CLASS DESCRIPTIONS—WEEK 5, MATHCAMP 2020

Contents

9:10 Classes

Complex analysis (Alan, Monday–Wednesday)
We'll define what a contour integral in the complex plane is, and prove a nonempty subset of the following fundamental theorems from complex analysis: Cauchy's integral theorem, Cauchy's integral formula, analyticity of holomorphic functions, residue theorem.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Multivariable calculus, specifically contour integrals (a.k.a. line integrals) and Green's theorem. Uniform convergence.

Continued fraction expansions and e (Susan, Monday–Wednesday)
The continued fraction expansion of e is

$$1+\cfrac{1}{0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{\ddots}}}}}}}}}}}}}}$$

Okay, but seriously, though, why?!?! Turns out we can find a simple, beautiful answer if we're willing to do a little integration. Or maybe a bit more than a little? No previous experience with continued fractions necessary. Come ready to get your hands dirty—it's gonna be a good time!

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: None.

Counting, involutions, and a theorem of Fermat (Mark, Friday)
Involutions are mathematical objects, especially functions, that are their own inverses. Involutions show up with some regularity in combinatorial proofs; in this class we'll see how to use counting and an involution, but no "number theory" in the usual sense, to prove a famous theorem of Fermat about primes as sums of squares. (Actually, although Fermat stated the theorem, it's uncertain whether he

had a proof.) If you haven't seen why every prime $p \equiv 1 \pmod 4$ is the sum of two squares, or if you would like to see a relatively recent (Heath-Brown 1984, Zagier 1990), highly non-standard proof of this fact, do come!

Chilis: 🌶🌶

Homework: None.

Prerequisites: None.

Dirac delta function (Alan, Thursday–Friday)
The Dirac delta function, a.k.a. the unit impulse function, is the "function" which satisfies

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

This may seem like nonsense, but this function shows up naturally in many physical problems.

In this class, we'll talk about the theory of distributions (note that "distribution" has many different meanings in mathematics), which will allow us to describe the delta function rigorously and make sense of statements such as $\frac{d^2}{dx^2}|x| = 2\delta(x)$. In fact, we'll learn how to differentiate *any* function. Then we'll see some applications of all this.

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: Single variable calculus, integration by parts.

Exploring the Catalan numbers (Mark, Thursday)
What's the next number in the sequence $1, 2, 5, 14, \ldots$? If this were an "intelligence test" for middle or high schoolers, the answer might be 41; that's the number that continues the pattern in which every number is one less than three times the previous number. If the sequence gives the answer to some combinatorial question, though, the answer is more likely to be 42. We'll look at a few questions that do give rise to this sequence (with 42), and we'll see that the sequence is given by an elegant formula, for which we'll see a lovely combinatorial proof. If time permits, we may also look at an alternate proof using generating functions.

Chilis: 🌶🌶

Homework: None.

Prerequisites: None, but at the very end generating functions and some calculus might make an appearance.

How Riemann *finally* understood the logarithms (Apurva, Thursday–Friday)
Logarithms are hard to define for complex numbers. (If you came to Jon's talk you know this all tpo well.) Euler settled the question by saying that the logarithm is a multi-valued function. But functions aren't allowed to be multi-valued! What's going on?

Riemann realized that the way to fix this is by not thinking about functions but instead studying graphs of functions. This led to the definition of a Riemann surface and resulted in the creation of half a dozen new branches of mathematics.

In this class, we will see how Riemann fixed the multi-valued logarithm problem and prove that an elliptic curve is a torus.

Chilis: 

Homework: Optional.

Prerequisites: You should know the polar decomposition of complex numbers and Euler's identity.

How to glue donuts (Apurva, Tuesday–Wednesday)
Mathematicians routinely encounter higher dimensional geometric objects. But our brains are incapable of imagining anything in higher dimensions. One way we circumvent this obstacle is by breaking up complex higher dimensional objects into simpler lower dimensional ones, like donuts.

In this class, we'll see how a donut can be a powerful tool in the hands of a mathematician and learn how to visualize four dimensional objects.

Chilis: 

Homework: Optional.

Prerequisites: None.

Skolem's paradox (Susan, Thursday–Friday)
Holy Axiomatizations, Batman! A ninja has snuck into the Museum of Real Numbers and stolen all but countably many of them. You, the curator, have a huge exhibition tomorrow. What are you going to do? Why, it's simple! You'll use the Lowenheim–Skolem theorem to build a countable model of set theory, complete with the real numbers. From inside the museum, no one will be able to tell that it's countable. To keep real number ninjas from interfering in your life, come to our class.

Chilis: 

Homework: Recommended.

Prerequisites: None.

The Sylow theorems (Mia, Tuesday–Wednesday)
Suppose I give you a mystery group and all I tell you about it is its order. What can you tell me? A surprising amount, actually! For example, if I tell you that a group has order 77, you can tell me that it has exactly one subgroup of order 11 and exactly one of order 7. In fact, you can even tell me that it is an Abelian group, isomorphic to $\mathbb{Z}_7 \oplus \mathbb{Z}_{11}$. All you need are the Sylow theorems.

In this class, we'll learn not only how to perform this group sleuthing, but also why it works. We'll start by developing two extremely useful tools for partitioning groups, cosets and conjugacy classes, which give deep insights into the structure of a group. Then, we'll move on to prove Lagrange's theorem, which states that the order of a subgroup divides the order of a group, and its partial converse, the Sylow theorems. Throughout the class, we'll look at what fascinating facts we can deduce about our mystery groups.

Chilis: 

Homework: Optional.

Prerequisites: Group theory.

10:10 Classes

Block designs (Emily, Thursday–Friday)
Suppose you are running a scrabble tournament and 13 people show up to play. You wish to structure the tournament so that each game consists of four people, and each pair of people plays against each other in some game exactly once. Is this structure possible? If so, how many games must be played?

Now suppose the year is 1850 and you are Thomas Kirkman. You wish to solve the following problem: "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast."

Both of these problems can be understood using block designs: a set together with a collection of subsets that satisfy some certain conditions. We will explore some properties of block designs and how we can construct them; this will involve some combinatorics and in some cases projective planes.

Chilis: 🌶🌶

Homework: Optional.

Prerequisites: None. Some basic linear algebra may be helpful at a few points, but it is not required.

Cantor's leaky tent (Ben, Thursday–Friday)

One of the notorious counterexamples in point-set topology is called "Cantor's Leaky Tent" or the "Knaster–Kuratowski Fan." This space is connected! But there's one particular point that, when removed, makes the space *totally disconnected.* In this class, we'll go over all of these terms, put up our tent, and prove that it does exactly what it's supposed to.

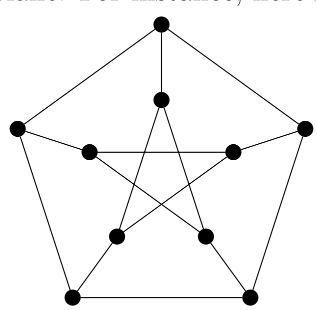
Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Some point-set topology. Knowing, or at least being willing to accept, the Baire Category Theorem.

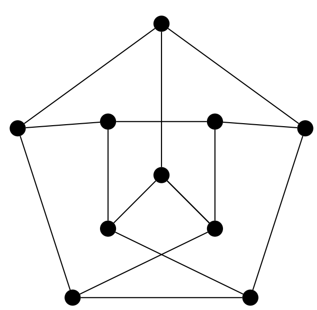
Crossing numbers (Yuval, Thursday–Friday)

We really like drawing graphs in the plane. For instance, here's a drawing of the Petersen graph.



Sometimes, when we draw graphs in the plane, some of the edges cross, which is a real bummer. Even worse, this is often unavoidable—if a graph is non-planar, then we will *always* have a crossing, no matter how we do our drawing.

Nevertheless, we can still try to do better. For instance, here's a different drawing of the Petersen graph:



As you can see, this drawing has only two crossings, which is better than the five crossings we had earlier. As it turns out, two crossings is the best we can do for the Petersen graph: its *crossing number* is 2.

Somewhat surprisingly, studying crossing numbers is an enormously fruitful activity. In this class, we'll prove a fundamental result about crossing numbers, and then use this result to say many interesting things about apparently unrelated areas of math. For instance, we'll use this to attack one of my favorite open problems in all of math: what is the largest number of unit distances that can exist among n points in the plane?

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Basic graph theory: you should know what it means for a graph to be planar. A bit of probability will be helpful but not required.

Homotopy colimits (Dennis, Thursday–Friday)
Let's take the circle S^1, and crush it down to a point. Now let's do it twice. Nothing changed, right? We just get the boring point as a result.

Well, let's look into this further. Crushing the circle down to a point, homotopically it's the same as gluing in a disk, so the circle is now filled in. After all, the disk is trivial homotopically; it's contractible! (This is exactly like with the fundamental group). If we do this process twice, we now have *two* disks, glued to the circle. This gives us an upper hemisphere and a lower hemisphere, glued at the equator, in other words S^2, the sphere! The sphere is definitely *not* a point. What happened?

The first case is a "strict" pushout, while the second is what's called a "homotopy" pushout. The second case is actually much better: for one, it remembers that we tried "crushing the circle to the point" *twice*, while the first one doesn't! Secondly, the second construction is a homotopy invariant! That means if you replace S^1 with a different, but homotopy equivalent, space, and perform the construction again, you'll get something homotopic to S^2 back, while the first one fails.

In this class, we'll go over some important constructions—like gluing, quotienting, taking special subspaces—of topological spaces. We'll also observe how they aren't great! They are almost *never* invariant under homotopy! We'll then go over the general process of correcting this, making everything "homotopical", or squishier, than it was before.

Chilis: 🌶🌶🌶🌶

Homework: Optional.

Prerequisites: Knowledge of topological spaces, homotopy, some gluing of topological spaces.

Infinitesimal calculus (Tim!, Monday–Wednesday)
If you've learned the definition of *continuous function*, you may have learned that a function f is continuous if an infintely small change in x results in an infinitely small change in $f(x)$. This is a pretty good definition: it's short, and you can picture it on a graph, and you can see the connection to more geometric descriptions ("a function is continuous if you can draw its graph without lifting your pen"). It's also how many of the pioneers of calculus thought about the subject.

But if you take a proof-based calculus class, you might see this definition instead: *A function f is continuous at c if for all $\epsilon > 0$, there is a $\delta > 0$ such that for all x with $|x - c| < \delta$, we have that $|f(x) - f(c)| < \epsilon$.* What an ugly definition! To be sure, it's correct, and is often useful, but nevertheless it's clunky and counterintuitive. Why would any class use it instead of the "infinitely small" definition? The problem is that there is *no such thing* as an infinitely small (or *infinitesimal*) real number.

Most proof-based calculus classes usually throw in the towel on infinitesimals at this point and haul out ϵ and δ instead. But not us. We'll just add some infinitesimal numbers to the real numbers to get the *hyperreal numbers*. And we'll get to have nice definitions like the one at the start of this blurb. We'll go through the process of defining the hyperreals. Then, we'll visit some of the the highlights of a calculus class, with proofs that are correct and often much simpler than the standard ones, but which nevertheless are alien and bizarre.

You see, when you start playing with with the fundamental building blocks of reality, things can start going totally bananapants. And perhaps we'll come to understand why most calculus classes shy away from infinitesimals.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Some calculus.

The Riemann zeta function (Mark, Monday–Wednesday)

Many highly qualified people believe that the most important open question in pure mathematics is the Riemann hypothesis, a conjecture about the zeros of the Riemann zeta function. Having been stated in 1859, the conjecture has outlived not only Riemann and his contemporaries, but a few generations of mathematicians beyond, and not for lack of effort! So what's the zeta function, and what's the conjecture? By the end of this class you should have a pretty good idea. You'll also have seen a variety of related cool things, such as the probability that a "random" positive integer is not divisible by a perfect square (beyond 1) and the reason that $-691/2730$ is a useful and interesting number.

Chilis: 🌶🌶🌶

Homework: Optional.

Prerequisites: Some single-variable calculus (including integration by parts) and some familiarity with complex numbers and infinite series; in particular, geometric series.

The matrix exponential and Jordan normal form (Dennis, Monday–Wednesday)

You've heard of the exponential function. You might have heard that the exponential function helps us solve differential equations, especially of the form $y' = ay$. What if you have not just one equation, but a whole system of them? What do you do then?

Well, first of all, if they are all linear, you'd probably think to use linear algebra; namely to write the system as one matrix equation $x' = Ax$, where now x is a vector and A is a matrix. How do we solve this? If you guessed, as in the above case, that we should use "e^{At}", whatever that means, you'd be right!!

In this class we'll explore exactly what this matrix exponential is and how it helps us solve differential equations. Along the way we'll need the Jordan normal form, which is a generalization of diagonalization, that at least puts matrices in upper triangular form (not necessarily diagonal form). But it's enough for us to actually *compute* the matrix exponential!

If we have time, we'll also go over how the matrix exponential ties together the Lie group GL_n and its Lie algebra, as well as all the other matrix groups. (Don't worry if you have no idea what this means!)

Chilis: 🌶🌶–🌶🌶🌶🌶

Homework: Optional.

Prerequisites: Basic linear algebra (familiarity with diagonalization), some calculus 2. For the proof of Jordan normal form only, we'll need some knowledge of polynomial rings.

Which things are the rationals? (Ben, Monday–Wednesday)

Do you know what the rationals look like, as a topological space? Can you recognize them in different guises? For example, which of the following spaces are homeomorphic to the rationals?

- $\mathbb{R}$, the real numbers?
- $\mathbb{Z}$, the integers?

- $\mathbb{Q}^2$, the space of rational points in the plane?
- The algebraic numbers (that is, the real numbers which are solutions to polynomial equations)?
- The Cantor set?

Oh, wait, some of those questions are really hard! Some of them we can deal with easily: the reals and the Cantor set, for example, are uncountable. The integers, on the other hand, are "discrete." Both of those let us tell that these things are not the rationals. But those don't let us do anything about the other two. These don't even let us figure out whether the sets $(0,1)\cap\mathbb{Q}$ and $[0,1]\cap\mathbb{Q}$ are homeomorphic!

In this course, we'll learn which things are the rationals, and which things are Cantor sets[1]. These questions are answered by a theorem of Sierpinski and a theorem of Brouwer.

Chilis: 🌶🌶🌶

Homework: Recommended.

Prerequisites: Some kind of point-set topology, and some introductory group theory.

12:10 Classes

A tour of Hensel's world (Mark, Thursday)
In one of Euler's less celebrated papers, he started with the formula for the sum of a geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and substituted 2 for x to arrive at the apparently nonsensical formula

$$1 + 2 + 4 + 8 + \cdots = -1.$$

More than a hundred years later, Hensel described a number system in which this formula is perfectly correct. That system and its relatives (for each of which 2 is replaced by a different prime number p), the p-adic numbers, are important in modern mathematics; we'll have a quick look around this strange "world".

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: Some experience with the idea of convergent series.

Complex dynamics: Julia sets and the Mandelbrot set (Neeraja, Friday)
If $p(z)$ is a polynomial, the sequence of *iterates* is the sequence

$$p(z), p(p(z)), p(p(p(z))), p(p(p(p(z)))), \ldots$$

for a fixed complex number z. For what values of z does this sequence converge? Diverge? For what values of z is it periodic? These questions led mathematicians Julia and Fatou to define certain sets, one of which is the filled Julia set, the set of all z for which the sequence of iterates is bounded. In this class, we'll draw some pictures and study some properties of filled Julia sets. In the process, we'll also come across the Mandelbrot set, which has been called "the most fascinating and complicated subset of the complex plane."

Chilis: 🌶🌶

Homework: None.

Prerequisites: Complex numbers (taking the modulus, writing a complex number in polar form). If you don't know this but would like to attend the class, please talk to me!

[1] Wait, where do Cantor sets come into it? Well, we'll see that in the course!

Computing trig functions by hand (Misha, Thursday)
When you learn about trig functions, you typically memorize a few of their values (for 30° or 45°, say) and if you want to know any of the other values, you get pointed to a calculator.

Has that ever seemed unsatisfying to you? If so, take this class, in which we'll see that finding some of these values is as easy as solving polynomials, and approximating all of them is as easy as multiplication. If time allows, we'll learn how to compute inverse trig functions, and also how to quickly find lots of digits of π.

Chilis: 🌶🌶

Homework: None.

Prerequisites: Be familiar with the formula $e^{ix} = \cos x + i \sin x$.

Extreme extremal graph theory (Mia, Friday)
A typical question in extremal graph theory asks, given a graph G with n vertices, how many edges does G need to guarantee that H is a subgraph? But what if I want not one graph H, but MANY? What if I want ALL of the cycles C_k, up to some fixed k? This class will look at a delightful proof of Bondy's theorem, which gives conditions that guarantee not one cycle, but all of them.

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: Graph theory.

Finding the center (Pesto, Monday)
Given n points in the plane, how can we find the center of the smallest circle containing them if:

(1) Programmer time is the main constraint;
(2) Worst-case runtime is the main constraint;
(3) Average-case runtime is the main constraint?

These have different answers!

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: Understand the statement "An algorithm runs in time $O(n^2)$".

How to ask questions (Eric, Thursday)
In this class you will learn about asking questions and also ask questions, though possibly not in that order. You will have the opportunity to learn practical wisdom on how to ask questions in a mathematical context and how to be intentional about your question asking.

Your homework will be to ask questions, in this class and others.

Chilis: 🌶

Homework: Required.

Prerequisites: None.

King chicken theorems (Marisa, Monday)
Chickens are incredibly cruel creatures. Whenever you put a bunch of them together, they will form a pecking order. Perhaps "order" is an exaggeration: the chickens will go around pecking whichever chickens they deem to be weaker than themselves, and whenever chickens encounter one another, it's a peck-or-be-pecked situation. Imagine you're a farmer, and you're observing the behavior of your chickens. You would like to assign blame to the meanest chicken. Is it always possible to identify the

meanest chicken? Can there be two equally mean chickens? Are there pecking orders in which all the chickens are equally mean?

Chilis: 🌶

Homework: None.

Prerequisites: None.

Many Counterexamples, Some Pathology (Some staff, Thursday)
Do you want to see your instructors talk about awful stuff? In this class, various teachers will make reasonable-sounding statements and then tell you why they were wrong.

Some of the MYRIAD, CONFUSING, STRANGE POSSIBILITIES are listed below!

- Katharine is happy to do pathological spaces.
- Linus will show you a combinatorial geometry conjecture 'the answer to this problem is exactly $n+1$,' and then how the answer is really, really not $n+1$.
- Mira will state statistics that will stupefy you.
- Susan will show you a ring that does a truly terrible thing.
- Come hear from Ben, about bad things in analysis!

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: None.

Perceptron (Linus, Tuesday)
For the third year in a row, I refuse to teach neural networks at Mathcamp. (There'd be almost nothing I can prove.)

But I'll skirt the edges, by teaching the simplest unit inside a neural network: a perceptron. These babies solve the following problem: given points in some n-dimensional space labeled + and −, how can we efficiently find a hyperplane separating all the + from all the − (assuming one exists)?

(What if only *most* of the + and − obey the rule, but there are some outliers? What if, instead of a hyperplane, a more complicated boundary (e.g. a polynomial) separates the + from the −?)

Chilis: 🌶🌶

Homework: None.

Prerequisites: Vectors, dot products.

Perfect numbers (Mark, Friday)
Do you love 6 (a big number!) and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called perfect, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes—a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.

Chilis: 🌶

Homework: None.

Prerequisites: None.

Random walks and electric networks (Misha, Tuesday)
In this class, I will tell you a few basic rules about how voltage and current in an electric network behave.

Then, we'll see that some properties of a random walk on such a network follow the same rules—and prove that any two objects that follow these rules must be the same object.

You don't need to have taken my Markov Chains class in week 2. If you did take that class, you will see completely new things about random walks, so you don't need to worry about being bored.

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: None.

Stirling's formula (Neeraja, Tuesday)
Stirling's formula gives an asymptotic estimate for $n!$, i.e. an approximation for $n!$ as n gets large. The formula first arose from correspondence between Stirling and de Moivre in the 1720s, when de Moivre used a version of the formula to discover what was essentially the central limit theorem in probability! (These days the central limit theorem is usually proved without using Stirling's formula.) In this class, we'll prove Stirling's formula and use it to solve some fun problems, such as showing the recurrence of the random walk on $\mathbb{Z}$ and $\mathbb{Z}^2$.

Chilis: 🌶🌶

Homework: Optional.

Prerequisites: Single-variable calculus (integration by parts).

The puzzle of the superstitious basketball player (Tim!, Friday)
Here's one of my favorite math puzzles. It's from Mike Donner, and it was published on FiveThirtyEight.

> A basketball player is in the gym practicing free throws. He makes his first shot, then misses his second. This player tends to get inside his own head a little bit, so this isn't good news. Specifically, the probability he hits any subsequent shot is equal to the overall percentage of shots that he's made thus far. (His neuroses are very exacting.) His coach, who knows his psychological tendency and saw the first two shots, leaves the gym and doesn't see the next 96 shots. The coach returns, and sees the player make shot No. 99. What is the probability, from the coach's point of view, that he makes shot No. 100?

I remember solving it. I had to do a bit of tedious calculation to arrive at the final answer. And when I saw the answer, I was astounded. It was so simple. I thought I was done with the puzzle, but really I was just beginning. Such a simple answer had to have a simple explanation, right? There are in fact a few simple explanations, each more satisfying than the previous.

In the end, I will make the following claim: even if we accept the scenario described by the puzzle, the basketball player's view of the world is totally wrong, and he is probably just superstitious. Perhaps there is a lesson here that we can take back with us to our real lives.

Chilis: 🌶🌶

Homework: None.

Prerequisites: None, but we'll spoil the answer to the puzzle pretty early in the class, so if you'd like to think about the puzzle yourself (which I wholly recommend), do it beforehand!

Voting theory 101 (Pesto, Tuesday)
"The only fair voting system is a dictatorship". What properties would make a voting system "fair'? What sorts of (non-dictatorship) voting systems are pretty good, even if they're not "fair"?

Chilis: 🌶

Homework: None.

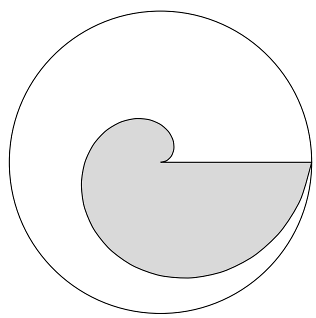
Prerequisites: None.

1:10 Classes

Ancient Greek calculus (Yuval, Thursday)
If you've ever seen a formal construction of the real numbers, you've probably heard of Dedekind cuts, named after the 19th-century German mathematician Richard Dedekind. However, he really doesn't deserve all the credit: 2000 years earlier, a Greek mathematician named Eudoxus of Cnidus came up with more or less the same definition. In my opinion, Eudoxus is the most important mathematician you've never heard of.

Even more, Eudoxus used his understanding of the real numbers to do what is essentially calculus; for instance, he was the first person to rigorously compute the volume of a cone. However, the mantle of ancient calculus was really picked up by Eudoxus's biggest fanboy, Archimedes. In my opinion, Archimedes is probably the most important mathematician you *have* heard of.

Building off of Eudoxus, Archimedes did some truly mind-blowing things. He computed the area of an arbitrary parabolic segment. He computed the volume and surface area of a sphere. He computed approximations of π. Perhaps most amazingly, he determined the area inside the following region, now called the *Archimedes spiral*.



If you've never seen this before, try it yourself—what fraction of the area of the circle is enclosed by the spiral? Even with modern integration techniques, the answer is not so easy to determine.

In this class, we'll get a sampling of ancient Greek proto-calculus. We'll start with Eudoxus's definition of the real numbers and we'll learn the "method of exhaustion", which was the proof technique he used to do calculus (it's more or less just evaluating an integral as a limit of Riemann sums). Then we'll move on to Archimedes and watch him do his magic, and we'll finish with his absolutely gorgeous argument for computing the area of a spiral.

Chilis: 🌶🌶

Homework: Optional.

Prerequisites: Having seen integrals and Dedekind cuts will be helpful, but not necessary.

Avoiding arithmetic triples (Misha, Monday)
Three-term arithmetic progressions like $1, 2, 3$ or maybe $89, 97, 105$ are the worst. If you hate them as much as I do, you might be on board with my plan to "fix" the number line, get rid of some natural numbers, and avoid all these arithmetic triples.

You might be less on board with my plan if it turns out that my strategy is to keep only the numbers

$$1, 2, 4, 8, 16, 32, 64, \ldots$$

and get rid of every number which isn't a power of 2.

Is there a less radical solution? Come to this class and find out!

Chilis: 🌶🌶

Homework: None.

Prerequisites: None.

Dominant eigenvalues and directed graphs (*Yuyuan*, Tuesday)
Suppose we have the following vector equation:

$$Ax + b = x$$

for some positive invertible matrix A and nonnegative vector b, does there exist a non-negative solution for x? This question can be answered with the help of the Perron–Frobenius theorem, which states that for an irreducible matrices, the dominant eigenvalue (i.e. the eigenvalue with the greatest magnitude) is real and positive, and its corresponding eigenvector has all positive entries. This theorem has many practical applications, such as in the fields of population modeling, statistical mechanics, and economic modeling. However, most proofs of this theorem require a lot of linear algebraic machinery. In this class, we will see a proof of this theorem that does not involve lots of algebraic manipulations; instead, we will assemble a proof through constructing directed graphs from matrices and converting the process of matrix multiplication into a process on graphs.

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: Familiarity with directed graphs, big-O notation, and eigenvalues.

Introduction to Coxeter groups (Kayla, Monday)
I want to begin by defining Coxeter systems and state the classification of Coxeter systems. Then I want the campers to play with a bunch of examples (I have about 10 in mind as of right now that would be good for them to work through). We will start with looking at what the Coxeter groups of simple Coxeter graphs are and work our way up to more technical examples like reflection groups and potentially discuss Weyl groups of roots systems. Time permitting and if there is interest, I would like to talk about some of the combinatorics of Coxeter groups introducing Bruhat order and weak order.

Chilis: 🌶🌶

Homework: Optional.

Prerequisites: Group theory.

Introduction to combinatorial topology (Kayla, Thursday)
Do you like combinatorics? Do you like topology? Ever wondered if there is any intersection between the two areas? It turns out that we can make topological spaces out of poset structures! Learn about what these topological spaces look like when we discuss the property of being shellable.

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: Having some familiarity with topology is nice! Also having seen posets and Hasse diagrams would be good.

Matrix completion (Linus, Thursday)
Can you find the pattern and fill in the question marks in the following matrix??

$$\begin{pmatrix} 10 & 10 & ? & 4 \\ 6 & 10 & 9 & ? \\ ? & 8 & 6 & 5 \\ 3 & 7 & 7 & 9 \end{pmatrix}$$

Did you get it? The numbers represent how much (columns) me, my boyfriend, and my roommates enjoy (rows) Nichijou, Dark Souls, The Lobster, and League of Legends. I haven't seen The Lobster, so if you figure out that ?, please let me know.

Well... okay. We can't hope to solve this exactly. But with enough 𝔅ig 𝔇ata, and a dose of linear algebra, we can find a good approximation. It's machine learning!

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: Linear algebra: you should be able to define rank, and prove that if $A : \mathbb{R}^m \to \mathbb{R}^k$ and $B : \mathbb{R}^k \to \mathbb{R}^n$ are linear maps then BA has rank at most k.

Posets and the Möbius function (Kayla, Tuesday)
Give an example-heavy introduction to posets and lattices in algebraic combinatorics following https://arxiv.org/pdf/1409.2562.pdf section 4. We will discuss types of posets and where they arise in math!

Chilis: 🌶🌶

Homework: Optional.

Prerequisites: None.

The Hilbert cube (*Harini*, Monday)
You're probably familiar with a one dimensional cube—the closed unit interval. And two dimensional cubes are also easy—unit squares. Three dimensional cubes are normal, and maybe you've seen hypercubes somewhere or the other. That's fine and good, because these are all finite dimensional. What happens when we try to take an infinite dimensional cube? How do things work there? This is called the Hilbert Cube, and it turns out that generalizing properties of finite dimensional cubes to it can be done, but with a little bit of cleverness. For example, we CAN define distance in this cube! Not only that, but when we do so, we can find a copy of ANY sufficiently small metric space inside it. And even more surprisingly, we can actually write down how to find this copy! Come to my class to find out how to do this!

Chilis: 🌶🌶🌶🌶

Homework: None.

Prerequisites: None.

The lemma at the heart of my thesis (Eric, Tuesday)
In the words of my thesis advisor "mathematics is not about proving theorems, it's about proving lemmas." I'll tell you the story (and prove the lemma!) of the lemma at the heart of my thesis. (Spoilers: it's a lemma about the structure of quotient rings of $\mathbb{Z}[\zeta]$ where ζ is a root of unity.)

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: You should know what a quotient ring is.

The redundancy of English (Mira, Thursday)
NWSFLSH: NGLSH S RDNDNT! (BT DN'T TLL YR NGLSH TCHR SD THT...)

The redundancy of English (or any other language) is what allows you to decipher the above sentence. It's also what allows you to decipher bad handwriting, or to have a conversation in a noisy (Zoom) room. The redundancy is a kind of error-correcting code: even if you miss part of what was said, you can recover the rest.

But can we quantify exactly *how* redundant English is? In other words, how much information is conveyed by a single letter of English text, relative to how much could theoretically be conveyed? We will answer this question in the way that Claude Shannon, the father of information theory, originally answered it: by (1) generating a bunch of amusing gibberish; (2) playing a word game that I call Shannon's Hangman, and using it as a way of communicating with our imaginary identical twins.

Chilis: 🌶

Homework: None.

Prerequisites: The definition of information, which was covered on Day 1 of my Week 3 Information Theory class. If you were not in that class, you can take a look at the slides for Day 1, and DM me if you have any questions. The definition is pretty straightforward, and you don't need to know anything else from that class to enjoy this one.

Tridiagonal symmetric matrices, the golden ratio, and Pascal's triangle (Emily, Monday–Tuesday)

Tridiagonal symmetric matrices are a type of Toeplitz matrix, which is a matrix in which every diagonal descending from left to right is constant. We will study a specific family of these matrices, namely $n \times n$ matrices with ones on the superdiagonal and subdiagonal and zeroes elsewhere:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Now where do the golden ratio and Pascal's triangle come in? It turns out that for certain values of n, the golden ratio (and its friends) appear as eigenvalues, and Pascal's triangle can tell us what the characteristic polynomials will look like! We will explore and prove these phenomena using a combination of linear algebra, trigonometry, and combinatorics.

Chilis: 🌶🌶🌶

Homework: None.

Prerequisites: Linear algebra (should know characteristic polynomials and eigenvalues), comfortable with summation notation and $\binom{n}{k}$ related to Pascal's triangle.