it will feel relaxed and stress-free. Do not worry at all about slowing the class down if you need to ask more questions to understand; the class plan is built to accommodate this.
- 🚗: The default pace of a Mathcamp class is 🚗: comfortable, yet brisk. Questions are welcome in all Mathcamp classes—but some questions may be postponed until TAU.
- ✈: A class with ✈ next to it will feel like an exhilarating sprint. The class will need to move from topic to topic quickly to get to the finish line; you might feel like you have to review your notes and/or talk to the instructor to fully grasp the material.

Contents

Alan Chang and Neeraja's Classes

The Kakeya problem (🏁🚗, Alan Chang and Neeraja, 4 days)
In this class, we will introduce the famous Kakeya problem, which asks, roughly: how small can a set be if it contains a line segment in every direction? This question can turn into a variety of math problems, depending on how we choose to interpret "smallness" of a set. For example: what is the smallest possible area of a set in the plane that contains a line segment in every direction? Or alternatively, what is the fractal dimension of such a set? In the class, we will also introduce a variant of the Kakeya problem that asks about the "smallness" of a set which contains a circle of every radius. Note: the construction of Kakeya sets that we'll do in this class will be different from those done in Charlotte's Perron trees class in Week 4.

Class format: Interactive lecture

Prerequisites: None

Homework: Optional

Arya's Classes

Optimising Starbucks in Burlington (🏴🚲, Arya, 1–3 days)
As you might have noticed, Burlington has 6 Starbucks locations that one can go to. You, a Starbucks enjoyer, are a lazy geometer, who wants to figure out which one is the closest. As we all know, Burlington lives on the surface of a torus. (By the flat earth theory, the Earth has to be a torus—the only compact Euclidean surface!) How do you find your closest Starbucks?

In this class, we shall talk about the geometry of surfaces, tilings and Dirichlet domains. The ideas behind "optimising Starbucks" showed up very recently in my research regarding diameters of surfaces, which I'd like to share.

Class format: Interaction based lectures and group work

Prerequisites: None

Homework: None

Unicorns and Poland (🏁🚲, Arya, 3–4 days)
"Unicorn paths" were defined by Polish mathematician Piotr Przytycki, because he initially wanted to define "one-cornered paths", and the word for "one-cornered" in Polish is very similar to the word "unicorns", and "unicorns" is way cooler. Polish people are so cool <3

But why do we care about these (and what are these)? Associated to every surface, there is a graph called the "curve graph", which very literally is a graph of curves on the surface. For deep reasons, this graph is very cool. Unicorns guide the way to travel along the unyielding terrains of this graph, and tell us a lot about the geometry of this graph.

Come to this class to get a feel about stuff people study in modern geometric topology!

Class format: Lectures (hopefully interactive? :P)

Prerequisites: None!

Homework: Optional

You CAN handle the truth (🏴🚲, Arya, 2–4 days)
All your life (or possibly, all this Mathcamp), we all have been wondering, "wth is a surface?" Kayla sure had a class about building donuts, but sometimes donuts are too spicy. And also what's this "topology" that everyone keeps accusing Arya of committing?

In this class, we shall talk about what are surfaces, how to build them, and how to classify them. In particular, you shall see that every surface is essentially built from a shopping bag by attaching handles. This class shall be a gentle introduction to topology for people who haven't seen the topic before, and be a hands-on exercise in proving the classification of surfaces (one of the first major theorems in topology!) for those who have.

Class format: Interaction based lectures—you shall basically come up with the proof yourselves!

Prerequisites: None

Homework: Optional

Ben's Classes

Everything Ben knows about nonmeasurable sets (, Ben, 2–3 days)
Here are a few questions. Some are easier and some are harder! Here, $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable sets of $\mathbb{R}$. Also, $\mathcal{N}$ denotes the set of Lebesgue non-measurable sets of $\mathbb{R}$. For convenience.

(1) How large is $\mathcal{L}$?
(2) Suppose we declare two sets A, B equivalent if their symmetric difference $(A \setminus B) \cup (B \setminus A)$ has measure zero. How many equivalence classes does $\mathcal{L}$ have?
(3) Same as the previous one. How many equivalence classes does $\mathcal{N}$ have?

This class will answer all of those in pretty abrupt fashion, by breaking out some comically overpowered tools, mostly from logic. Some other topics might include the fact that we can use nonprincipal ultrafilters to build nonmeasurable sets and Bernstein sets, the sets which don't contain or miss any perfect sets.

Class format: Lecture

Prerequisites: Having seen measure (e.g. in Week 3) is probably necessary for this to make sense. It will help to have seen transfinite induction before.

Homework: Recommended

A couple things Ben kinda knows about measure zero sets (, Ben, 3 days)
Everyone[1] knows that the *countable* union of measure zero sets has measure zero. It's not hard to convince yourself that you can take the union of $|\mathbb{R}|$-many measure zero sets and build a set that doesn't have measure zero.

Continuum Hypothesis fans may now be wondering: what happens if you take the union of a number of sets that is *more* than countable and *less* than the size of the real numbers? This class will explore that topic! This will require us to ponder some apparently unrelated content—filters! dominating functions! posets!—and sit nicely between the worlds of set theory and analysis.

As advertised in the title, this is something I've seen once before, but haven't brushed up on in a while!

Class format: Lecture

Prerequisites: Having seen measure theory will help motivate one day of this class, but nothing is really necessary except having seen induction.

Homework: Recommended

[1] Or at least everyone who has taken Tanya/Charlotte's Week 3 class, or other measure theory.

Something Ben doesn't know about measurable sets (🏁🚗, Ben, 1–2 days)
A fun fact, known to J-Lo, is that if every subset of $\mathbb{R}$ is measurable, then there is a partition of $\mathbb{R}$ into a bunch of nonempty classes, and "more" classes than elements of $\mathbb{R}$, in a precise sense.

The intuitive thing is that throwing out this much of the axiom of choice makes cardinality get weird. The unintuitive thing is that I shouldn't be able to gerrymander my state into more districts than people, no matter how hard I try.

Ben doesn't currently know any of this material, but he will learn it real fast if this class runs.

Class format: Disorganized and hectic lecture

Prerequisites: Measure theory in some form

Homework: Recommended

How to rob your friends (part piracy) (🏴🚲, Ben, 1 day)
This is a class about small and somewhat silly games that I think are fun to think about. My favorite one is the Pirate Game!

We have 10 pirates and, say, 100 coins to divide up betwixt them. These greedy, bloodthirsty malcontents require a careful code of rules to regulate their conduct, and (for reasons best known to themselves) they've settled on the following:

(1) First, the top-ranked pirate[2] proposes a way to divide the gold coins among the pirates.
(2) Second, the pirates vote! If more than half of them (at first, five) vote in favor, the division is implemented. If fewer than half of them do so, the top-ranked pirate is killed, the other pirates all promote themselves, and we return to step (1) with one fewer pirate.

It's reasonably entertaining to think about what "should" happen—and that'll be the first question in this class!—but I'll also invite people to consider why it "should happen"—what are the assumptions, what are we finding, how much does it generalize? Then I'll ask a few related questions and generally try to have a fun, if silly, introduction to sequential game theory.

Class format: Almost entirely small group or individual work

Prerequisites: None

Homework: Recommended

Manifold ways to get into shape (🏁✈, Ben, 3–4 days)
In the calculus of variations class that Steve and I taught, one example I sometimes turned to was minimizing lengths on a sphere—this let us get some geometric intuition for the rather abstract business we had there!

Why don't we make that precise? This class will very quickly go through the usual notions of Riemannian geometry, focusing specifically on finding geodesics, when they are minimizing, and how these questions relate to the geometry of surfaces.

Class format: Lecture

Prerequisites: Either calculus of variations, or some analysis, some linear algebra

Homework: Recommended

The fractal zoo (🏁🚗, Ben, 2 days)
In Krishan's week 1 class, we saw a way to find fixed points of certain kinds of functions from a metric space to itself.

Here's a way to familiarize yourself with all kinds of fun and fancy fractals: Find a function that fixates on such sets, by formulating it to go from a *metric space of sets* to itself. Forgoing some

[2]Pirates are linearly ranked, from 1 to 10, with 1 being the captain and 10 being the least senior.

formalities, you can figure out the symmetries of your fixed point from the function you found, and this lets you find the fractal properties of the aforementioned fixed point.

Some particular fractals that are formed in this way are the Sierpiński triangle, the Sierpiński carpet, the Menger sponge, and the Cantor–Lebesgue function.

Class format: Lecture, hopefully supplemented with pretty pictures

Prerequisites: Metric spaces; the Banach fixed point theorem

Homework: Recommended

Ben, Ian, Kevin, Krishan, Narmada, Neeraja, Raj, Tanya, and Travis's Classes

Not the math we need, but the math we deserve (🏁🚗, Ben, Ian, Kevin, Krishan, Narmada, Neeraja, Raj, Tanya, and Travis, 2–3 days)
Do you ever feel like math is the most beautiful, transcendent, magical discipline that connects disparate ideas in a seemingly seamless manner? This class is here to shatter that illusion—unfortunately, math is filled with unintuitive, unwieldy, even monstrous counterexamples that will make you question if anything would ever make sense again. If this somehow doesn't scare you off, come join us on a tour of the ugly side of math the teaching staff has been trying to shield you from, sampled from across different fields according to the specialties of the instructors.

Class format: Lecture

Prerequisites: None

Homework: None

Charlotte and Tanya's Classes

Lebesgue integration, the chili multiverse edition (🌶, 🌶🌶, and 🌶🌶🌶, Charlotte and Tanya, 2 days)
Do you ever feel like Riemann integration sucks? We have a remedy for you—Lebesgue integration, the superior mode of integrating functions over the reals. This is a choose-your-own adventure class—if you prefer a more relaxed experience, on the first day we will provide 1-chili worksheets for you and if you still have the energy for a challenge in Week 5 of camp, you can endeavor on a 3-chili journey into the theory of Lebesgue integration with us. On the second day, we will have a 2-chili and a 3-chili option. If you enjoyed either of the Week 3 classes on measure theory, consider taking this course to see why we spent days on measure theory alone!

Class format: IBL/group work

Prerequisites: either of the week 3 measure theory classes, Riemann integration, and pointwise convergence of functions

Homework: Optional

David Roe's Classes

A crash course in Galois theory (🏁✈, David Roe, 1 day)
I believe that the most beautiful parts of mathematics are those that build a connection between two completely different topics. These connections can create breakthroughs in longstanding problems. For example, from the 9^{th} through the 16^{th} century mathematicians found formulas for solving equations of degree 2, 3, and 4, but no progress was made in degree 5 for 300 years. Galois theory provides a conceptual framework for why there are no analogous formulas for larger degrees, as well as an

explanation for why you can't trisect an angle with straightedge and compass. It is a central part of modern number theory.

In one day, we won't have time to prove everything, but we will give a precise statement of the Galois correspondence, work through some examples, and use the correspondence to prove that there are no solutions to a general quintic equation.

Class format: Interactive lecture

Prerequisites: Introduction to group theory. Know the definition of a field and be comfortable with some examples.

Homework: Optional

A crash course on class field theory (🏳✈, David Roe, 1–2 days)
The first three times I saw class field theory in a course, I had trouble wrapping my head around it. So I want to offer you the chance to start in on your own sequence of attempts! Considered the crowning achievement of early 20$^{\text{th}}$ century number theory, it provides a description of number fields with abelian Galois group. Our goal will be to understand the statements of the main theorems, and a few examples.

Class format: Interactive lecture

Prerequisites: Galois theory (as offered in Week 5, either by me or by Mark), How to count rings

Homework: None

From the Sato–Tate conjecture to murmurations (🏳✈, David Roe, 2–4 days)
The Sato–Tate conjecture was made in the 1960s based on numerical experimentation with elliptic curves, and proven in 2011 (though generalizations to higher genus curves are still open). It focuses on counting solutions to equations like $y^2 + y = x^3 - x$ modulo different primes p. We prove Hasse's theorem, which tells us that the number of solutions is about $p+1$, with an error of at most $2\sqrt{p}$. The Sato–Tate conjecture focuses on the distribution of the error term as p varies, providing two possible specific limiting distributions which you can see numerically with cool histograms.

Last year, a related phenomenon was observed by several authors, including an undergraduate at the University of Connecticut. Instead of fixing an elliptic curve and allowing p to vary, they ordered elliptic curves with rational coefficients in a specific way and looked at how a certain average value of the error term varied. The resulting plots showed an unexpected oscillation, and behave differently based on whether the elliptic curve has finitely many or infinitely many rational points (rather than points modulo p).

Class format: Interactive lecture

Prerequisites: Elliptic curves and Introduction to group theory. Finite fields and Functions of a complex variable will be helpful for a few parts of the course, but not required.

Homework: Optional

The finite group game (🏴🚲, David Roe, 1 day)
In the traditional finite group game, players take turns describing groups, ordered by size. A player is eliminated if their group is isomorphic to a group that has already been named, or if another player challenges with a smaller group that has not yet been named. Success at the finite group game relies on knowing lots of groups of small order. To help hone your skills, I've been working on a database of groups (it now has 544,831 groups, and you can find it at `beta.lmfdb.org/Groups/Abstract`). We'll talk about how you can use the database to beat your friends and about how to describe a finite group in the first place. In addition, we'll brainstorm alternative rulesets for the game, which can sometimes

get bogged down by the fact that most groups have order a power of 2 (for example, 99.15% of all groups with order between 1 and 2000 have order equal to 1024).

Class format: Mixed: bring a laptop/tablet/smartphone if you have one to explore the database, and we'll mix interactive lecture with group work (i.e. game-playing).

Prerequisites: You'll get more out of the class if you took Introduction to Group Theory, but it will be very example based so you're welcome to come even if you haven't played around with groups before.

Homework: Play the finite group game.

Della's Classes

Let $\varepsilon_0 > 0$ be sufficiently small (🏳🚗, Della, 2 days)
Ordinal numbers are what you get if you decide to count and never stop, not even at infinity. A lot of introductions to ordinal numbers go through the ordinals up to ε_0, and then skip straight past all the countable ordinals to ω_1. But in between ε_0 and ω_1, there's a lot of stuff happening—in fact, an uncountable amount of stuff!

In this class, I'll introduce the ordinal numbers in a somewhat unconventional way, and talk about ways to represent countable ordinals. Then we'll try to construct larger and larger countable ordinals, and get a sense of the flavor of the uncountable amount of stuff that's waiting for us. Of course, we'll only be able to understand a tiny (countable) portion of it, but we can get much further than ε_0.

Class format: Lecture

Prerequisites: 🚗 becomes ✈ if you haven't seen ordinal numbers before.

Homework: Recommended

Recognizing regularity (🏁🚗, Della, 2 days)
Imagine you're running a game of Word Zendo: you need to determine whether a given word fits the rule you have in mind. But the players are asking about some rather long words, and you're very forgetful, so you read the word one letter at a time and can only remember some finite amount of information between letters. Which rules would be safe to use, meaning that you can determine whether a word fits the rule under these constraints?

It turns out the rules that work are precisely the *regular languages*, and there's math to be done about them! We'll define regular languages, prove the pumping lemma (a powerful tool for proving a rule *can't* be computed this way), and show that different characterizations of the regular languages are equivalent.

Class format: Lecture

Prerequisites: None

Homework: Recommended

Seven trees in one (🏁🚲, Della, 1 day)
A (binary) tree is either a single node, or a root with a subtree on the left and a subtree on the right. Algebraically, if T is the set of all trees, this can be expressed as $T = 1 + T^2$.

If I gave you the equation $x = 1 + x^2$, you would use the quadratic formula to get $x = \frac{1\pm\sqrt{-3}}{2}$. But that doesn't make any sense in the context of trees, right? And the fact that the solutions satisfy $x = x^7$ can't possibly tell us anything about 7-tuples of trees, can it?

Class format: Lecture

Prerequisites: None

Homework: None

The inner lives of outer automorphisms (⚑🚗, Della, 2 days)
An *outer automorphism* is an isomorphism from a group to itself, which isn't just conjugation by some element. We'll primarily talk about outer automorphisms in the context of symmetric groups: it turns out that S_n doesn't have any outer automorphisms, except that S_6 has exactly one (up to inner automarphism). We'll prove this fact, and see some cool pictures of the outer automorphism of S_6, including a relationship with the group of rotations of the dodecahedron.

Class format: Lecture

Prerequisites: Group theory—you should understand the words in the blurb.

Homework: Optional

~~**Large**~~ **Small cardinal axioms** (⚑✈, Della, 1 day)
You might have heard set theorists talking about weird things like 'inaccessible', 'measurable', 'ineffable', and 'almost huge' cardinals. These are called *large cardinal axioms*, and they assert that some really big numbers exist—in particular, big enough to imply that normal set theory (ZFC) is consistent.

I'm not going to tell you about any of those. Instead, I will argue that 0 is a large cardinal, and the axiom that the empty set exists is a large cardinal axiom. Then we'll move on to the axioms that let us build even bigger cardinals like 6! The point of this class is to introduce all of the axioms of ZF, from the perspective of large cardinals.

Class format: Lecture

Prerequisites: Basic set theory

Homework: None

Eric's Classes

Clopening colloquium (⚐✈, Eric, 1 day)
This class will prove the same result as the opening colloquium, that there's no quintic formula built from continuous functions and radicals, but we'll actually try to do **all** of the details that I elided during the opening colloquium.

Okay, that's a lie, we won't do all the details in one day; but I promise an actual technical version with many more details than the opening colloquium!

Class format: Interactive lecture

Prerequisites: Group theory, at the level of the week 1 group theory class. Psychological preparation for tons of different branches of math showing up.

Homework: None

Enumerating symmetric groups (⚑🚗, Eric, 1 day)
In this class we'll devise several algorithms for enumerating permutations! It's easy to write down a simple algorithm, but we'll go through several iterations to produce algorithms with increasingly nice recursive properties. These algorithms have "real world" applications that we'll mention along the way (the giant Clock of the Long Now installed in a mountain uses one! Tim! and I use one on a regular basis!).

Class format: Worksheets!

Prerequisites: None

Homework: Optional

From high school arithmetic to group cohomology (Eric, 3–4 days)
Adding two digits numbers is something we can do mostly automatically, even though we might have to do an annoying "carrying" operation. In this class we will think **incredibly hard** about what carrying is. It turns out that carrying is an instance of group cohomology! We'll explore that connection and use it as a pathway into learning about the subject of group cohomology.

Class format: Learning through worksheets!

Prerequisites: Basic group theory, to the point of understanding that the groups $\mathbb{Z}/100\mathbb{Z}$ and $\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ are not isomorphic.

Homework: Recommended

What is your favourite bug? (Characteristic polynomials) (Eric, 1 day)
I'll tell you the story of my favourite (software) bug, which involves computing characteristic polynomials of matrices in some silly and not-so-silly ways.

Class format: Interactive lecture

Prerequisites: You've seen a definition of the characteristic polynomial of a matrix. This class involves a story about a software bug, so some programming experience will help you understand context.

Homework: None

Zeroes of recurrence sequence through p-adics (Eric, 2 days)
In this class we'll aim to prove (a simple case of) the Skolem–Mahler–Lech theorem: for an integer recurrence sequence a_n, the set of indices n where $a_n = 0$ can be at worst the union of a finite set and some arithmetic progressions. This theorem is really cool because the only known proofs rely on p-adic analysis in a crucial way. I'll introduce some minimal amount of necessary facts about the p-adic numbers and we'll prove the Skolem–Mahler–Lech theorem using p-adic infinite series.

Class format: Interactive lecture

Prerequisites: Linear algebra to the point of being comfortable with diagonalizing 2×2 matrices. Comfort with the idea of defining functions by infinite series, having seen series expansions of e^x and $\log(x)$ before is useful. Modular arithmetic, at the level of a is invertible mod n if and only if $\gcd(a, n) = 1$.

Homework: Recommended

Ian's Classes

L^p-space: What's the norm? (Ian, 1–2 days)
Since habituating norms in our society can be overwhelming, we will instead talk about norms in L^p-spaces. These are generalizations of the usual notions of distance, like Euclidean distance or Manhattan distance. With this tool, we aim to prove cool theorems like Holder's inequality and Minkowski's triangle inequality. Dependent on time, we will discuss both discrete and integral version.

Class format: Lecture + IBL

Prerequisites: None

Homework: Optional

Geometry Gala (Ian, 1 day)
Welcome to the Geometry Gala, the epilogue of the Geometry Galore! In the Gala, we will discuss Jacobi's Theorem, an application of Trig Ceva and/or radical axis theorem which were discussed in the Galore. Do not worry if you haven't taken the Galore class, since I will briefly mention Trig Ceva

and radical axis theorem in the beginning so that everyone is on the same page. Are you ready for the Grand geometric finale?

Class format: 20 minute lecture + 20 minute problem solving + 10 minute wrap up

Prerequisites: Geometry Galore recommended (though not required)

Homework: Recommended

Pseudorandomness—this is totally not random! (, Ian, 1 day)
What if every encryption was based on randomness? Then the scheme will be secure for sure! However, using random functions or generators eat up way too much memory. To solve this issue, we introduce the idea of pseudorandomness, which is not actually random, but it looks random. For instance, a pseudorandom generator takes a short string as an input and outputs a long string that seems random from a perspective of an unauthorized person. How secure are the schemes that use pseudorandom tools instead of random? Do such schemes even exist? We will investigate these questions in this class.

Class format: Lecture

Prerequisites: Introduction to Cryptography recommended

Homework: None

Symmetric Functions and their Combinatorics (, Ian, 4 days)
The topic of symmetric functions has a deep connection with combinatorics. In this course, our goal is to describe the symmetric functions with combinatorial objects such as Young tableaux and a set of lattice paths.

The main topics include:

- Monomial symmetric polynomials/functions;
- Elementary, homogeneous, Power sum symmetric functions;
- Young tableaux;
- Schur function;
- Jacobi–Trudi identity.

Class format: Lecture

Prerequisites: A good understanding of basic combinatorics

Homework: Recommended

Jane Wang's Classes

Fair division using topology (, Jane Wang, 2 days)
How can we fairly divide a cake among multiple people when each person values frosting, edges, etc. differently? We can answer this question using tools from topology, the study of continuous functions and properties that are preserved under continuous deformation. It turns out that topology has many surprising applications to fields ranging from economics to combinatorics to data science. In this short course, we will survey some applications to problems of fair division (of cakes, necklaces, rent, and more). No prior knowledge of topology will be assumed.

Class format: Interactive lecture

Prerequisites: None

Homework: Optional

Kayla's Classes

Dimers and webs (, Kayla, 1–2 days)
In chemistry, a *dimer* is a polymer with only two atoms. A *dimer covering* of a graph G is a collection of edges that covers all the vertices exactly once. One can think of vertices of G as univalent atoms that bond to exactly one neighbor. This is more commonly known as a perfect matching! A dimer model for a graph G is the set of all perfect matchings or dimer coverings of G.

In this class, we will be generalizing the notion of a dimer model to double and triple dimer models that satisfy some "web connectivity". What this boils down to is superimposing single dimer models such that their underlying graphs reduce to objects called non-elliptic webs.

If you like graphs, coloring edges of graphs and a lot of math with picture, this is the class for you!

Class format: Interactive lecture with activities!

Prerequisites: None

Homework: None

Kayla and Mia's Classes

Taming the grouchy Grassmannian (, Kayla and Mia, 1–3 days)
You know a mathematical object must be notorious if two teaching staff independently proposed classes on it and both hinted at its spookiness...

Mia and I are co-teaching a class all about the **Grassmannian** grrrr!

Here are the independent blurbs!

Mia: "The second sentence of the first paper my advisor gave to me read, "We embed this space as a linear slice of the totally nonnegative Grassmannian." And I thought to myself, "The totally nonnegative Grassmanni-who? That sounds like some sort of math boogeyman that lives under the bed!" Well, it's possible that the Grassmannian does dwell under my bed, but it turns out he's not so scary.

The goal of this class is to demystify the Grassmannian, and shed light on this space that has proven valuable to algebraic combinatorialists, physicists, differential geometers, and algebraic geometers."

Kayla: "The Grassmannian is a mathematical object that is natural place to work in. As a set, it is comprised of k-dimensional vector space in some n-dimensional space. However, even with its innocent definition, this space is notoriously grouchy for its complicated associated structures.

Namely, the Grassmannian has topological and geometric structure we can endow it with. For instance, we can view it as a *projective variety* using a set of coordinates called Plücker relations or as a compact smooth manifold by looking at something called its Schubert decomposition.

We will be exploring the dark sides of the Grassmannian and taming its grouchiness with *combinatorics.* Come see some tableaux, plabic graphs and perfect matchings tame this grouchy Grassmannian!"

Class format: Interactive lecture

Prerequisites: None (some linear algebra exposure is good e.g. reduced row echelon form, determinants)

Homework: None

Kevin's Classes

a^p **calculus** (, Kevin, 2 days)
When $p = 0$, the derivative of a^p is 0. This leads to all sorts of headaches because this makes it possible for an irreducible polynomial to have multiple roots!

This class will be an introduction to inseparable polynomials and inseparable field extensions, which only happen in characteristic p. The end goals will be to describe the structure of inseparable extensions and to show how these relate to the structure of field extensions in general.

Class format: Interactive lecture

Prerequisites: Ring theory

Homework: None

The Chevalley–Warning theorem (⚑🚲, Kevin, 1 day)
The Chevalley–Warning theorem is a nice little result that says that the number of solutions to a system of equations mod p must be divisible by p, given that the number of variables is sufficiently high relative to the degrees of the polynomials. We'll see a slick proof of the theorem using Fermat's little theorem, which we'll also introduce in the class.

Class format: Interactive lecture

Prerequisites: Modular arithmetic, polynomials

Homework: None

Kevin and Krishan's Classes

Why 0 is the biggest prime (⚐🚗, Kevin and Krishan, 2 days)
This class will be about a neat application of model theory to algebra called the Lefschetz principle.

In the first day we'll use model theory to prove the Lefschetz principle, which states that a first order property is true in $\mathbb{C}$ if and only if its true in algebraically closed fields of with arbitrarily large characteristic. We'll use a bunch of concepts from the model theory class in the second week, but a camper who didn't take the class will be able to understand the big ideas.

In the second day, Kevin will talk about how the Lefschetz principle can be used to be prove some cool results about algebra and algebraic geometry, such as (1) the fact that any injective polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is surjective (2) Hilbert's Nullstellensatz, which describes all maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$.

Class format: Interactive lecture

Prerequisites: Ring theory (required). 🚗 becomes ✈ if you haven't taken model theory.

Homework: None

Kevin, Eric, and Narmada's Classes

🍒 **Nuclear** 🍒 **physics** 🍒 **via** 🍒 **fruit** 🍒 **puzzles** 🍒 (⚑🚲, Team KEN (Kevin 🍈 Eric 🍈 Narmada), 1–4 days)
Math is hard, let's go shopping! Shopping for 🍉 fruit 🍉, that is. You may have seen 🍉 fruit 🍉 puzzles online that look deceptively simple, but turn out to be related to algebraic geometry. Did you know that many other advanced topics can be formulated in terms of 🍉 fruit 🍉? 🍉 Fruit 🍉 is everything. Physics is just 🍉 fruit 🍉.

In this class, we'll cover several different applications of fruit to nuclear physics:

🍇 Quantum 🍇 chromodynamics 🍇 with red, green, and blue 🍇 grapes 🍇 (okay, maybe 🍇 grapes 🍇 are more purple than blue). 🍊 Atomic 🍊 hybridization 🍊: did you know that besides modern examples like the 🍊🍊🍊🍊 tangelo 🍊🍊🍊🍊, more classical fruit like many kinds of 🍏🍏 apples 🍎 are also the result of hybridization? 🍋 Nuclear 🍋 fusion 🍋: what fruit does it taste like if you blend a bunch of 🍋 fruit 🍋? The splitting of the atom via 🍌 banana 🍌 splits. The plum pudding model of the atom 🥝 (sure, that's a kiwi, but it looks sort of like an atom).

Class format: IMAX

Prerequisites: Nuclear physics, and your best attitude

Homework: Required

Krishan's Classes

How the compactness theorem got its name (Krishan, 1 day)

The compactness theorem is one of the most important theorems in the field of model theory. Surprisingly, it gets its name from the topological notion of compactness. In this class we'll see why this is the case. We'll define a certain topological space associated with a theory, and we'll prove that the compactness theorem is equivalent to the fact that this space is compact.

Class format: Interactive lecture

Prerequisites: Model theory

Homework: None

Mark's Classes

A tour of Hensel's world (Mark, 1 day)

In one of Euler's less celebrated papers, he started with the formula for the sum of a geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and substituted 2 for x to arrive at the apparently nonsensical formula

$$1 + 2 + 4 + 8 + \cdots = -1\,.$$

More than a hundred years later, Hensel described a number system in which this formula is perfectly correct. That system and its relatives (for each of which 2 is replaced by a different prime number p), the p-adic numbers, are important in modern mathematics; we'll have a quick look around this strange "world".

Class format: Interactive lecture

Prerequisites: Some experience with the idea of convergent series

Homework: None

Counting, involutions, and a theorem of Fermat (Mark, 1 day)

Involutions are mathematical objects, especially functions, that are their own inverses. Involutions show up with some regularity in combinatorial proofs; in this class we'll see how to use counting and an involution, but no "number theory" in the usual sense, to prove a famous theorem of Fermat about sums of squares. (Actually, although Fermat stated the theorem, it's uncertain whether he had a proof.) If you haven't seen why every prime $p \equiv 1 \pmod 4$ is the sum of two squares, or if you would like to see a relatively recent (Heath-Brown 1984, Zagier 1990), highly non-standard proof of this fact, consider taking this class.

Class format: Interactive lecture

Prerequisites: None

Homework: None

Elliptic functions (Mark, 4 days)

Complex analysis, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k)\,,$$

where $\sigma_i(k)$ is the sum of the i^{th} powers of the divisors of k. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1^3 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1],$$

which you are welcome to check if you run out of things to do.)

Class format: Interactive lecture

Prerequisites: Functions of a complex variable; in particular, Liouville's theorem

Homework: Optional

Galois theory crash course (🏳✈, Mark, 4 days)
In 1832, the twenty-year-old mathematician and radical (in the political sense) Galois died tragically, as the result of a wound he sustained in a duel. The night before Galois was shot, he hurriedly scribbled a letter to a friend, sketching out mathematical ideas that he correctly suspected he might not live to work out more carefully. Among Galois' ideas (accounts differ as to just which of them were actually in that famous letter) are those that led to what is now called Galois theory, a deep connection between field extensions on the one hand and groups of automorphisms on the other (even though what we now consider the general definitions of "group" and "field" were not given until fifty years or so later).

If this class happens, I expect to be rather hurriedly (but not tragically) scribbling as we try to cover as much of this material as reasonably possible. If all goes well, we might conceivably be able to get through an outline of the proof that it is impossible to solve general polynomial equations by radicals once the degree of the polynomial is greater than 4. (This depends on the simplicity of the alternating group, which we won't have time to show in this class.) Even if we don't get that far, the so-called Galois correspondence (which we should be able to get to, and probably prove) is well worth seeing!

Class format: Interactive lecture

Prerequisites: Group theory; linear algebra; some familiarity with fields and with polynomial rings

Homework: Optional

Quadratic reciprocity (🏁✈, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is q a square modulo p?"

(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And the next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, with or without technology!

Class format: Interactive lecture

Prerequisites: Some basic number theory (if you know Fermat's little theorem, you'll probably be OK)

Homework: Optional

The Cayley–Hamilton Theorem (🏁✈, Mark, 1 day)
Take any square matrix A and look at its characteristic polynomial $f(X) = \det(A - XI)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(X) = X^4 - 6X^3 - X^2 + 17X - 8$, then compute $f(A) = A^4 - 6A^3 - A^2 + 17A - 8I$. The answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.

Class format: Interactive lecture

Prerequisites: Linear algebra, including a solid grasp of determinants (if it happens, the "Magic of Determinants" class would definitely take care of that)

Homework: None

The magic of determinants (🏁🚗, Mark, 3 days)
In many introductions to linear algebra, determinants are either hardly mentioned at all, or introduced using a very non-intuitive recursive definition (the Laplace expansion). If that has left you feeling dissatisfied, this might be a good class for you! If all goes well, we'll give a definition of determinant that's both motivated and rigorous, along with proofs of its main properties (such as Laplace expansion). We may also get to a few applications, such as general formulas for the inverse of a matrix and for the solutions of n linear equations in n unknowns ("Cramer's Rule").

Class format: Interactive lecture

Prerequisites: Some linear algebra, including linear transformations and matrix multiplication

Homework: Optional

Mia's Classes

Sophie Germain primes (🚩🚲, Mia, 1 day)
Born in 1776, Sophie Germain was a French mathematician, physicist, and philosopher. Despite having to publish under a male pseudonym (in order to have her work recognized) she spent much of her life at the forefront of mathematics and did ground-breaking work studying the Fermat equation $x^q + y^q = z^q$ for primes q with the property that $p = 2q + 1$ is also prime. Such primes are now known as *Sophie Germain primes*.

In this class, we'll study Sophie Germain primes, prove a surprising fact about primitive roots modulo a Sophie Germain prime, and celebrate a seriously awesome female mathematician!

Class format: Interactive lecture

Prerequisites: Introduction to number theory

Homework: Optional

The Dinitz problem (🏁🚗, Mia, 1 day)
In the theory of Latin squares, a first question you might ask is, "given an $n \times n$ grid and a set of n

symbols, is it possible to fill the cells such that each row and column contains each number exactly once?” “Easy,” you declare and scribble down a grid.

But not so fast. What if each cell has strong *preferences* about the symbols it will accept? That is, what if there are $m \geq n$ symbols and each cells has a list of n allowable symbols? Is it still possible to fill in the $n \times n$ grid such that no symbol is repeated in any column or row? In this class, we'll answer this question, seeing elegant applications of oriented graphs, list coloring, and the Gale–Shapley matching algorithm along the way.

Class format: Interactive lecture

Prerequisites: List coloring (I'm happy to explain it before class!)

Homework: None

Perfection (, Mia, 1–2 days)

Commutative algebraists have their excellent rings and algebraic geometers have their wonderful compactifications, but no one achieves perfection like graph theorists. In this class, we will prove none other than the perfect graph theorem which, in addition to having an excellent[3] name, has an exceedingly clever proof. So, what is perfection? In Graph colorings, we proved that $\omega(G) \leq \chi(G)$ and then asked, can we push those two invariants arbitrarily far apart. An alternative question one might ask is, what graphs achieve equality? Or even better, which graphs achieve equality and have that their subgraphs achieve equality too? The answer, perfect graphs! And what's more, the perfect graph theorem gives us an elegant characterization of these graphs.

Note: Graph colorings is not a prerequisite.

Class format: Interactive lecture

Prerequisites: Graph theory

Homework: Optional

Mia and Nathan's Classes

Imperfection (, Mia and Nathan, 1 day)

While things might be all sunshine, daisies, and perfection in the world of graph theory, life gets yuckier in the land of analysis. Exhibit A: $0.9999\ldots = 1$. Who allowed this nonsense?!?! Maybe they should have thought about *not* including so many darn digits. Why can't 0 and 1 be enough? Which brings us to Exhibit B: Have you ever successfully represented 2 in base 10 using just the numbers 0 and 1? We didn't think so. (And if you have, do let us know.) In short, there is an inherent tension between representing real numbers uniquely and representing every real number. AKA bases suck. In this class, we'll prove why.

Class format: Interactive lecture

Prerequisites: Epsilons and deltas

Homework: Optional

Misha's Classes

Computing trig functions by hand (, Misha, 1 day)

When you learn about trig functions, you typically memorize a few of their values (for 30° or 45°, say) and if you want to know any of the other values, you get pointed to a calculator.

Has that ever seemed unsatisfying to you? If so, take this class, in which we'll see that finding some of these values is as easy as solving polynomials, and approximating all of them is as easy as

[3]Excellent under the English definition, not the algebraic one.

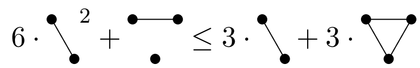
multiplication. If time allows, we'll learn how to compute inverse trig functions, and also how to quickly find lots of digits of π.

Class format: Interactive lecture

Prerequisites: Be familiar with the formula $e^{ix} = \cos x + i \sin x$.

Homework: None

Flag algebra marathon (, Misha, 1 day)
Do you want to write inequalities where our variables are graphs?



Do you want to see the most roundabout proof that $R(3,3) = 6$?

Do you want to prove graph theory results from 1907 by methods developed in 2007?

Do you want to have gigabytes of multiplication tables to throw at supercomputers?

And do you want to do this **all in one day**?

Then this is the class for you.

This class is being taught in **marathon** format. This means that it will appear in the schedule vertically: we'll go 10:10am–12pm, take a break for lunch, and reconvene 1:10pm–3pm.

Note: Travis's class on graph inequalities through magic will have a similar flavor and prove a few of the same applications, though we're fundamentally doing different things. To avoid confusing you with completely different sets of inequalities where the variables are graphs, at most one of our classes will appear in the Week 5 schedule.

Class format: Marathon! (It will not all be lecture; you will spend some time in class solving problems.)

Prerequisites: You should be comfortable with matrix multiplication, linear transformations, and an assortment of basic concepts from graph theory.

Homework: None

Going in cycles (, Misha, 2 days)
A *knight* is a chess piece that jumps from a square to any other square exactly $\sqrt{5}$ units away. Put one of these in the corner of an 8×8 chessboard. Can it visit every other square of the board exactly once, then come back to the start?

This is an instance of the Hamiltonian cycle problem. In general, it's very hard to solve. We will talk about some ways we can guarantee a solution exists—or quickly demonstrate that it doesn't.

Class format: Interactive lecture

Prerequisites: Graph theory

Homework: Recommended

How to pronounce "Lucas" (, Misha, 2–3 days)
The French mathematician Édouard Lucas is kind of awesome. In between studying Fibonacci numbers, finding patterns in Pascal's triangle, and inventing an algorithm to look for Mersenne primes and perfect numbers, he invented the paper-and-pencil game Dots and Boxes, as well as the Tower of Hanoi puzzle (which he marketed under an anagram name, like Voltaire or Voldemort: N. Claus de Siam).

He's also probably tied with Euler for being the mathematician with the most mispronounced name. To pronounce his last name, say "Lu" to rhyme with "flew" and "cas" as though you're saying "car" with a strong Bostonian accent.

In this class, we'll focus on a subset of the cool things that Lucas did; we'll find out why the Lucas sequence deserves to be studied alongside the Fibonacci sequence, see its connection to primality testing, and finally find out how Lucas proved that

$$2^{127} - 1 = 170\,141\,183\,460\,469\,231\,731\,687\,303\,715\,884\,105\,727$$

is prime (a process that famously took him 19 years).

Class format: Interactive lecture

Prerequisites: Some modular arithmetic. Specifically, you should be comfortable with Fermat's little theorem about $a^{p-1} \bmod p$, and it will help if you've been exposed to quadratic reciprocity, though it's fine if you don't remember the statement.

Homework: Optional

Problem solving: quick and quirky questions (, Misha, 1 day)
Lots of problem-solving classes at Mathcamp are focused on the "respectable" kinds of olympiad problems: the ones where you sit down for a while and write a proof.

In this class, we'll instead look at relays-style problems that you have to solve under time pressure.

(If you took a class with the same name last year, the format will be the same but the problems will be new.)

Class format: Bursts of timed problem-solving, followed by going over problems.

Prerequisites: None

Homework: None

Narmada's Classes

A donut has four holes (, Narmada, 2 days)
This class will introduce us to the wonderful world of combinatorial topology (which only became wonderless once people started calling it "algebraic" topology.) We'll turn the donut into a directed 'graph' and use combinatorics to find four vector spaces that correspond to the four holes of the donut.

Class format: Lecture, but we will work through examples in groups.

Prerequisites: Know what a vector space is, and what the range and kernel of linear maps are.

Homework: Recommended

Axiom of choice (, Narmada, 2 days)
Why do mathematicians get so fussy about the axiom of choice? We'll talk a little bit about why the axiom of choice isn't just obviously true. We'll look at some obviously fake statements that are equivalent to the axiom of choice. We'll see why math without the axiom of choice might be sad sometimes. And by the end of this class, *you* get to be a mathematician who's fussy about the axiom of choice!

(If you are a returning camper, this is the same class I ran last year.)

Class format: Lecture

Prerequisites: None

Homework: None

Neverending study Hall (, Narmada, 2 days)
Can't get enough of Studying Hall? In this class we'll look at Hall's theorem for infinite graphs,

Hall's theorem for infinite graphs with topologies on them, and Hall's $(1+\epsilon)$-theorem that implies the Banach–Tarski paradox.

Class format: Lecture

Prerequisites: study Hall by Travis

Homework: Optional

Outro to linear algebra (, Narmada, 2 days)
If you loved linear algebra and really want to get your money's worth, boy do I have the class for you! We'll take a quick peek into the world of infinite-dimensional vector spaces with the Riesz representation theorem for Hilbert spaces. This class will focus more on seeing the details of proofs, so the pace will be faster than my week 1 linear algebra class. Since we are working in a more abstract setting, I will also leave some details as homework (with steps to guide you through them!)

Class format: Lecture

Prerequisites: Intro to linear algebra (linear maps and inner products)

Homework: Optional

Neeraja's Classes

Heisenberg's uncertainty principle (, Neeraja, 3 days)
Heisenberg's uncertainty principle, as used in physics, states that

$$\Delta p \cdot \Delta x \geq \frac{h}{4\pi},$$

where Δp is the error in the measurement of the momentum of a particle, Δx is the error in measurement of the position, and h is Planck's constant. The principle is derived from its mathematical version by interpreting momentum and position, or frequency and time, as "Fourier transform duals", or "conjugate variables" with respect to the Fourier transform.

For certain well-behaved functions $f\colon \mathbb{R} \to \mathbb{C}$, we can define a related function known as the *Fourier transform* of f. The mathematical version of the uncertainty principle states that the "spread" of a function is inversely proportional to the "spread" of its Fourier transform, so if the mass of a function is tightly concentrated in a region, the its Fourier transform is much more widely spread out. In this class we'll do some Fourier analysis to state and prove the mathematical version of the uncertainty principle.

Class format: Interactive lecture

Prerequisites: Single-variable calculus and complex numbers (Euler's formula)

Homework: Optional

Philosophy of Math (, Neeraja, 1 day)
What is the relationship between a mathematical proof and our knowledge of the theorem that it proves? Do some mathematical proofs go beyond establishing the truth of their theorems and actually explain why the theorems are true? How has our mathematical knowledge grown throughout history? We'll discuss some or all of these questions. Most of the questions and examples motivating the discussion come from the book *Proofs and Refutations* by Imre Lakatos.

Class format: Some lecture, mainly discussion

Prerequisites: None

Homework: None

Raj's Classes

Introduction to Schubert calculus (Raj, 4 days)
"How many lines generically meet four given lines in space?" One goal of modern Schubert calculus is to solve such enumerative problems using tools from linear algebra and algebraic geometry. This course will be an introduction to this subject. The main components we will discuss are Schubert polynomials and the Grassmannian of linear subspaces of fixed dimension in space.

Class format: Interactive lecture

Prerequisites: Linear algebra, group theory

Homework: Recommended

Steve's Classes

Gödel's completeness theorem (Steve, 3 days)
Suppose we write down a set of rules for mathematical arguments. We want things like "From A and B you can infer [A AND B]," as well as more complicated rules for handling quantifiers. Obviously you don't want to add any rules that are false (e.g. "From [A OR B] infer A"). But what about the opposite side of things: how do you know that you haven't missed any rules?

In this class we'll prove Godel's completeness theorem: that a particular set of rules for mathematics is complete in the sense that, if they don't let you derive A from Γ, there must be a model of Γ in which A fails. This is a beautiful result, and we'll see two of its proofs: the first proof we'll see (and the one which is usually taught these days) is due to Henkin, and the second will be Godel's original proof.

If time permits, we'll see further elaborations of this result—in particular, we'll talk about what happens once we allow *infinitely long* sentences in our language!

Class format: Lecture

Prerequisites: Model theory (or willingness to accept some informality); we'll need to say things like "statement X is true in structure $\mathfrak{M}$."

Homework: None

How not to integrate (Steve, 1 day)
The function e^{-x^2} has no elementary antiderivative, but the doubly-improper "Gaussian integral"

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

can be easily computed: it's just $\sqrt{\pi}$. To see this, square it, rewrite the result as a double integral, and change to polar coordinates—this shifts from "$dx\,dy$" to "$r\,dr\,d\theta$," letting us wrap up with a simple u-substitution.

This is a really clever trick, and it's natural to hope that it can be used to solve other integrals. Unfortunately, it can't: in a precise sense, any integral which can be solved this way is just a trivial modification of the Gaussian integral itself. This is a simple argument due to Robert Dawson, and we'll see it in detail. If time remains, we'll examine a follow-up argument by Denis Bell, that (1) there is actually a more general version of the Gaussian integral trick and (2) it's also useless.

Class format: Lecture

Prerequisites: Multivariable calculus; be comfortable with the solution of $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ discussed in the blurb.

Homework: Recommended

Not theory (Steve, 2–3 days)
Classical propositional logic is pretty boring; there isn't a lot you can do with just two truth values. At various points more interesting logical systems have been proposed, with varying levels of success/interestingness. Eventually a coherent "theory of propositional logic**s**" emerged, treating in a very general way the properties that any "reasonable" deductive system can have. For example, in a precise sense it *is* possible to add a "half-negation" operator to classical logic without breaking it, but it is *not* possible to add an "un-negation" operator to intuitionistic logic (= classical logic without double negation cancelling) without breaking it.

(That is, and in contrast to things Raj may have told you, a square not is simpler than an un-not.)

We'll explore this area of logic, with particular focus on negation.

Class format: Lecture

Prerequisites: Comfort with Boolean algebra (basically, you should know what a truth table is, and what the truth table for "If A then B" looks like). You do **not** need to have heard of intuitionistic logic before!

Homework: Recommended

The pseudoarc (Steve, 1 day)
A *planar continuum* P is basically a shape drawn in the plane. The formal definition is "connected compact subset of $\mathbb{R}^2$," but "connected" is pretty self-explanatory and "compact" in this case can be taken to mean "closed and bounded." There are some weird planar continua out there—consider the graph restricted to $x \in (0,1]$ of $\sin(\frac{1}{x})$ together with the line segment $\{0\} \times [-1,1]$—but overall one might suspect that planar continua are pretty nice things.

In this class we'll see that that is WRONG WRONG WRONG! We'll construct a *hereditarily indecomposable* planar continuum; essentially, this is a planar continuum which cannot be "cut in half" in any reasonable way. Interestingly, it turns out that "most" planar continua are of this form; while we won't prove this, we will state it precisely.

Class format: Lecture

Prerequisites: None

Homework: Recommended

a Metric with Compact Spaces as Points (Steve, 1–2 days)
I heard you like metric spaces, so I put some metric spaces in your metric spaces!

It turns out that there is a meaningful way to define the "distance" between two (compact) metric spaces. This gives rise to something called the Gromov–Hausdorff space. In this class we'll construct the Gromov–Hausdorff space and examine some of its properties.

Class format: Lecture

Prerequisites: Metric spaces

Homework: Recommended

Susan and Ben's Classes

Ben teaches Susan's class (Susan and Ben, 1 day)
Five minutes before class time, Susan will send Ben a slide deck. Good luck, Ben!

Class format: Slideshow Telephone

Prerequisites: None. Prerequisites would be too big a hint.

Homework: None

Tanya's Classes

Honey, I shrunk the vectors (, Tanya, 1 day)
Imagine you have a high dimensional dataset (e.g. how much each camper likes every possible LN2 ice cream flavor) that you're trying to analyze, however, every algorithm that you attempt to use ends up being much too slow. There are several possible ways to address this issue—use a cleverer, faster algorithm, get a more powerful computer to run your code on, or find a way to shrink your data without losing too much of the original structure. The focus of this class will be the latter technique. We will see a simple randomized procedure for dimensionality reduction while (almost) preserving pairwise distances between the points in your dataset with high probability. If you enjoyed the second half of my Week 2 class comparing randomized and deterministic computation, you will most likely have fun here as well!

Class format: Interactive lecture

Prerequisites: Knowing how matrices act on vectors

Homework: None

Percolating through percolation theory (, Tanya, 1 day)
Percolation theory is an area of probability theory that studies the structure of infinitely large graphs after edges get randomly removed. We will see that certain properties for such graphs will either hold with probability 0 or 1, however, perhaps surprisingly, we may not be able to easily distinguish between the two situations. We will also learn about some open questions in this field, as they are often rather straightforward to state, yet incredibly challenging to resolve.

Class format: Interactive lecture

Prerequisites: Having seen a bit of probability before might be helpful.

Homework: None

Tim!'s Classes

Calculus without calculus (, Tim!, 1–2 days)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Eleni is 5 cubits tall and Krishan is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Eleni's head to the top of Krishan's head that touches the ground in the middle. What is the shortest length of string you can use?
- Della rides a bike around an elliptical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an very straight section of the Lake Champlain shoreline. The dog's person stands 20 meters away along the shoreline, and throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?

- When Mathcamp rented out the movie theater to see the *Barbie* movie, you had the chance to choose the optimal seat. Which seat should you have chosen so to make the screen take up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Class format: Interactive lecture

Prerequisites: We won't use calculus (that's the point), but it would be good if you've seen it for context.

Homework: Recommended

Intersecting polynomials (, Tim!, 2 days)
You might think that everything there is to know about one-variable real polynomials has been known for hundreds of years. Except, in 2009, while bored at a faculty meeting, Kontsevich scribbled down a brand new fact about polynomials. You'll discover it.

Class format: IBL—you'll discover this result from start to finish in groups.

Prerequisites: None

Homework: None

Sperner's lemma (, Tim!, 2 days)
Suppose that after Mathcamp you and your new friends decide to hold a reunion in Alaska. You rent an igloo to share, and you will all chip in to pay for it. But how do you decide who gets to sleep where, and how much each person should pay? You'd be willing to pay more to sleep in a bed rather than a couch, and that's still worth more than the frozen floor. Different people have different preferences—some may rather get a nice bed, while others might not care where they sleep as long as they save money. Can you arrange it so that nobody is jealous of another's sleeping spot (with its corresponding price tag)?

Especially if there's a lot of people, it's not clear that you can come up with envy-free room assignments. But it's possible! The justification relies on an aesthetically-pleasing result about coloring points in a triangle. And this unassuming lemma (whose proof is particularly cute) can prove a whole host of other facts too! It gives a way to fairly divide a cake among friends. It can prove Brouwer's fixed-point theorem (if I crumple up a map of Champlain College and throw it on the ground, some point on the map will be on top of the real-life point it represents). It can prove that a square cannot be divided into an odd number of equal-area triangles, but you have to see it to believe it.

Come experience what this lemma can do!

Class format: Interactive lecture

Prerequisites: None

Homework: None

The puzzle of the superstitious basketball player (, Tim!, 1 day)
Sometimes I encounter a math problem, go though a bunch of work to solve it, then arrive at an answer too simple or elegant for the mess of work I did. At that point, I know that there is something deeper and more interesting going on, and I have to know what it is! If you took my class on Guess Who?, you've gone on that journey with me. I also had such an experience with the math puzzle below. It's from Mike Donner, and it was published on FiveThirtyEight.

> A basketball player is in the gym practicing free throws. He makes his first shot, then misses his second. This player tends to get inside his own head a little bit, so this isn't

> good news. Specifically, the probability he hits any subsequent shot is equal to the overall percentage of shots that he's made thus far. (His neuroses are very exacting.) His coach, who knows his psychological tendency and saw the first two shots, leaves the gym and doesn't see the next 96 shots. The coach returns, and sees the player make shot No. 99. What is the probability, from the coach's point of view, that he makes shot No. 100?

I remember solving it. I had to do a bit of tedious calculation to arrive at the final answer. And when I saw the answer, I was surprised. It was so simple. I thought I was done with the puzzle, but really I was just beginning. Such a simple answer had to have a simple explanation, right? There are in fact a few simple explanations, each more satisfying than the previous.

In the end, I will make the following claim: even if we accept the scenario described by the puzzle, the basketball player's view of the world is totally wrong, and he is probably just superstitious. Perhaps there is a lesson here that we can take back with us to our real lives.

Class format: Interactive lecture

Prerequisites: None

Homework: None

Travis's Classes

Bonus back to basi(c)s (, Travis, 1 day)
For alliteration purposes, I should say that this class will be bigger and better, but it will actually be smaller and sassier. We'll be back proving combinatorial theorems using vector spaces, but this time vector spaces over *finite fields*!

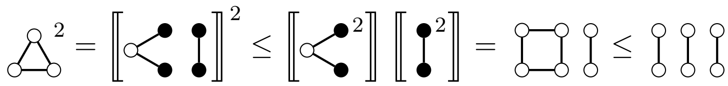
For example: If $F_1, \ldots, F_m$ are subsets of $\{1, 2, \ldots, n\}$ such that every set contains an odd number of elements, but the intersection of any two sets contains an even number of elements, then $m \leq n$. But if instead every set has an even number of elements, then it is possible to have up to $m = 2^{\lfloor n/2 \rfloor}$ different sets (but no better)! We'll prove both of these statements using linear algebra.

Class format: Interactive lecture

Prerequisites: Know that $\mathbb{Z}/2\mathbb{Z}$ is a finite field or be willing to take some stuff on faith. **You do not need to have taken *Back to basi(c)s* to take this class.**

Homework: Optional

Graph inequalities through magic (, Travis, 2 days)
In this class, we will:
prove inequalities
about graphs
using magic.
We'll get to the point where this



is a complete proof of a foundational result in extremal graph theory. It'll be completely rigorous, and we'll see not just why it's true (and what it means) but also lots of cool examples.
Note: See Misha's note in *Flag algebras marathon.*

Class format: Interactive lecture

Prerequisites: It would probably help if you've thought about graphs before.

Homework: Optional

Lastly, choose randomly (, Travis, 1 day)
Have you been taking a random walk on the Champlain campus, pining over the loss of random choices in Mathcamp classes since week 2? Have you despaired over the dire dearth of combinatorial chaos? Then rejoice as we revive that which you thought you had lost.

In this class, we talk about yet one more method in the probabilistic arsenal for solving combinatorial problems, one which that dastardly devil Travis has thus far hidden from your prying eyes. We'll see how to use the Lovász Local Lemma to solve problems in graph theory, or to color the real number line to break *every single pattern in the world*.

Class format: Interactive lecture

Prerequisites: **You do not need to have taken *First, choose randomly* to take this class.** Prior experience with graphs (the combinatorial ones) will help make one application easier to understand.

Homework: Optional

The Ra(n)do(m) graph (, Travis, 1 day)
In *First, choose randomly*, we limited ourselves to randomly producing finite graphs. But it turns out that strange things happen once we start to choose *infinite* random graphs, and in this class, I will tell you the story of these graphs. It is surprisingly short—short enough to fit in 50 minutes—but we'll hit upon some serious mathematics, including the 0–1 law for graphs, which says that every "first-order" graph property is either true for almost every graph or false for almost every graph, with no middle ground.

So settle in, my friends, while I tell you the tale of the infinite random graph. Lean back while I weave for you the disparate threads of its history, from probability theory and model theory to the enduring legacy of popular folk singer-songwriter Arlo Guthrie's most enduring song. Come experience the full range of human emotion, shouting with excitement, gasping with amazement, and weeping over what might have been, as you revel in the dramatic legend of the infinite random graph.

Note: I taught the first half of this class under the same name last year; this version will cover more material.

Class format: Storytime.

Prerequisites: Be familiar with basic laws of probability. **You do not need to have taken *First, choose randomly* to take this class.**

Homework: Optional

The transcendence of a single number (including Liouville's constant) (, Travis, 1 day)
Proving the transcendence of many numbers is good, but proving the transcendence of one number is good enough. Plus, for this, we can prove transcendence simply and quickly: in one day only!

In this class, we'll take the quick route to finding an explicit transcendental number (Liouville's constant) and then see why the title of this class is actually not telling the truth and consequently find an uncountable set of real numbers and prove that all of them are transcendental.

Class format: Interactive lecture

Prerequisites: You don't need nuthin'!

Homework: None

study Hall (, Travis, 1 day)
Are you tired of a week of study hall without any new mathematical content? Have I got some news for you! In this class, instead we'll study Hall('s matching theorem about bipartite graphs and systems of distinct representatives, and we'll look at a few applications. Possible applications include the characterization of stochastic matrices, Latin squares, and various competition-style problems).

Class format: Interactive lecture
Prerequisites: Know what a graph is
Homework: Optional