CLASS DESCRIPTIONS—WEEK 1, MATHCAMP 2020

Contents

9 am Classes

An inquiry-based approach to group theory. (Katharine, Monday–Friday)
Groups are settings in which we can perform an operation, like addition or multiplication. Many groups are things you're already familiar with (we can add integers, or multiply real numbers), but we can also "add" symmetries of shapes, arrangements of objects, or states of a game. If you like Sudoku, logic grid puzzles, or arranging your bookshelf in creative ways, you might like groups!

We'll start by defining binary operations and groups, then look at tons of examples, prove some properties of groups, learn to present information about them in different ways, and explore relationships between groups. The "inquiry-based" part of the title means here that you can expect to use your time outside of class proving theorems and working out examples, and a lot of class time sharing your ideas and discussing ideas with the rest of the class (as a group, one might say).

Chilis: 🌶🌶

Homework: Required.

Prerequisites: None.

Cluster: Group theory.

Required for: Combinatorics of tableaux (W2); Representation theory (W3–4); Classifying complex semisimple Lie algebras (W3); So you like them triangles? (W4)

Cut that out! (*Zach Abel*, Monday–Friday)
Let's cut shapes into other shapes and assemble them into even more shapes! We'll have a plethora of pretty pictures and a panoply of perplexing puzzles, including:

- Can you divide a square into (any number of) polygons that are similar to each other but all have different sizes? Can you do it with just 2 pieces? 3? Try it!
- Can you cut a square into (any number of) polygons and rearrange them exactly into an equilateral triangle? What about funkier polygons? If we allow curved cuts, can you rearrange the square into a circle? What about a cube into a tetrahedron using 3D pieces? What if we allow infinitely many cuts? Or fractal cuts? or . . .

Shaaaapes!

Chilis: 🌶🌶

Homework: Optional.

Prerequisites: None.

Determinantal formulas. (Kayla, Monday–Friday)
How can we use linear algebra to help us answer counting problems? For example, how many ways can we stack boxes into the corner of an $n \times n \times n$ room? Why do we mathematicians care about such problems? In this course, we will be exploring how to express solutions to counting problems as determinants of matrices!

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: None

Cluster: Counting things.

Introduction to graph theory. (Misha, Monday–Friday)
A graph is an object with a bunch of things (called "vertices"), some of which have connections between them (called "edges"). You could argue that just about anything is a graph. So graph theory is the most important subject in all of mathematics.

There are some problems which it's more useful to study with graphs than others. For example, `https://en.wikipedia.org/wiki/Switzerland#Cantons` has a map of Switzerland's 26 cantons, colored with 6 colors. Can we use fewer, without coloring adjacent cantons the same color? (We'd like to at least get it down to 5, since it's confusing that both Wallis and the adjacent Lake Geneva are similar shades of light blue.)

This class is an overview of some of these problems. We won't spend too long on any of them, but we'll try to catch a glimpse of many different topics that are really graph theory under the hood.

Chilis: 🌶🌶

Homework: Recommended.

Prerequisites: None.

Cluster: Graph theory.

Required for: Graphs on surfaces (W2); Ramanujan graphs and number theory (W2); Conflict-free graph coloring (W2); Spectral graph theory (W3); Brooks' theorem blues (W4); Extremal graph theory (W4)

Teaching math to computers. (Apurva, Monday–Friday)
Wouldn't it be great if computers could automatically generate proofs of complex mathematical theorems? Or even come up with new math? But right now computers can't even *understand* basic undergraduate math curriculum. There has been a lot of progress in the recent years to improve this appalling situation and there is a lot more that needs to be done.

In this class, we will teach math to computers using the Lean proof assistant. Lean is a programming language that is almost human readable and looks very close to math. Example Lean code:

```
theorem fermats_last_theorem
(n : N)
(n_gt_2 : n > 2)
:
not(exists x y z : N,
    (x^n + y^n = z^n) and
    (x > 0) and (y > 0) and (z > 0))
:=
begin
    sorry,
end
```

We will teach computers some basic number theory results like the infinitude of primes and that $\sqrt{2}$ is irrational. The goal of this course is to simply introduce you to this fascinating world of computer assisted theorem proving in Lean so that you might do a project on it afterward. If you get *really* interested, then you can eventually even contribute to the official Lean math library. (This isn't far fetched as many contributions to the Lean math library are being made by undergrads.)

This will a DIY course. You'll learn the language by coding yourself. I'll simply provide the assignments and give short 10-minute introductions each day.

See the class website `https://apurvanakade.github.io/courses/lean_at_MC2020/introduction.html` to explore more.

Chilis:

Homework: Required.

Prerequisites: You do not require any special software to run Lean. You can run it in a web browser. You also do not require any serious coding experience but you should be interested in coding and find it enjoyable.

You *do* need to be very comfortable with the following terms: logical operators (and, or, if ... then ..., not, iff), quantifiers (for all, there exists), proof by contradiction, contrapositive, principle of mathematical induction.

Cluster: Math and computers.

10 am Classes

Cubic curves. (Mark, Monday–Friday)
A curve in the x, y-plane is called a cubic curve if it is given by a polynomial equation $f(x, y) = 0$ of degree 3. Compared to conic sections (which have degree 2), at first sight cubic curves are unpleasantly diverse and complicated; Newton distinguished more than 70 different types of them, and later Plücker made a more refined classification into over 200 types. However, as we'll see, by using complex numbers and points at infinity we can bring a fair amount of order into the chaos, and cubic curves have many elegant and excellent properties. One of those properties in particular, which is about intersections, will allow us to prove a beautiful theorem of Pascal about hexagons and conic sections, and it will also let us define a group structure on any cubic curve—well, almost. We may have to leave out a singular ("bad") point first, but a cubic curve has at most one such point (which may be well hidden; for example, $y = x^3$ has one!), and most of them don't have any. Cubic curves without singular points are known as elliptic curves, and they are important in number theory, for example in the proof of the Fermat–Wiles–Taylor theorem (a.k.a. "Fermat's Last Theorem"). However, in this week's class we probably won't look at that aspect at all, and no knowledge of number theory (or even groups) is required. With any luck, along the way you'll pick up some ideas that extend beyond cubic curves, such as how to deal with points at infinity (using "homogeneous coordinates"), what to expect from intersections, and where to look for singular points and for inflection points.

Chilis:

Homework: Recommended.

Prerequisites: A bit of differential calculus, probably including partial derivatives; complex numbers; a bit of experience with determinants.

Cluster: Algebra and geometry.

Hyperplane arrangements. (Emily, Monday–Friday)
They sound fancy, but hyperplane arrangements are pretty simple to define. In $\mathbb{R}^2$, they are collections of lines; in $\mathbb{R}^3$, they are collections of planes (and we can keep going into higher dimensions!). For example, cutting a pizza into slices produces a hyperplane arrangement, where the cuts are the

hyperplanes. We will discuss how to classify the different pieces of hyperplane arrangements, and how to do operations on them.

Another thing that we will explore is how to count the number of slices that hyperplanes cut $\mathbb{R}^n$ into. This is obviously very easy in the case of a pizza, but in general it is not always so nice (especially when we are constructing arrangements that we cannot easily visualize). Some tools that we will use are posets, the Möbius function, and characteristic polynomials.

Chilis: 🌶

Homework: Recommended.

Prerequisites: None.

Cluster: Counting things.

Integration on manifolds. (Neeraja, Monday–Friday)

In single-variable calculus, integrals look something like this: $\int_a^b f(x)\,dx$. In this class, we will define and learn to evaluate integrals of the form $\int_M f$, where M is a manifold (e.g. a sphere or a torus). In order to define this integral, we will introduce differential forms, which turn out to be the "correct" objects to integrate on a manifold. After this we will prove the generalized Stokes' theorem, from which many classical integration results, such as Green's theorem, the classical Stokes' theorem and the divergence theorem, can be easily derived. We will derive at least one of these classical integration results and evaluate some integrals on manifolds.

Chilis: 🌶🌶🌶🌶

Homework: Recommended.

Prerequisites: Familiarity with vectors (addition, scalar multiplication, dot product) and single variable calculus up to the fundamental theorem of calculus.

Cluster: Real analysis.

Introduction to linear algebra. (Linus, Monday–Friday)

How indeed? It turns out one can rotate an image 45 degrees by using Paint's Resize tool to (1) skew 45 degrees horizontally, (2) skew −26 degrees vertically, and (3) stretch 50% horizontally. (Try it!) (But what if you want to rotate 30 degrees. . . ?)

These skews and stretches are 2-dimensional examples of *linear transformations*. And it is impossible to escape these beasts. So far this quarantine, I, personally, have used linear transformations for:

algebraic number theory; machine learning; Markov chains; problem 5c on the qualifying quiz; and helping the EHT decide where to build a \$2 million telescope. (Brag.)

This class will consist of $\sim 33\%$ lecture and $\sim 66\%$ learning through problem-solving.

Topics: vector spaces, rank, eigenvalues and eigenvectors, determinant, diagonalization, special matrices (e.g. symmetric, positive semidefinite), and more.

Chilis:

Homework: Recommended.

Prerequisites: Complex numbers.

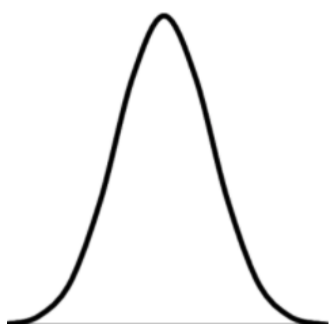
Required for: Combinatorics of tableaux (W2); Spectral graph theory (W3); Representation theory (W3–4); Classifying complex semisimple Lie algebras (W3); So you like them triangles? (W4); Solving equations with origami (W4); Functions you can't integrate (W4)

The bell curve. (Mira, Monday–Friday)

> I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error." The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason.
>
> Sir Francis Galton, 1889

Human heights; SAT scores; errors in scientific measurements; the number of heads you get when you toss a million coins; the number of people per year who forget to write the address on a letter they mail. . . . What do all of these (and numerous other phenomena) have in common?

Empirically, all of them turn out to be distributed according to "the bell curve":



The bell curve, known in the 19th century as the "Law of Error", is now usually called the *normal* or *Gaussian* distribution. It is the graph of the function $e^{-x^2/2}/\sqrt{2\pi}$ (scaled and translated appropriately). We will see how Gauss derived this function from a completely backward argument—a brilliant leap of intuition, but pretty sketchy math. We'll see how the great probabilist Laplace explained its ubiquity through the Central Limit Theorem. (Maybe you've learned about CLT in your statistics class . . . but do you know the proof?) We'll talk about how the normal distribution challenged the nineteenth century concept of free will. Finally, we'll look at some other mathematical contexts in which the normal distribution arises—it really is everywhere!

Chilis:

Homework: Required.

Prerequisites: Integral calculus. (There will be a *lot* of integrals!)

Cluster: Probability and statistics.

Required for: Either this class *or* Causal inference (W2) is required for Information theory (W3)

Noon Classes

Don't worry, these cats don't bite! (Basic category theory). (Dennis, Monday–Friday)
Categories have always been seen as an extremely high level concept, requiring a basic knowledge of several fields of mathematics already to understand. In this class, we'll try to explain some basic categorical concepts without anything more than a very rudimentary knowledge of what "sets" are. After all, the categorical language is incredibly intuitive once you get the hang of it; it's much more aligned to our usual intuition than how classical set theory constructs things. We'll be going over many examples, at a leisurely pace, of classical set theoretic constructions and how they are more naturally expressed in terms of categorical means.
Chilis:
Homework: Recommended.
Prerequisites: Intuition about sets as collections of things, of functions/maps as a rule assigning things of one set to another. Hopefully this makes it as accessible as possible.

Fourier something something boolean functions. (Tim!, Monday–Friday)
At its most basic, a computer program takes in a string of ones and zeros, and outputs either *accept* or *reject*. Such a program evaluates a *boolean function*; that is, a function $f\colon \{0,1\}^n \to \{0,1\}$. Perhaps because of the computer connection, and also because the definition is so fundamental, people have been asking questions about boolean functions for a long time.

But how do you study boolean functions? How do you even represent them? Consider the majority function on five variables; this is the function $\mathrm{Maj}_5\colon \{0,1\}^5 \to \{0,1\}$ given by

$$f(x) = \begin{cases} 0 & \text{if the input } x \text{ has three or more 0s} \\ 1 & \text{if the input } x \text{ has three or more 1s} \end{cases}$$

Instead of this succinct description, you could instead write the majority function as the polynomial $\mathrm{Maj}_5(x_1,x_2,x_3,x_4,x_5) = 3x_1x_2x_3x_4x_5 - 3x_1x_2x_3x_4 - 3x_1x_2x_3x_5 - 3x_1x_2x_4x_5 - 3x_1x_3x_4x_5 - 3x_2x_3x_4x_5 + x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 + x_3x_4x_5$. Some people would call this a waste of time. But others would call it *Fourier Analysis*.

Somehow, this actually has a bunch of applications. You can study election systems: Which voting systems (majority, electoral college, etc.) give voters the most power? Which voting systems are more vulnerable to tampering? You can prove Arrow's Impossibility Theorem that there is no "fair" voting system for an election with three candidates. You can study learning theory and cryptography.

A few notes:

- We will be doing Fourier analysis on $\mathbb{F}_2^n$. A lot of folks like to do Fourier analysis on $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ instead (go to Alan's class in Week 3!). They are connected but different. It's nice to see both, because there are beautiful underlying ideas that aren't necessarily apparent if you just see the real version.
- Fair warning that this class will have quite a few definitions to learn. But it's worth it! You'll want to do the homework so that the definitions don't become a jumble.
- I would have called this class "Fourier Analysis of Boolean Functions" but the Academic Ordinators have made it clear that I should not call this "analysis" to avoid confusion with actual analysis. They've used such strong language as "maybe leave out the word analysis?" and "to be clear, it's really not a big deal to put the word analysis in".

Chilis:
Homework: Required.
Prerequisites: None.
Cluster: Math and computers.

Introduction to analysis. (Alan, Monday–Friday)
This class is a rigorous introduction to limits and related concepts in calculus. Consider the following questions:

(1) Every calculus student knows that $\frac{d}{dx}(f+g) = f' + g'$. Is it also true that $\frac{d}{dx}\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f_n'$?
(2) Every calculus student knows that $a + b = b + a$. Is it also true that you can rearrange terms in an infinite series without changing its sum?

Sometimes, things are not as they seem. For example, the answer to the second question is a resounding "no." The Riemann rearrangement theorem, which we will prove, states that we can rearrange the terms in infinite series such as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ so that the sum converges to π, e, or whatever we want!

To help us study the questions above and many other ones, the key tool we'll use is the "epsilon-delta definition" of a limit. This concept can be hard to work with at first, so we will study many examples and look at related notions, such as uniform convergence. Being comfortable reasoning with limits is central to the field of mathematical analysis, and will open the door to some very exciting mathematics.

Chilis:

Homework: Required.

Prerequisites: Single-variable calculus (you should know what derivatives and integrals are)

Cluster: Real analysis.

Required for: Hilbert's space-filling curve (W2); Weierstrass approximation (W2); Cantor, Fourier, and the first uncountable ordinal (W2); Bairely complete (W3)

Majorizing-Comparisons Solving of Problems. (Pesto, Monday–Friday)
High-school olympiads usually try to choose problems relying on as little prior knowledge as possible. In inequalities problems, they usually fail completely; training is necessary to solve most and sufficient to solve many of them. We'll go over the common olympiad-style inequalities, and solve problems like the following:

(1) Prove that if a, b, and c are positive and $ab + bc + cd + da = 1$, then $\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}$.
(2) [USAMO 2004] Prove that if a, b, and c are positive, then $(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to olympiad-style problems you'll've solved as homework the previous day.

Chilis:

Homework: Required.

Prerequisites: None

Mathcamp crash course. (Susan, Monday–Friday)
There are two fundamental parts to doing mathematics: the toolbox of notation and techniques that go into proofs, and the ability to communicate your ideas through writing and presentation. Most math books, papers, and classes (including at Mathcamp!) take these things for granted; this is the class designed to introduce and reinforce these fundamentals. We'll cover basic logic, basic set theory, notation, and some proof techniques, and we'll focus on writing and presenting your proofs. If you are new to advanced mathematics, or just want to make sure that you have a firm foundation for the rest of your Mathcamp courses, then this class is *highly* recommended. If you want to build up confidence

in working with others mathematically, from simply asking questions in class to writing proofs for others to read to presenting at a blackboard, this class may also be right for you.

Here are some problems to test your knowledge of this fundamental toolbox:

(1) Negate the following sentence without using any negative words ("no", "not", etc.): *"If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."*

(2) Given two sets of real numbers A and B, we say that A *dominates* B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two disjoint, nonempty sets A and B such that A dominates B and B dominates A.

(3) Prove that there are infinitely many prime numbers.

(4) Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)

(5) Define rigorously what it means for a function to be increasing.

(6) Prove that addition modulo 2013 is well-defined.

(7) What is wrong with the following argument (aside from the fact that the claim is false)?

> **Claim:** On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.
>
> **Proof:** We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k+1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. QED.

(8) Explain what it means to say that the real numbers are uncountable. Then prove it.

If you would not be comfortable writing down proofs or presenting your solutions to these problems, then you can probably benefit from this crash course. If you found this list of questions intimidating or didn't know how to begin thinking about some of them, then you should *definitely* take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Chilis: 🌶

Homework: Required.

Prerequisites: None.