CLASS DESCRIPTIONS—WEEK 1, MATHCAMP 2023

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9:10 Classes

Cubic curves (, Mark, $\boxed{\text{TW}\Theta\text{FS}}$)
A curve in the x, y-plane is called a cubic curve if it is given by a polynomial equation $f(x, y) = 0$ of degree 3. Compared to conic sections (which have degree 2), at first sight cubic curves are unpleasantly diverse and complicated; Newton distinguished more than 70 different types of them, and later Plücker made a more refined classification into over 200 types. However, as we'll see, by using complex numbers and points at infinity we can bring a fair amount of order into the chaos, and cubic curves have many elegant and excellent properties. One of those properties in particular, which is about intersections, will allow us to prove a beautiful theorem of Pascal about hexagons and conic sections, and it will also let us define a group structure on any cubic curve—well, almost. We may have to leave out a singular ("bad") point first, but a cubic curve has at most one such point (which may be well hidden; for example, $y = x^3$ has one!), and most of them don't have any. Cubic curves without singular points are known as *elliptic* curves, and they are important in number theory, for example in the proof of the Fermat–Wiles–Taylor theorem (a.k.a. "Fermat's Last Theorem"). However, in this week's class we probably won't look at that aspect at all, and no knowledge of number theory (or even groups) is required. With any luck, along the way you'll pick up some ideas that extend beyond cubic curves, such as how to deal with points at infinity (using "homogeneous coordinates"), what to expect from intersections, and where to look for singular points and for inflection points.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: A bit of differential calculus, probably including partial derivatives; complex numbers; a bit of experience with determinants.

Fourier series (, *Jonathan Tannenhauser*, $\boxed{\text{TW}\Theta\text{F}}$S)
A Fourier series is an expansion of a periodic function as a sum of sines and cosines. After studying how and why Fourier series work and seeing some basic examples, we'll look at two applications.

(1) The heat equation. Joseph Fourier introduced Fourier series in 1807 to solve the differential equation that models how heat diffuses through a given region.
(2) The isoperimetric problem. Among closed plane curves of fixed perimeter, which curve maximizes the enclosed area? Adolf Hurwitz in 1901 used Fourier series to show that the answer is a circle.

Homework: Recommended

Class format: Interactive lecture
Prerequisites: None.

Hlod onto yoru ahts![1] (🌶🌶, Tim!, TWΘF$\boxed{\text{S}}$)
Here's a game for fifteen campers: Every camper will have a random color of hat placed on their head. Each hat is either black or white with 50% probability, independently chosen. A camper can't see their own hat, but they can see the other fourteen hats. Simultaneously, all fifteen campers make a choice: they can either guess the color of their own hat, or remain silent. They must all make this decision simultaneously, they can't use anyone else's guess (or lack thereof) as part of their decision. The game is cooperative; they all win the game if at least one camper guesses their hat color correctly and nobody guesses incorrectly. They all collectively lose the game if *anybody* guesses wrong, or if everybody remains silent.

The campers are allowed to strategize before they receive their hats. What strategy should the campers use to maximize their probability of winning? What is that probability?

Certainly, they can win 50% of the time: Before the game, they agree on one camper who will make an arbitrary guess, while the other fourteen campers remain silent. That camper has no information whatsoever about their own hat—they know the other campers' hat colors, but those were chosen independently so are of no help, and they can't communicate with the other campers in any way. So their guess is just a shot in the dark; they have a 50% of being right by pure luck.

Perhaps unsurprisingly, they can't win 100% of the time. Again, no camper has any information about their own hat color, and can't communicate with anyone before they guess. So if anyone guesses at all, there is some chance they will be wrong. And if nobody guesses, they just lose.

But perhaps very surprisingly, they *can* win more than 90% of the time. When I first heard about this, it sounded impossible: Every camper has a 50% chance of being wrong if they guess, and if multiple campers guess, the situation can only get worse, right?

Homework: None
Class format: Interactive lecture
Prerequisites: None

[1]P.S. I want to apologize about the title of this class; it was supposed to be "Hold onto your hats!" But I guess you figured that out. I only slightly misspelled my class title; it was *approximately* correct (at least it had all the right letters). So you were able to figure out *approximately* what the title was supposed to be. Wait, actually, you were probably able to figure out *exactly* what the title was supposed to be. You're able to understand exactly what my typos were and how to fix them. That's sort of wild and exciting, isn't it? That I give you something that's only *approximately* right, but you can figure out *exactly* what I mean? What a useful superpower. Think about this in other contexts. Let's say you are sending photos back to earth from a Mars rover, or just sending a text message to a friend. The physical world is an imperfect place, so some of those electronic signals will inevitably get a bit distorted. But even if the *signal* is a little distorted, you don't want the *message* (the photo or the text message) to be corrupted even a little bit. You want to encode the message in such a way that even the signal is distorted, you can (or really, your computer/phone can) correct the error and figure out what the original message was. To allow the recipient to correct errors in your message, you want to encode your message using an **error-correcting code**. That basically means you want to add a little redundancy to the message you send. English is inherently redundant; that's why when I type "yoru" you can guess that I meant "your". English is kind of inconsistently redundant though; if I tell you I moved one letter of a word and got "zeabr" you can guess that the word was "zebra", but if instead I give you "daer", you don't know if the word was "dear" or "dare". So English is slightly inefficient as an error-correcting code in that sense. And you would have a harder time if I were allowed to move multiple letters. Different error correcting codes are able to handle different amounts of data correction, and different error correcting codes are more consistent/efficient than others in terms of how much error correcting they can do in different situations. For example, **Hamming codes** are codes that can only do a bit of error-correction, but they are very consistent about how much error correction they can do; in that sense they make very efficient use of their signal. Sorry for rambling; I could go on about this for a while. Actually, I think I'll stop rambling, and instead of that silly hat game, I'll teach a class about error-correcting codes and Hamming codes in particular.

Inspecting gadgets (, Della, $\boxed{\text{TW}\Theta\text{FS}}$)
Imagine you're designing a level in a video game like Mario, and want to construct something the player can only use in certain ways: maybe a hallway that they can go through in only one direction, or a hallway that they can't go through until they visit some other location. How would you do this? If you build one of them, could you use it to build the other?

A good way to model this kind of problem is with 'gadgets', which are an abstraction for self-contained components which lets us stop thinking about the details of the game the gadgets were built in. Once you have such gadgets, you can sometimes use them to build other, cooler gadgets—but not always. How can you know when this sort of 'simulation' is and isn't possible?

This class will be about a particular genre of gadgets with a lot of fun simulations. You'll spend the first half working together to tryng to find as many simulations as possible, which feels like solving puzzles or designing video game levels. Towards the end, I'll show you general results about when simulations are or aren't possible: we'll see a gadget that can simulate *every* possible gadget!

Gadgets originally come from hardness proofs in computational complexity (e.g. NP-hardness), but this class will study gadgets in their own right without that context. If you've seen and enjoyed NP-hardness proofs in the past, you'll probably have fun simulating gadgets.

I intend to offer a project on gadgets—there's a lot more to learn that what I can cover in a class, including a lot of approachable open problems.

Homework: Recommended

Class format: Days 1-3 will be mostly IBL; days 4-5 will be lectures.

Prerequisites: None.

Introduction to linear algebra (, Narmada, $\boxed{\text{TW}\Theta\text{FS}}$)
Linear algebra is ostensibly the study of linear equations, but that's like calling writing the study of the alphabet. The real beauty of the subject lies in the techniques we'll learn.

This course will take a non-traditional approach by focusing on matrices: the star players in finite dimensions. By the end of this course, you'll understand what it means to say a matrix behaves like a real number, and what that tells us about the geometry of high-dimensional spaces.

Homework: Recommended

Class format: Lecture with some group work

Prerequisites: None.

Required for: Representation theory of the symmetric groups (W2); The transcendence of many numbers (including π and e) (Week 2 of 2) (W2); Polytopes (Week 1 of 2) (W2); Solving equations with origami (W3); How to count rings (W3); Polytopes (Week 2 of 2) (W3); Linear algebra through knots (W3); Finite fields (W4); Vhat are your vectors vorth? or, part of the part of combinatorics and discrete geometry that ve can do easily vith linear algebra (W4); Gaussian magic (W4); Quantum computing (W4)

Khinchin's constant and the ergodic theorem (, Ben, $\boxed{\text{TW}\Theta\text{FS}}$)
Every irrational number can be written as a "continued fraction," that is, as an expression like

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $a_1, a_2, \ldots$ are positive natural numbers and a_0 is an integer. Here's a fun fact: if you take the geometric mean of $a_1, a_2, \ldots, a_n$, and let n go to infinity, this will almost always converge to a specific number, which is about 2.685. This is Khinchin's constant.

In this class, we'll investigate how to prove this kind of result. To do so, we'll talk a bit about integration, a bit about limits, and just a little bit about bakers. Why bakers? Why, because one of the things we'll be talking about is this mysterious thing called "ergodicity," and one of the classic examples of an "ergodic map" is called the baker's transformation!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Familiarity with integration and limits—e.g. have taken some calculus class covering both of these topics. No knowledge about continued fractions is needed!

10:10 Classes

Discreet calculus (shh!) (, Travis, $\boxed{\text{TW}\Theta\text{FS}}$)
There's a parallel world of calculus out there—the one Newton and Leibniz don't want you to know about! Why bother with limits when you can just plug in $\epsilon = 1$?

This class is about a discrete version of calculus: we'll see versions of the derivative, integral, Fundamental Theorem of Calculus, Taylor series, and differential equations. This is a really nice topic, because on the one hand, it's more concrete than calculus (no limits), but it still maintains much the same story. And it has its own rewards, as we'll find out when we evaluate tricky sums simply using "summation by parts" or solve recursions lickety-split.

A note about the chilis: This class is 1 chili because it'll be fairly relaxed. We won't be speeding through a lot of math, but we'll see some interesting things and have a good time along the way.

Homework: Optional

Class format: Interactive lecture

Prerequisites: You don't need to know calculus to take this class. That said, if you have seen the definition of a derivative or the Fundamental Theorem of Calculus, you'll be able to appreciate how discrete calculus compares to "traditional" calculus, and some of the things we do in class might seem more natural.

Homotopy groups of spheres (, Kevin, TWΘ$\boxed{\text{FS}}$)
The spheres are some of the simplest spaces to define, but the mysteries of their topology have stumped mathematicians for nearly a century. In this class, we'll take a look at the open problem of computing the homotopy groups of spheres, which describe the various ways spheres of different dimensions can wrap around one another. We'll talk about how the problem is set up, why it's so difficult, and what we currently know.

I'll introduce all the key objects along the way, so this class will be a short preview of algebraic topology for those of you who think homotopies sound cool. This class will feature a ton of pictures: pictures of spheres, maps, and homotopies on Day 1 and pictures/demos of spectral sequences on Day 2.

Homework: Optional

Class format: Interactive lecture

Prerequisites: Some familiarity with the notion of continuity would be helpful. Group theory could also be helpful but is definitely not required. Don't worry if you haven't seen any topology before!

Introduction to number theory (, Mia, $\boxed{\text{TW}\Theta\text{FS}}$)
At some point in your life, you might have wondered about the following *very common* questions:

(1) Given that yaks cost \$23 and xylophones cost \$5, is it possible to spend exactly \$303 on yaks and xylophones?
(2) If Pete the Pirate claims that he split a giant square chocolate bar among 7883 of his minions and had 13 squares left-over, should you believe him? Or is he lying?
(3) If an evil rhinovirus (named Vincent) is planning to take over the world and the only way to stop him is to unlock the VirusVanquisher 2001 which asks you to compute the units digit of 7^{1001} mod 12 with only a pencil and paper, is all lost or is there a way??

Yet underpinning these goofy questions is the elegant structure of the integers. And the goal of this class is to better understand that structure. We'll start with the study of linear Diophantine equations, then transition to the world of modular arithmetic where we'll build from the study of order modulo n to primitive roots to quadratic reciprocity.

Although this class will follow an interactive lecture format, it will be inquiry-focused. We'll start each day with a simple-to-state (yet not necessary easy!) question, build the theory to answer it, and then use that as a springboard to generate our next line of inquiry.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Metric spaces (🌶🌶, Krishan, TWΘFS)
Choose any number between 0 and 1 (non-inclusive) and use a calculator to find its cosine. Then take that number and find its cosine. Repeat this process again and again. Eventually you'll find that the results are approaching a number starting with 0.739085. . . if they aren't you haven't done it enough times. This number is an irrational number known as the Dottie Number, which is the unique fixed point of the cosine function. Surprisingly, the reason this process works is that —cos(x) - cos(y)— is less than —x-y—, due to a result known as the Banach Fixed Point Theorem. In this class we will work toward proving this theorem in a general setting. We'll discuss metric spaces, which are sets together with a way to measure the distance between points. We'll see how to tell when a metric space has "holes", and we'll see how to generalize the extreme value theorem from calculus. The class will borrow some ideas from calculus, but don't worry if you haven't taken it yet. We will approach these ideas from a different (ε-free) direction, so no knowledge of calculus is required.

Homework: Optional

Class format: Interactive lecture

Prerequisites: None.

Multivariable calculus crash course (🌶🌶🌶, Mark, TWΘFS)
In real life, interesting quantities usually depend on several variables (such as the coordinates of a point, the time, the temperature, the number of campers in the room, the real and imaginary parts of a complex number, ...). Because of this, "ordinary" (single-variable) calculus often isn't enough to solve practical problems. In this class, we'll quickly go through the basics of calculus for functions of several variables. As time permits, we'll look at some nice applications, such as: If you're in the desert and you want to cool off as quickly as possible, how do you decide what direction to go in? What is the total area under a bell curve? What force fields are consistent with conservation of energy? One reason, and maybe the best reason, to take this crash course right now rather than waiting until you encounter the material naturally after BC calculus and/or in college, is to be able to take the course on functions of a complex variable (which have many amazing features) that starts in week 3.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic knowledge of single-variable calculus (both differentiation and integration)

Required for: Functions of a complex variable (Week 1 of 2) (W3); Calculus of variations (W3); Functions of a complex variable (Week 2 of 2) (W4); Gaussian magic (W4)

Reverse mathematics (, Steve, $\boxed{\text{TW}\Theta\text{FS}}$)
Usually in mathematics we proceed from *axioms* to *theorems*. But axioms are really heavy and bulky, and RyanAir doesn't allow carry-on bags. In order to figure out what axioms we really need, we're going to have to go backwards: start with the theorems we care about and figure out what axioms are needed to prove them.

This general line of thought has a long history; it relates, for example, to early investigations in the foundations of geometry (what can you prove without the parallel postulate?), abstract algebra (what algebraic properties of the integers are behind unique factorization?), and set theory (where does the axiom of choice show up?). **Reverse mathematics** is a more recent aspect of this. In reverse mathematics, we focus on theorems which are about natural numbers or sets of natural numbers. Some examples of such statements include Ramsey's theorem, weak Konig's lemma ("every infinite binary tree has an infinite path"), and determinacy principles (such as "Every finite-length two-player perfect-information game is determined"). We measure "axiom strength" along two axes:

- How much **induction** do we need?
- What sort of **set formation** (= **comprehension**) do we need?

In the development of this subject, connections with computability theory and (very) theoretical computer science emerge. Reverse mathematics has recently been described as "the playground of logic"—let's have fun!

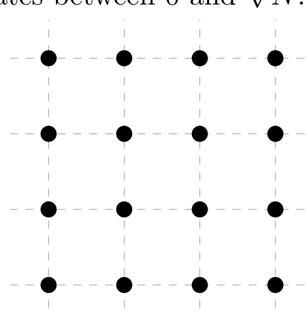
Homework: Recommended

Class format: Lecture

Prerequisites: None

11:10 Classes

Erdős's distinct distance problem (, Neeraja, $\boxed{\text{TW}\Theta\text{FS}}$)
If P is a set of N distinct points in the plane, the set of distances between points in P is called the distance set $\Delta(P)$. The size of the distance set is at most $\binom{N}{2}$ and we can find examples where this upper bound is realized (for instance, choose the endpoints of a scalene triangle). A much more difficult question is to ask how small the distance set can be. For example, $P = \{(1,0),(2,0),\ldots,(N,0)\}$ gives $|\Delta(P)| = N$. Paul Erdős discovered a better example by taking his points in a square lattice, that is, taking all points with integer coordinates between 0 and $\sqrt{N}$:



For this set, $|\Delta(P)|$ works out to be about $N/\sqrt{\log(N)}$. Based on the lattice example, Paul Erdős conjectured in 1946 that $|\Delta(P)| \geq N/\sqrt{\log N}$. This conjecture was proved by Guth and Katz in 2015 (or rather almost proved, as they showed a lower bound of $N/\log N$). In this course we will look at their proof, which uses topological tools such as the polynomial ham sandwich theorem, algebraic geometry tools and incidence geometry arguments.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: vector geometry (vectors in $\mathbb{R}^3$, rotations in $\mathbb{R}^3$, what it means for vectors to be orthogonal).

Geometry, under construction (🌶, Arya, $\boxed{\text{TW}\Theta\text{FS}}$)
From the age of dinosaurs to the age of Gauss, straight-edge and compass constructions have been peak enjoyment activities. But like, what stuff can you really construct? How much information do you really need to construct, say, a triangle? Other polygons? How much algebra can you build geometrically? Get your hard hats and your thinking caps, as we don appropriate headgear and explore the wonders of constructive geometry!

Homework: Recommended

Class format: Interactive lectures and group work.

Prerequisites: None.

Information theory and the redundancy of English (🌶🌶, *Mira Bernstein*, $\boxed{\text{TW}\Theta\text{FS}}$)
NWSFLSH: NGLSH S RDNDNT!! (BT DN'T TLL YR NGLSH TCHR SD THT)

The redundancy of English (or any other language) is what allows you to decipher the above sentence. It's also what allows you to decipher bad handwriting or to have a conversation in a crowded room. The redundancy works as a kind of error-correcting code: even if you miss part of what was said, you can recover the rest.

How redundant is English? There are two ways to interpret this question:

- How much information is conveyed by a single letter of English text, relative to how much could theoretically be conveyed? (But what is information? How do you measure it?)
- How much can we compress English text? If we encode it using a really clever scheme, can we reduce the length of the message by a factor of 2? 10? 100? (But how will we ever know if our encoding is the cleverest possible one?)

Fortunately, the two interpretations are related. In this class, we will first derive a mathematical definition of information, based on our intuitive notions of what this word "should" mean. Then we'll prove the Noiseless Coding Theorem: the degree to which a piece of text (or any other data stream) can be compressed is governed by the actual amount of information that it contains. We'll also talk about Huffman codes: the optimal way of compressing data if you know enough about its source. Finally, we'll answer our original question—how redundant is English?—in the way that Claude Shannon, the father of information theory, originally answered it: by playing a game I call *Shannon's Hangman* and using it as a way of communicating with our imaginary identical clones.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Introduction to group theory (🌶🌶🌶, Eric, $\boxed{\text{TW}\Theta\text{FS}}$)
Groups are a super common type of mathematical object. Whenever you've got a way of sticking two mathematical things together to make a third in a reasonable way there's very likely a group in the picture! Familiar processes like addition and multiplication will live in groups, as well as many more wild and strange ones (shuffling cards! performing music on bells! twisting strands of yarn!).

This particular course on group theory will aim to emphasize two things:

- group theory itself is an introduction to the subject of *abstract algebra* so we'll devote time to learning the style of axiomatic arguments used in this subject,

- one of the things that makes group theory interesting is the absolutely insane variety of groups in existence, and so we'll get hands on experience with both some traditional and less traditional examples, and prove or state some results which hint towards the plethora of groups you'll encounter after this class.

Homework: Recommended

Class format: Interactive lectures! We'll move at a fairly brisk pace as there's a lot of exciting material to cover, but there will be activities interspersed throughout helping us drill down into some core concepts.

Prerequisites: None!

Required for: Representation theory of the symmetric groups (W2); Introduction to model theory (W2); Take it to the limit (W2); Coxeter groups (W3); The outer life of inner automorphisms (W4); Braid groups (W4)

Knot invariants (, Raj, TWΘFS)
In this course, we will learn all about how to distinguish knots. Knot invariants are tools we use to distinguish knots, and some major examples include tricolorability and the Jones polynomial. Hopefully by the end of the course, you will be able to tell your friends why the Kinoshita- Teresaka mutant knots are different!

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

1:10 Classes

Bhargava's cube (, Kevin, TWΘFS)
This class is about a mathematical story more than a thousand years in the making. In the 7th century, Brahmagupta discovered his identity

$$(x^2 - Ny^2)(z^2 - Nw^2) = (xz + Nyw)^2 - N(xw + yz)^2.$$

In 1801, Gauss generalized Brahmagupta's identity to polynomials of the form $ax^2 + bxy + cy^2$ (a.k.a. binary quadratic forms) by figuring out how to compose different binary quadratic forms. In 2014, Manjul Bhargava won the Fields Medal, partly for figuring out how to generalize Gauss composition to higher dimensional spaces like $2 \times 2 \times 2$ cubes.

In this two day class, we'll learn about the history and theory of binary quadratic forms and the elegant mathematics that goes into understanding them.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Number theory (e.g. you should be comfortable with modular arithmetic)

Infinite arithmetic (, Susan, TWΘFS)
Infinity is a funny concept. Many infinite sets that look like they ought to be different sizes actually turn out to be the same, like the natural numbers, the integers, and the rational numbers. But there are definitely different sizes of infinity—there are more real numbers than rational numbers, and we can prove that any set is smaller than the set of its subsets.

This class deals with different sizes of infinity and explores how they're organized and what happens when we attempt to add, multiply, and exponentiate these infinite numbers. We'll see that cardinal addition and multiplication are beautifully simple (though for complicated reasons), and that cardinal exponentiation is beastly (but can be tamed given the right assumptions). Our journey will take us through another way of categorizing infinities—the ordinal numbers—and a discussion of the generalized continuum hypothesis.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: None.

Required for: Infinite Ramsey theory (W2)

Is it possible to gamble successfully? (🌶️🌶️, Tanya, TWΘFS)

The subject of this course is incredibly dangerous - it almost got banned from Mathcamp 2023! Here you will learn the mathematics behind why gambling can leave you in financial ruin or the richest person on Earth. Along the way, you will be introduced to the foundational concepts of probability theory, which is an area of study that seeks to discover patterns in the inherently unpredictable behavior of random phenomena. Some of the initial attempts to develop a robust formal theory of probability were made in 17th century France thanks to a gambler who was interested in improving his chances of winning and happened to be friendly with Blaise Pascal. In this course, we take inspiration from probability theory's historical roots and investigate a simple gambling scheme that can be mathematically represented by a random walk. Methods of analysis we will be learning will be applicable to the study of a much broader class of stochastic processes known as Markov Chains (unfortunately beyond the scope of the curriculum, but I hope you will be inspired to learn about these objects in the future). This class is inquiry-based, so you will be discovering all these exciting concepts and techniques on your own, with gentle nudging and occasional hints.

Homework: Recommended

Class format: Interactive lecture

Prerequisites: Basic set theory

Mathcamp crash course (🌶️, Charlotte, TWΘFS)

Math is useless unless it is properly communicated. Most of math communication happens through a toolbox of terminology and proof techniques that provide us with a backbone to understand and talk about mathematics. These proof techniques are often taken for granted in textbooks, math classes (even at Mathcamp!) and lectures. This class is designed to introduce fundamental proof techniques and writing skills in order to make the rest of the wonderful world of mathematics more accessible. This class will cover direct proofs from axioms, proofs using negation, proofs with complicated logical structure, induction proofs, and proofs using cardinality. If you are unfamiliar with these proof techniques, then this class is highly recommended for you. If you have heard of these techniques, but would like to practice using them, this class is also right for you. Here are some problems that can assess your knowledge of proof writing:

- Negate the following sentence without using any negative words ("no", "not", etc.): "If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."
- Given two sets of real numbers A and B, we say that A dominates B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two disjoint, nonempty sets A and B such that A dominates B and B dominates A.
- Prove that there are infinitely many prime numbers.
- Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)
- Define rigorously what it means for a function to be increasing.

- What is wrong with the following argument (aside from the fact that the claim is false)? On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

  Proof. We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k + 1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. □

If you would not be comfortable writing down proofs or presenting your solutions to these problems, then you can probably benefit from this crash course. If you found this list of questions intimidating or didn't know how to begin thinking about some of them, then you should definitely take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Homework: Required

Class format: Mix of lecture and group work.

Prerequisites: None.

Problem solving: geometry galore (🌶🌶🌶🌶, Ian, $\boxed{\text{TW}\Theta\text{FS}}$)

Welcome to the Geometry Galore! Here we befriend circles, lines, and angles to complete several quests. The quests involve empowering points, searching for radical axes, and saving our majesty Ceva, Menelaus, Desargue, and Pascal!...

Since the Galore involves problem solving, it is strongly encouraged that the participants work on problems outside of the 50-minute time limit, and complete prerequisite quests before each class to make the most of the Galore...

Once you are equipped with the enthusiasm to complete all the quests, the Galore is yours - good luck...

==Start Quest==>

Homework: Required

Class format: The first few minutes will have a lecture format, where we go through theorems and solve problems together. Then the rest of the class will be working on problems individually or in groups.

Prerequisites: A good understanding of basic geometry (some experience with Math Olympiad is recommended)

The transcendence of many numbers (including π and e) (Week 1 of 2) (🌶🌶🌶, *Dave Savitt*, TW$\boxed{\Theta\text{FS}}$)

We'll prove a result called the Schneider–Lang theorem, which will allow us to show that a great many numbers are transcendental. For example, we will be able to show that e^α is transcendental whenever α is algebraic. (Exercise: why does this show that π is transcendental?) The overarching idea is that since the functions z and e^z are algebraically independent (there's no polynomial relationship between them), they shouldn't be simultaneously algebraic very often. During the three days of week 1, we'll prove they cannot be simultaneously rational very often, proving in particular that e and π are irrational. Then in week 2 we'll explain how to upgrade this to the statement about transcendence; this part will additionally need some linear algebra as a prerequisite.

Homework: None

Class format: lecture
Prerequisites: multivariable calculus (week 1), linear algebra (week 2)

Colloquia

Voting Theory, Burlington, VT, and the Gibbard–Satterthwaite Theorem (*Mira Bernstein*, Tuesday)
Happy 4th of July! To celebrate the birthday of the world's oldest democracy, let's talk about voting. It's much more complicated than the US Founding Fathers thought: whenever you have more than two candidates, the outcome of an election depends on the voting system you use, and all voting systems are flawed. There's a whole branch of mathematics called voting theory that deals with these issues.

In this colloquium, I'll give an example of the kinds of problems that can arise in practice by discussing a famous example of a mayoral election held in Burlington, VT in 2009. At the time, Burlington was using a voting method called Instant Runoff Voting (IRV). Over the past few decades, many election reform advocates have been lobbying for IRV (which they inaccurately call "RCV") as a solution to all our voting problems. But in this example, we'll see that while IRV is certainly better than plurality (the most common voting system in the US, in which whoever gets the most votes wins), it is hardly a silver bullet. We will then prove the Gibbard–Satterthwaite (GS) theorem, which says that any reasonable voting system is vulnerable to strategic voting. (When people want to argue that all voting systems are flawed, they usually cite the much more famous Arrow's Theorem; but the GS Theorem is just as interesting and has much more real-world relevance.)

Mediants, circles, and Stern–Brocot patterns (*Assaf Bar-Natan*, Wednesday)
An ant is at the corner of a tetrahedron, and starts walking along a face. When she hits an edge, she goes over it, continuing in a "straight" path, and when she hits a vertex, she explodes. Can the ant return to her starting vertex without exploding? Can you explicitly construct a countable collection of circles in the plane whose centers converge to any real number? How would you find the continued fraction expansion of π? All of these questions, and more, can be answered using "kindergarten addition" of fractions: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. In this talk, we'll see how!

Hacking heads off hydras (Susan, Thursday)
A hydra is a mythical being with many heads. Whenever a head gets cut off two grow back in its place! A mathematical hydra is a game that we play on trees. On the n^{th} turn, we cut off a "head," causing n new "heads" to grow back in this place. In this colloquium we'll discuss the rules of the Hydra game, and explore some questions that arise. The rules of the game make it seem hopeless, but on small enough graphs the game is clearly winnable. For which graphs is there a winning strategy? And what do the ordinal numbers have to do with it?

The only formula it can be! (*Noah Snyder*, Friday)
You can often guess the right formula for a mathematical problem, not by deriving the formula directly, but instead by showing it's the only formula that has the right properties. In this talk I'll explain one of the simplest examples of this phenomenon: Heron's formula for the area of a triangle. This formula says that if you have a triangle whose sides have length a, b, and c and if you let $s = (a + b + c)/2$ be the semiperimeter, then the area is $\sqrt{s(s-a)(s-b)(s-c)}$. We will guess (and eventually prove!) Heron's formula by explaining why it's the only formula that can be right. In particular, we will use much less geometry but a bit more algebra than you might expect.