CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2023

Contents

9:10 Classes

Beyond inclusion/exclusion (🌶🌶🌶, *John Mackey*, TWΘFS)
Inclusion/Exclusion is a useful counting method wherein one successively corrects overapproximations and underapproximations. We'll spend one day reviewing Inclusion/Exclusion and then consider Möbius Inversion on Posets and Sign Reversing Involutions.

Möbius Inversion will provide a useful look inside the dual nature of accruing and sieving objects, and cast an algebraic context onto Inclusion/Exclusion. Sign Reversing Involutions will lend a matching perspective to the calculation of alternating sums.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Experience with elementary counting and matrix algebra recommended.

Epsilons and deltas (🌶🌶, Ben & Charlotte, TWΘFS)
This class is a rigourous introduction to limits and related concepts in calculus. Consider the following questions:

(1) Every calculus student knows that $\frac{d}{dx}(f+g) = f' + g'$. Is it also true that $\frac{d}{dx}\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f_n'$?
(2) Every calculus student knows that $a+b=b+a$. Is it also true that you can rearrange terms in an infinite series without changing its sum?

Sometimes, things are not as they seem. For example, the answer to the second question is a resounding "no." The Riemann rearrangement theorem, which we will see, states that we can rearrange the terms in infinite series such as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ so that the sum converges to π, e, or whatever we want!

To help us study the questions above and many other ones, the key tool we'll use is the "epsilon-delta definition" of a limit. This concept can be hard to work with at first, so we will study many examples and look at related notions, such as uniform convergence. Being comfortable reasoning with limits is central to the field of mathematical analysis, and will open the door to some very exciting mathematics.

Homework: Required

Class format: A mix of lecture and group work.

Prerequisites: A calculus class of some kind

Required for: Non-standard analysis (W3); Why do we need measure theory? (W3); aspacefillingcurve (W4)

Infinite Ramsey theory (🌶🌶🌶, Susan, TWΘFS)
Suppose you throw a party and invite six people, and some of those people know each other and some don't. A well known result from graph theory tells us that we can guarantee a group of three mutual friends, or three mutual strangers. But suppose you wanted to throw a much, much bigger party... like an infinite party? Can you guarantee an infinite group of friends or strangers?

More generally, suppose we want to color the edges of a graph with κ colors, where κ is an infinite cardinal number. How many vertices do we need in order to guarantee a monochromatic clique? In this class, we'll find a bound for the answer to this question, and prove that our bound is sharp.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Representation theory of the symmetric groups (🌶🌶🌶🌶, Raj, TWΘFS)
Representation theory is a field designed to understand how groups act on vector spaces. In this course, we will focus specifically on the symmetric groups and understand how their representation theory is connected with partitions of natural numbers and combinatorial gadgets known as Young tableaux. Importantly, you will learn how to use combinatorics to tackle abstract problems.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Linear algebra. Group theory.

What are your numbers worth? (🌶, Eric, TWΘFS)
In this class we will figure out what numbers are worth. Some numbers will be worth a lot, other numbers will be worth negative amounts, yet still others will be worth fractional amounts. We will learn the difference between knowing a number (very extremely mind-bogglingly hard) and knowing what a number is worth (surprisingly incredibly magically easy). We will gain the power of being able to figure out the worth of numbers (and many cool corollaries of this power) by making line doodles. In actuality this is a course about the local part of algebraic number theory, but those words won't show up much until the last day. (If you want more technical words: we'll be learning about the p-adic valuation(s) on $\mathbb{Q}$ and finite extensions thereof, through the lens of Newton polygons. But don't worry if you don't know any of these words yet!)

Homework: Recommended

Class format: Mixture of interactive lecture and worksheets. Tentatively: day 1 will be lecture, days 2 through 4 will be mostly worksheets, day 5 will be back to lecture.

Prerequisites: You'll need to be comfortable with modular arithmetic, at the level of knowing that you can translate between statements in modular arithmetic and statements in divisibility of integers; Mia's week 1 introduction to number theory class will prepare you super well. Towards the end we will do some linear algebra with coefficients in $\mathbb{Z}/p\mathbb{Z}$, but at a 🌶 pace so don't be scared. There will be a few very optional homework problems that use background in ring theory and linear algebra.

10:10 Classes

Green's Theorem (🌶🌶🌶, Mark, TWΘFS)
How can you measure the area of a lake without ever venturing into the water? If you have the right

mechanical device (called a *planimeter*), you can do this by simply moving the device all around the lakeshore. Green's Theorem, which gives a fundamental connection between line integrals over closed curves in the plane and double integrals, explains how this is possible. The theorem (to be stated in class) has other useful applications, as well as interesting generalizations. It will be used (without proof) in the weeks 3-4 class on functions of a complex variable. Meanwhile, in this one-day class we'll quickly review the necessary concepts, state and sketch a proof of the theorem, give a paradoxical proof that $2\pi = 0$, and if time permits, mention a quite general result that has Green's Theorem as a special case. If you've taken the multivariable calculus crash course in week 1, you'll definitely be ready for this class, because we'll start where that course left off.

Homework: None

Class format: Interactive lecture

Prerequisites: Multivariable calculus (if possible, having encountered line integrals as well as double integrals)

Introduction to cryptography (, Ian, TWΘFS)
CIPHERTEXT: LIBRX FDQQR WGHFL SKHUW KLVWH AWWKH QBRXV KRXOG WDNHW KLVFO DVVYZ

plaintext: if you cannot decipher this text then you should take this class

Contents:

- Art of encryption & decryption
- Danger of attacks & security

Homework: Recommended

Class format: Mostly lecture with some activities

Prerequisites: Basic probability

Introduction to model theory (, Krishan, TWΘFS)
A large part of math boils down to studying "sets with structure." These could be groups, fields, vector spaces, any of the different varieties of graphs, or even more exotic examples like valuation rings or ordered groups. In this class we'll see how all of these are manifestations of the same idea and we'll study them all simultaneously at a meta level. We'll learn how to view mathematical objects like the real number line and the field of complex numbers in a new light. We'll encounter strange objects like infinite groups which behave as if they are finite. We'll see why large random graphs all tend to look the same, and we'll even talk about how you can build models of set theory where the reals are countable!

*Thanks to Aaron Anderson for the blurb inspiration!

Homework: Recommended

Class format: Interactive Lecture

Prerequisites: Group theory (a week at mathcamp would be enough)

Mechanics of fluid flow (, Neeraja, TWΘFS)
The Euler and Navier–Stokes equations are partial differential equations that describe the motion of fluid particles (gases and liquids) in $\mathbb{R}^3$. The equations are derived from Newton's second law of motion, with the force equal to a sum of contributions by pressure, external forces and in the Navier–Stokes case, viscous stress. Though the Euler equations were first written down in 1757, and the Navier–Stokes equations were formulated as a generalization of the Euler equations in the 19^{th} century, theoretical understanding of the solutions to these equations remains incomplete. Mathematicians have neither proved that smooth solutions to the equations always exist, nor have they found any counterexamples.

The goal of this class is to develop enough of the theory of fluid mechanics to derive the Euler and Navier–Stokes equations and to appreciate (to whatever extent possible) the difficulty of these open problems. If time permits, we will also discuss the existence of solutions to Euler's equation.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: single-variable calculus

Polygons, friezes, and snakes — oh my! (🌶, Kayla, TWΘFS)
What do polygons, friezes and snakes all have in common? It turns out that they can all be related using algebraic combinatorics! We will begin the class by exploring the connection between triangulations of polygons and finite arrays of positive integers called frieze patterns. Over the course of the week, we will ramp up this construction to involve not just arrays of positive integers, but eventually, rational functions. When we get into these weeds, we will see snakes! The types of snakes we will see are called snake graphs—a tool for understanding which rational functions can appear in our frieze patterns and are constructed from our triangulations of polygons. Come see how all these cute objects are all connected! Do NOT beware of the snakes, they are friendly and don't bite!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Problem solving: triangle geometry (🌶🌶🌶, *Zach Abel*, TWΘFS)
Come explore the rich, diverse, and endlessly surprising world of triangle geometry! Triangles have loads of named "center" points, and we'll venture well beyond the classical centroid and orthocenter into some lesser-known yet unreasonably beautiful ones. Why has the symmedian point been called "one of the crown jewels of modern geometry"? Why is the existence of Feuerbach's point even reasonable (I'm still not convinced. . .), and how might we approach its construction synthetically (i.e., without inversion)? What are the (literally!) more than $10,000$ triangle centers listed in the Encyclopedia of Triangle Centers, and how can this encyclopedia be interpreted?

Homework: Required

Class format: This class is largely problem based: there will be some lecturing, but much of the time you will present your solutions to the previous day's olympiad-style homework problems.

Prerequisites: Some familiarity with synthetic geometry (similar triangles, cyclic quadrilaterals, etc.).

11:10 Classes

Elliptic curves (🌶🌶🌶, *Ruthi Hortsch*, TWΘFS)
Let n be a positive integer. We call n a congruent number provided that there is a right triangle that has all rational sides and area n. Can we find an easy way of determining whether n is a congruent number? Tunnell's theorem gives us a way, but its methods of proof use techniques more advanced and modern than the statement of the question itself: the theory of elliptic curves and modular forms, which are fundamental objects of study in advanced number theory. It is also dependent on a weak form of an unsolved conjecture called the Birch and Swinnerton-Dyer Conjecture. (A proof—or counterexample—of this conjecture is worth a million dollars!)

In this class we will: discuss congruent numbers and Tunnell's theorem, define elliptic curves and their basic properties, explain why answering questions about elliptic curves will tell us about congruent numbers, and get some idea about how this leads us to Tunnell's result.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic modular arithmetic. Group theory is not required but is useful context.

Gödel's incompleteness theorems (🌶🌶🌶, Steve, TWΘFS)
Consider the following very silly argument:

"Let f_i be the i^{th} function $\mathbb{N} \to \mathbb{N}$ which we can prove is well-defined. There are obviously infinitely many such functions. Now consider the function

$$g : \mathbb{N} \to \mathbb{N} : x \mapsto f_x(x) + 1.$$

Obviously we can prove that this function g is well-defined. But g can't be any of the functions that we can prove are well-defined. So, [explosion]."

You may be tempted to reject this immediately as "too informal," or "hiding some subtleties," or "really frikkin' silly." However, with work it can in fact be made perfectly precise! When we do so what we get is a proof that no "reasonable" system of mathematics is able to resolve every question. This is (a version of) **Gödel's first incompleteness theorem**, and in this class we'll explore it, its strengthenings, and its corollaries (note the word "first" there...). This will involve looking under the hood of mathematics, so to speak, and getting really precise about what exactly a definition/theorem/proof in mathematics is; think about the difference between an algorithm and a computer program in some specific language. It will also involve breaking reality a wee bit, and learning how to write sentences that refer to themselves.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Required for: Consistency of arithmetic (W3)

Introduction to ring theory (🌶🌶, Kevin, TWΘFS)
Welcome to the world of rings! First discovered by Dedekind, Hilbert, and Noether in the late 19^{th} century, rings are algebraic structures that capture the abstract ideas of addition and multiplication. It's no exaggeration to say that ring theory touches all of mathematics. The number theorists are still here, but nowadays we have lots of geometers, topologists, and combinatorialists. Even analysts from faraway lands will make the occasional trip.

In this week-long tour, we'll get acquainted with their basic axioms and properties and see some of the sights (the integers, fields, polynomial rings, some weirder stuff near the end). A major theme of the class will be ideals in rings. As one of the ring theorists' key innovations, ideals vastly generalize the notion of divisibility and provide new perspectives on modular arithmetic and prime factorization.

Homework: Recommended

Class format: Interactive lecture

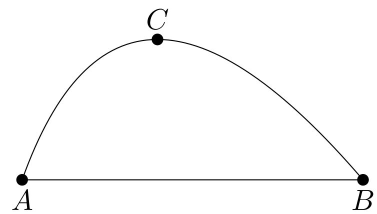
Prerequisites: Some familiarity with basic number theory (e.g. modular arithmetic) would be helpful but is not required. Group theory would also be helpful but is definitely not required.

Required for: How to count rings (W3); Finite fields (W4)

Parabolic curves (🌶🌶, Misha, TWΘFS)
First of all, this class has nothing to do with Ruthi's class later in the week. Elliptic curves are algebraic curves with hugely important applications in number theory and cryptography. Parabolic curves are a silly joke—my class is just about parabolas.

You may be familiar with parabolas as the graphs of functions like $y = x^2$. In this class, we'll look at parabolas the way mathematicians thought about them before coordinate geometry was invented.

On the first day of class, we'll just take a tour of some properties of parabolas. The second day of class is a theorem of Archimedes: the "quadrature of the parabola". We will find out how to compute the area of a shape bounded between a line segment and a parabola:



Homework: Optional

Class format: Interactive lecture with lots of diagrams.

Prerequisites: Some Euclidean geometry, but nothing fancy: if you're comfortable with parallel lines, congruent and similar triangles, and the Pythagorean theorem, you should be fine.

The Wythoff array (🌶️🌶️, Della, TWΘFS)
Here's a simple game! There's a single Chess queen on a large board. Two players take turns moving it any distance down, left, or diagonally down and left. Whoever moves the queen to the bottom left square wins. What's the optimal strategy?

You've probably written numbers in base ten before, and you might have even used base two. But you can also write numbers in base Fibonacci, where digits represent Fibonacci numbers! Under what conditions do numbers have unique representations of this form? What operation does adding a 0 to the end correspond to?

Draw a ray from the origin with slope x, and write down the sequence of horizontal and vertical grid lines it passes. What can we say about this sequence? How is it related to the sequence you'd get for other numbers, such as $x+1$?

Imagine I beat a drum every x seconds (x isn't an integer). You watch a timer, and record which 1-second intervals have drum beats in them. How does the sequence of numbers you get depend on x? What's interesting about the intervals which *don't* have drum beats in them?

In this class, I'll answer all of these questions! Then we'll see how all of these problems are intrinsically related, and use their relationships to construct a beautiful grid of numbers called the Wythoff array, and learn about some of the magical properties it has: for instance, every positive integer appears exactly once, and each row satisfies the Fibonacci recurrence.

The first two or three days will cover several disjoint topics, but by the end of the class you'll see how it all fits together.

Homework: Recommended

Class format: Lecture

Prerequisites: None.

Wedderburn's Theorem (🌶️🌶️🌶️, Mark, TWΘFS)
Have you seen the quaternions? They form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$, with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1,\ \ i\,j = k,\ \ j\,i = -k,\ \ j\,k = i,\ \ k\,j = -i,\ \ k\,i = j,\ \ i\,k = -j.$$

Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Homework: None

Class format: Interactive lecture

Prerequisites: Some group theory; the idea of vector spaces and bases from linear algebra; familiarity with complex roots of unity would be helpful.

When will this end??? (🌶🌶, Arya, TWΘFS)
You can run from your fate, but can you really escape it? On the Earth, sadly not. On the real line, sure - just run off to infinity on either side! How many ways can you escape fate on an infinite plane? In a group? On an arbitrary surface?

In this class, we shall study ends of spaces, which are in some sense, different ways to go off to infinity. A lot of information can be obtained about a space or a group given how its ends behave. At some point, I shall be drawing the Loch Ness monster, and possibly the Eiffel Tower. What's not to love? :)

Homework: Recommended

Class format: Lectures.

Prerequisites: You should know what a group is, and possibly what a graph is. If you don't, chat with me before class!

1:10 Classes

First, choose randomly (🌶🌶🌶, Travis, TWΘFS)
How do you prove that something exists without knowing what it is? In math, there are many ways to do this, and the probabilistic method is, at least in combinatorics, one of the most popular. Counterintuitively, even though it doesn't provide any specific example, it often can still be used to generate examples in practice fairly quickly.

In this class, we'll use probability to prove things about combinatorics and discrete geometry. If you've never seen it before, it's very surprising that probability is an effective tool in combinatorics; and if you have seen it before, even many times, it's surprising how effective it is: The probabilistic method can often turn otherwise very difficult problems into child's play.

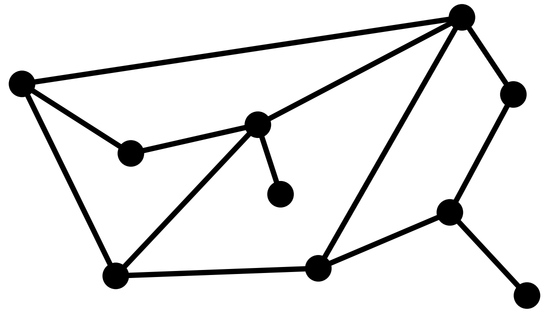
In this class, we'll start by seeing how simple methods in probability can already prove important (and sometimes still cutting-edge) results in combinatorics. Then we'll build up more advanced and refined techniques throughout the week to prove even more. Applications will be far and wide, from graphs, set families, and permutations; to number theory; to discrete geometry. The probabilistic method touches on most areas of combinatorics, and the goal of this class is to give rapid tour of these connections and applications.

Homework: Optional

Class format: Interactive lecture

Prerequisites: familiarity with basic probability is helpful

Introduction to graph theory (🌶🌶, Tim!, TWΘFS)
A graph consists of some *vertices*, which we will draw as dots on our paper, and some *edges*, which we will draw as segments connecting pairs of vertices, like this:



Because the definition of a graph is so lightweight, graphs can represent many different things. For example, each vertex could represent a person at camp, and we could draw an edge between two people if they are close contacts for the purposes of contact tracing. Or, each vertex could represent a path on a manifold, with two vertices being joined by an edge if the corresponding paths don't intersect. Therefore, many problems (both in the real world and in other areas of math) can be transformed into questions about graphs! If you can prove things about graphs, you can solve problems in many different areas!

With the understanding that graphs have lots of important applications, we can feel even more comfortable studying graphs for their own sake. Again because graphs have such a simple definition, there are many different lines of investigation we can take, and we will look at different topic each day. We'll spend one day thinking about *planarity*. One question we will ask: Draw three dots on a piece of paper, representing an electric utility, a water utility, and a gas utility. Draw three more dots, representing three houses. Can you draw a path connecting each utility to each house, without any of those paths crossing? This is a famous puzzle; it's even been printed on merchandise! Here it is on a coffee mug:

From `https://mathsgear.co.uk/`

The more general question is: Which graphs can we draw in the plane without edges crossing? Is there a nice way to describe which ones can be and which ones can't be?

Topics that may appear on other days of class include: trees, connectivity, matchings, and graph colorings.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None

Required for: Graph colorings (W3); Matroids and the chromatic polynomial (W4)

Packing permutation patterns (🌶🌶🌶, Misha, TWΘ[FS])
Prepare by picking a permutation π and a pattern P. Probabilistically pick $|P|$ pieces of π: perhaps putting them together produces P? Let $\rho_P(\pi)$ be the probability of producing P.

To pack P in π, puff up this probability, making P as plentiful as possible. We will ponder the packing problem for $P = 132$ (and plenty of its pals) using a progression of powerful problem-solving procedures.

Homework: Recommended

Class format: Fill-in-the-blank lecture: to help you along, I'll provide lecture notes at the beginning of class, but some details will be left out for you to supply as we go. Homework is sparse, but I highly recommend taking the time to make sure you are happy with all the examples if you want to get the most out of this class.

Prerequisites: Nothing we'll do will require more background than knowing what $n!$ and $\binom{n}{k}$ count. Ask me after the end of the week if you're curious about the heavy-prerequisite version of this topic (which can involve graph theory, linear and polynomial algebra, logic, and semidefinite programming).

Polytopes (Week 1 of 2) (🌶🌶🌶, Susan, [TWΘFS])
A polygon is a two-dimensional shape bounded by line segments. A polyhedron is a three-dimensional shape bounded by polyhedra. A polytope is a d-dimensional shape, bounded by $(d-1)$-dimensional polytopes. In this class we'll explore these high-dimensional objects.

Our first task is to define them—what is a polytope? One possible definition is that a polytope is the convex hull of its vertices. Another is that a polytope is the bounded intersection of a collection of half-spaces. These two definitions seem to produce the same collection of objects, but it's not at all obvious why this should be the case!

This week we'll build a small library of example polytopes, see what types of theorems can be proved with each of our two definitions, and show that the definitions are equivalent.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Randomized vs deterministic computation (🌶🌶🌶, Tanya, [TWΘFS])
If you're using a computational method to test whether your submersible vehicle can withstand high pressure it would experience several miles underwater, you would like to use simulation techniques as precise and detailed as possible to avoid a potential catastrophe. However, if you are doing so on your personal laptop which doesn't have very much processing power, you may be inclined to take some shortcuts and perhaps run a simpler, less computationally expensive, but potentially error prone algorithm. Ideally, we would like to have algorithms that are efficient in runtime and memory usage, solve our desired question exactly and are easy to analyze theoretically so the risks posed by the aforementioned scenario never have to be taken. However, in practice, there are often trade-offs between all of these desired qualities. In this course we will explore what happens if we introduce some small error to our solution by allowing random operations in the example of the classical problem of finding a minimum cut of a graph. We will start with a deterministic approach which yields an exact solution after some non-trivial analysis and compare it to an almost painfully simple to state and analyze, however error-prone, randomized solution.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

The transcendence of many numbers (including π and e) (Week 2 of 2) (🌶🌶🌶🌶, *Dave Savitt*, TWΘFS)
This is a continuation of my week 1 class. Linear algebra is an additional prerequisite for the second week.

Homework: None

Class format: Lecture

Prerequisites: linear algebra, multivariable calculus

Colloquia

One-half factorial from scratch (*Zach Abel*, Tuesday)
Even though factorials are all about counting orderings of people, and the idea of permuting just half of a person makes you queasy, you may nevertheless have some vague suspicion that the factorial of 1/2 can somehow be made sense of. You just can't shake this hazy impression that if you squint real hard and pray to Gauss to please please please let ϵ be greater than zero, somehow a value for (1/2)! pops out in some reasonable manner. You may even hear voices on the wind that this value is somehow connected to the square root of π, which is kinda hard to believe, tbh.

Armed only with this hunch, and without looking anything up, and without calculus in any form, how might you rediscover what value should be assigned to (1/2)! on your own, and convince yourself that it's the right one?

Some stories about squares (mod p) (*Dave Savitt*, Wednesday)
Which tend to be larger, on average: the squares mod p, or the non-squares mod p? And by how much? This simple question turns out to have surprising depth.

Computer-aided mathematics and satisfiability (*John Mackey*, Thursday)
During this talk we'll review some of the successes over the last 40+ years in computer-aided mathematics, and then focus on recent applications of SAT solvers to problems in discrete math.

A SAT solver considers a logical formula in conjunctive normal form, for example (A or -B) and (B or C) and (-A or -B or C) and (-C), and attempts to determine whether it is possible to assign truth values to the variables such that the formula is true.

We'll show how to encode some discrete math problems in conjunctive normal form and see how modern SAT solvers process them.

Project selection fair (Staff, Friday)
A special event scheduled instead of a Friday colloquium: at the Project Selection Fair, you'll get a chance to ask staff about the projects they've proposed (project blurbs will be available on Thursday at sign-in). If you're interested in participating in a staff-supervised project at camp, whether it's already been proposed, or you have an idea of your own that you're excited about, make sure to come to the project selection fair and fill out a preference form.