CLASS DESCRIPTIONS—MATHCAMP 2019

Classes

A Convoluted Process. (Ben)

The convolution is an important way of combining two functions, letting us "smooth out" functions that are very rough. In this course, we'll investigate the convolution and see its connection to the Fourier transform. At the end of the course, we'll explain the Bessel integrals: a sequence of trigonometric integrals where the first 7 are all equal to $\frac{\pi}{2}$... and then all the rest are not.

Why does this pattern start? Why does it stop? One explanation for this phenomenon is based on the convolution.

Prerequisites: Know how to take integrals (including improper integrals), some familiarity with limits

Algorithms in Number Theory. (Misha)

We will discuss which number-theoretic problems can be solved efficiently, and what "efficiently" even means in this case.

We will learn how to tell if a 100-digit number is prime, and maybe also how we can tell that $2^{82\,589\,933} - 1$ is prime.

We will talk about what's easy and what's hard about solving (linear, quadratic, higher-order) equations modulo n. We may also see a few applications of these ideas to cryptography.

Prerequisites: Number theory: you should be comfortable with modular arithmetic (including inverses and exponents) and no worse than mildly uncomfortable with quadratic reciprocity.

All About Quaternions. (Assaf + J-Lo)

On October 16, 1843, William Rowan Hamilton was crossing the Brougham Bridge in Dublin, when he had a flash of insight and carved the following into the stone:

$$i^2 = j^2 = k^2 = ijk = -1.$$

This two-week course will take you through a guided series of exercises that explore the many implications of this invention, and how they can be used to describe everything from the rotations of 3-D space to which integers can be expressed as a sum of four squares.

Prerequisites: None for Week 1; linear algebra for Week 2.

All Things Manifoldy. (Apurva)

Who said that mathematicians are not real doctors? We perform surgeries all the time. In this class, we'll take baby steps towards understanding manifolds. We'll learn some of the uber awesome techniques invented by topologists to study manifolds, perform surgeries on them, and do origami using simplices. By the end of the class you'll be able to visualize (some) manifolds in higher dimensions.

Incidentally, when Einstein tried to combine special relativity with Newton's gravity, nothing seemed to work. It took him a decade to finally realize a beautiful solution to the conundrum: our universe is a 4-dimensional manifold, and gravity is a measure of how the manifold curves. But what is a manifold?

The first two days are aimed at people who have never seen manifolds before. In the next two days we will cover Heegaard splittings and Dehn twists.

Prerequisites: None.

Analysis with Prime Numbers. (Eric)

I'm a number theorist, so I just want to do modular arithmetic all day long. But I had to take all these calculus and analysis classes instead of doing things I actually wanted to do! Wouldn't it be nice if I could do something that looks like calculus, but is secretly just number theory?

In this class, we will learn about p-adic numbers and analytic functions. We'll talk about the strange world of non-Archimedean spaces, where all triangles are isosceles and the distance you travel on a hike is the same as the biggest step you took. With the help of some new versions of our old friends e^x and $\log(x)$, we'll prove a cool theorem about the structure of linear recurrence sequences (e.g. things like Fibonacci numbers) that doesn't look like it should have anything to do with calculus.

Prerequisites: None! If you've seen functions defined by power series and the exponential and logarithmic functions before that's great, but it's not at all necessary. Come talk to me if you have questions!

A Proof of the Sensitivity Conjecture. (Tim!)

The Sensitivity Conjecture was a big open problem in theoretical computer science for thirty years. I spent some time thinking about it earlier in grad school. I would have offered to show you the proof earlier in camp (you know, back when I was at camp), but when we were making the four-week schedule, there was no proof yet (as far as the world knew)! In fact, Hao Huang posted his proof during Week 2 of camp. The proof is amazingly short, tantalizingly simple and very beautiful, yet also clever enough that it eluded prominent mathematicians for decades. It uses ideas from graph theory and linear algebra, by looking at eigenvalues.

More about the conjecture itself: The Sensitivity Conjecture is (or I should say, *was*) a conjecture about the complexity of boolean functions. Theoretical computer scientists care a lot about boolean functions (in part) because computers operate with boolean values. They care about complexity because they care about how complicated their programs are (in various senses of the word "complicated" — how long they take to run, how much memory they use, etc.).

There are many ways to measure complexity. Many of these complexity measures are known to be equivalent (or more precisely, *polynomially related*) to each other; these include *degree*, *(deterministic) decision tree complexity*, *bounded-error randomized decision tree complexity*, *certificate complexity*, *block sensitivity*, *approximate degree*, and *quantum decision tree complexity with bounded error*. This is both surprising (because these complexity measures have very different-sounding definitions) and a relief (because if all these complexity measures disagreed with each other, then who would you trust?). However, there was one complexity measure — called *sensitivity* — that was not known to be in the club. The Sensitivity Conjecture asserted (sensibly) that sensitivity is polynomially related to all those other complexity measures.

It seems fortuitous that this problem, which I've spent time on and which has been open for so long, was solved during Mathcamp and has such an elegant proof. If I were still at camp, I'd definitely propose a Week 5 class on it. Right now, though, I am in the Midwest, land of cows and rolling cornfields. But you know what… someone hold my cheese; I'm on my way!

In this class, we'll follow the story of the Sensitivity Conjecture from its development up to and including its proof.

Prerequisites: You should know what *degree* means in graph theory, and what an *eigenvalue* is in linear algebra.

A Very Difficult Definite Integral. (Kevin)

In this class, we will show

$$\int_0^1 \frac{\log(1+x^{2+\sqrt{3}})}{1+x}\,dx = \frac{\pi^2}{12}(1-\sqrt{3}) + \log(2)\log(1+\sqrt{3}).$$

We'll start by turning this Very Difficult Definite Integral into a Very Difficult Series. Then we'll sum it!

We'll need an unhealthy dose of clever tricks involving some heavy-duty algebraic number theory. It will be Very Difficult.

(This class is based on a StackExchange post by David Speyer.)

Prerequisites: If you have taken all of the algebraic number theory classes at camp, you will recognize some of the ingredients in results that we will assume.

Beyond Inclusion/Exclusion. (*John Mackey*)

Inclusion/Exclusion is a useful counting method wherein one successively corrects overapproximations and underapproximations. We'll spend one day reviewing Inclusion/Exclusion and then consider Mobius Inversion on Posets and Sign Reversing Involutions.

Mobius Inversion will provide a useful look inside the dual nature of accruing and sieving objects, and cast an algebraic context onto Inclusion/Exclusion. Sign Reversing Involutions will lend a matching perspective to the calculation of alternating sums.

Prerequisites: Experience with elementary counting and matrix algebra recommended.

Bhargava's Cube. (*Dave Savitt*)

One of the fundamental problems in number theory is to determine which integers are represented by a given polynomial: which integers have, say, the form $x^n + y^n$, or $x^3 + y^3 + z^3$, or $x^2 + xy + 2y^2$, and so forth. In this class we will focus on polynomials of the latter kind: quadratic polynomials in two variables. First studied systematically by Gauss, remarkably such polynomials are still generating new mathematics of the highest caliber, for example leading recently to the Fields-medal-winning work of Manjul Bhargava. In this class we will learn enough of the theory of binary quadratic forms to gain an appreciation for one of Bhargava's key contributions, the remarkable "Bhargava cube" – an idea that Gauss missed!

Prerequisites: None.

Breaking Bad (RSA Encryption). (Michael)

This class will be a walk through various methods in primality testing and prime factorization, and how many of them relate to each other via group theory. We'll discuss these in detail:

Pollard Rho and Pollard p-1 The Quadratic Sieve Williams p+1 and Primality testing with Lucas numbers

We will also talk about how two of these are strongly related via group theory (one of them is just for fun), and use this to understand why elliptic curves are useful for factorization and primality testing. We will, however, assume some of these common traits when we take a look at elliptic groups. We will look at how the elliptic group is constructed, the vast majority of a proof (i.e. we will look at the generic case of the Cayley-Bacharach theorem, but ignore the corner cases) that the elliptic group is actually a group, and how to perform operations in it that are useful to making it jump through the hoops we want it to on a computer.

Prerequisites: None!

Building Mathematical Sculptures. (*Zach Abel*)

Come transform ordinary items into extraordinary geometric sculptures! In these small yet intricate construction projects (that you'll take home when complete), we will assemble flexible paperclip cubes, precarious pencil weavings, gorgeous drinking straw jumbles, and more! Browse `http://zacharyabel.com/sculpture/` for examples of the types of projects this course may feature. Assembling these mathematical creations requires scrutiny of their elegant mathematical underpinnings from such areas as geometry, topology, and knot theory, so come prepared to learn, think, and build!

Prerequisites: None.

Calculus on Young's Lattice. (Kevin)

Young's lattice is a beautiful poset capturing the structure of partitions, with connections to combinatorics, geometry, and representation theory. Young's lattice is almost unique among posets[1], with the remarkable property that we can do calculus on it. We'll discuss how, and we'll see how doing calculus on Young's lattice yields powerful enumerative results, including some famous identities involving Young tableaux. Plus, if you like exponentiating things, we'll even get a chance to exponentiate xD!

Prerequisites: Differential calculus. Familiarity with partitions or any of Shiyue's Young Tableaux classes will be helpful.

Calculus without Calculus. (Tim!)

If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Lizka is 5 cubits tall and Eric is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Lizka's head to the top of Eric's head that touches the ground in the middle. What is the shortest length of string you can use?
- Assaf rides a bike around an elliptical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 20 meters way along the shoreline throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- Where in a movie theater should you sit so that the screen takes up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Prerequisites: Some calculus will be useful for context, but we won't actually use calculus (that's the point).

Cap Sets, SET, and ProSet. (*Elizabeth Chang-Davidson*)

One formulation of the cap set problem asks: how many points[2] can you have in a set before you are guaranteed three collinear points? The question is interesting for a number of reasons, one of which is that the current best general upper bound was proved quite recently, in 2016 by Ellenberg and Gijswijt, and another is that we only know the exact answer for up to 6-D space. In this class, we will talk about how the cap set problem relates to the card game SET, prove some of the known exact answers, discuss the 2016 result briefly, and then talk about a different card game, projective set.

Prerequisites: Some familiarity with arithmetic mod 3 is necessary. Linear algebra experience is helpful but not necessary.

Cauchy–Davenport with Combinatorial Nullstellensatz. (Bill)

The Cauchy–Davenport Theorem states that for a prime number $p > 2$, and for $A, B \subseteq \mathbb{Z}_p$,

$$|A + B| \geq \min(p, |A| + |B| - 1).$$

In this class, we'll prove the Cauchy–Davenport Theorem using a powerful technique known as Combinatorial Nullstellensatz, in which one carefully defines a polynomial whose set of zeros captures some combinatorial problem; and then one uses a (somewhat peculiar) theorem about the zeros of multivariate polynomials in order to extract properties of the original combinatorial problem.

Prerequisites: None.

Chaos in Voting. (Ben)

Suppose you and your friends want to figure out how many croutons to order for your event "Having Lots of Croutons Around." One way to pick a result that you're all kinda OK with is to have a sequence of majority votes. Eventually, you'll probably end up choosing to have the amount of croutons that the person in the middle of your group wants, the "median crouton wanter."

Now, you might think that this is also a good way to decide how many croutons and how much soda to buy for your “Having Lots of Crouton Salads Around” event. What we will show in class is that this is not only a bad idea, but it is a bad idea in the worst possible way—it’s possible (depending on a few things) that you’ll end up ordering a number of croutons AND an amount of salad that makes everyone very sad.

Prerequisites: None.

Cluster Algebras. (Véronique)

The goal of this class is to learn what cluster algebras are and prove a few properties about them. We will focus on cluster algebras from surfaces since we can easily visualize them and work on them. The first part is a short introduction to cluster algebras; the second, third and fourth parts are devoted to cluster algebras from surfaces, especially to the expansion formulas for the cluster variables. Expression formulas allow us to easily compute a cluster variable (that is usually a tedious computation). Finally, we will construct a canonical base in terms of snake graphs.

Prerequisites: None.

Conway–Coxeter Frieze Patterns. (Vronique)

Conway and Coxeter introduced frieze patterns in 1973: a frieze pattern over a field K is a map that associates to specific points in $\mathbb{Z}^2$ elements of K following a given rule. In this class, you will study properties of frieze patterns (periodicity, construction from a given diagonal, construction from a given row). More important, you will study the connection between frieze patterns and triangulations of an n-gon. Indeed, the number of triangles incident to each vertex appear in a frieze pattern, as well as a labeling of the vertices of the triangulated n-gon based on the triangles they share.

Prerequisites: None.

Counting, Involutions, and a Theorem of Fermat. (Mark)

Involutions are mathematical objects, especially functions, that are their own inverses. Involutions show up with some regularity in combinatorial proofs; in this class we’ll see how to use counting and an involution, but no “number theory” in the usual sense, to prove a famous theorem of Fermat about primes as sums of squares. (Actually, although Fermat stated the theorem, it’s uncertain whether he had a proof.) If you haven’t seen why every prime $p \equiv 1 \pmod 4$ is the sum of two squares, or if you would like to see a relatively recent (Heath-Brown, 1984, Zagier 1990), highly non-standard proof of this fact, do come!

Prerequisites: None.

Counting Points over Finite Fields. (*Aaron Landesman*)

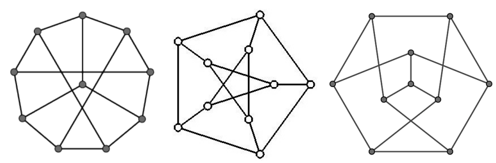
How many invertible matrices are there? How many subspaces of a vector space are there? How many squarefree polynomials of degree d are there? How many points does a curve have? You might guess the answer to all these questions is “infinitely many”. And indeed, you’d be correct over the real or complex numbers. But what if you work over a finite field, such as the integers modulo a prime p? Then there are only finitely many of these objects. In this class, we’ll learn how to count them.

After answering the above questions, we'll take the occasion to learn some algebraic geometry in order to define and explore the notion of zeta functions over finite fields. Roughly speaking, zeta Functions are a tool to count points over all finite fields simultaneously, and are closely related to the Riemann zeta function from number theory.

Prerequisites: Linear algebra, group theory, and finite fields

Crossing Numbers, from 1954 to 2012. (Marisa)

Suppose our goal is to draw graphs with as few edges crossing as possible. Take, for instance, the Petersen graph. We can draw it in a nice way, displaying very pretty symmetry, with 6 edge crossings. (Terrible!) Or, exhibiting different symmetry, with five crossings. (Still terrible!) Or, exhibiting yet another symmetry, with three crossings. (Closer.) The best possible is actually two crossings; can you do it?



In this class, we'll prove nice bounds (and look for exact answers) for the minimum number of edge crossings of a graph, and we'll find very quickly that many questions about crossing numbers are wide open. In fact, even the crossing number of $K_{m,n}$ on the plane is open! (An answer to this question was conjectured—and a proof claimed—in the 1950s by Zarankiewicz, and was followed by a 1969 paper entitled "The decline and fall of Zarankiewicz's theorem.")

Prerequisites: None.

Discrete Derivatives. (Tim!)

Usually, we define the derivative of f to be the limit of $\frac{f(x+h)-f(x)}{h}$ as h goes to 0. But suppose we're feeling lazy, and instead of taking a limit we just plug in $h = 1$ and call it a day. The thing we get is kind of a janky derivative: it's definitely not a derivative, but it acts sort of like one. It has its own version of the power rule, the product rule, and integration by parts, and it even prefers a different value of e. We'll take an expedition into this bizarre parallel universe. Then we'll apply what we find to problems in our own universe: we'll talk about Stirling numbers, and we'll solve difference equations and other problems involving sequences.

Prerequisites: Calculus (derivatives)

(Dys)functional Analysis. (*Viv Kuperberg*)

Let's say you want to do linear algebra, but you're working with an infinite dimensional vector space. Many things get much worse very quickly! We will talk about some of those things, including: different notions of length and how they interact with each other; bases and spanning sets, and how the span of a set depends on the order you write it in; dual spaces, and when you can and can't say that a vector space is its dual's dual; linear transformations and how they can misbehave; and more! We'll also talk about some examples of these spaces as they're used and, in particular, the joy of calling functions vectors.

Prerequisites: This doesn't need much calculus, but we will write down some infinite series and even an integral or two.

Eigenstuff and Beyond. (Mark)

If after a sunny day, the next day has an 80% probability of being sunny and a 20% probability of being rainy, while after a rainy day, the next day has a 60% probability of being sunny and a 40% probability of being rainy, and if today is sunny, how can you (without taking 365 increasingly painful steps of computation) find the probability that it will be sunny exactly one year from now? If you are given the equation $8x^2 + 6xy + y^2 = 19$, how can you quickly tell whether this represents an ellipse, a hyperbola, or a parabola, and how can you then (without technology) get an accurate sketch of the curve? These are two of many problems that can be solved using "eigenstuff" – more formally, eigenvalues and eigenvectors of square matrices. After defining those and seeing how one might in principle find them (although there are complications for 5×5 and larger matrices), we'll look at orthogonal (distance-preserving) linear transformations, at applications such as the above, and if time permits, at a seemingly magical, and fundamental, fact about square matrices called the Cayley–Hamilton Theorem. By the way, we'll probably also see how one might *come up with* Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\Big((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n \Big)$$

for the Fibonacci numbers. (If someone gives you the formula, you can prove it by induction.)

Prerequisites: Beginning linear algebra; for example, Apurva's week 1 class.

Electrifying Random Trees. (Will)

Consider a graph, and a particular edge in this graph. A natural question to ask is: how important is this edge in connecting the graph together? Here are two ways we might quantify this importance:

(1) The spanning trees of the graph give all the ways of minimally connecting the vertices using a subset of the edges. So look at all the spanning trees, and figure out how many of them contain our edge. Put another way, what is the probability that a random spanning tree contains the edge?
(2) Build your graph out of 1-ohm resistors. Hook up a 1-volt battery to the ends of your edge, and then measure how many amps of current are passing through that edge (as opposed to going through other paths in the graph).

These two numbers are actually the same! In this class, we'll see why. On the way, we'll see how one might generate a random spanning tree in the first place.

Prerequisites: Familiarity with the language of graphs (i.e., what a tree is; what the degree of a vertex is).

Elegant Applications of Linear Algebra to Combinatorics. (Bill)

In this class we'll use ideas from linear algebra to prove beautiful (and surprising!) theorems in discrete math. For example, we'll prove:

- For any irrational number x, it's impossible to tile a $1 \times x$ rectangle with finitely many squares (*even* if the squares are permitted to have both irrational and rational side-lengths).

The theme of the class will be that basic ideas in linear algebra (e.g., linear independence, dimension, bases) can be immensely useful when analyzing seemingly unrelated problems (e.g., tilings of a rectangle, or set systems satisfying certain intersection properties).

Prerequisites: Basic familiarity with linear algebra. Understanding of terms: vector space, basis, dimension, linear function, linear independence. No familiarity with eigenstuff is required.

Everything You Ever Wanted to Know About Finite Fields. (Eric)

Finite fields! They pop up in many places across mathematics and are a great place to get experience thinking about abstract algebra. In this class you'll work through all of the basic structural properties of finite fields, see how to construct them in a variety of hands on and not hands on ways, and see some Galois theory for infinite extensions in action by determining the full list of subfields of the algebraic closure of a finite field.

Prerequisites: You should know ring theory, to the level where you're comfortable quotienting a ring by an ideal.

Factoring in the Chicken McNugget Monoid. (Gabrielle)

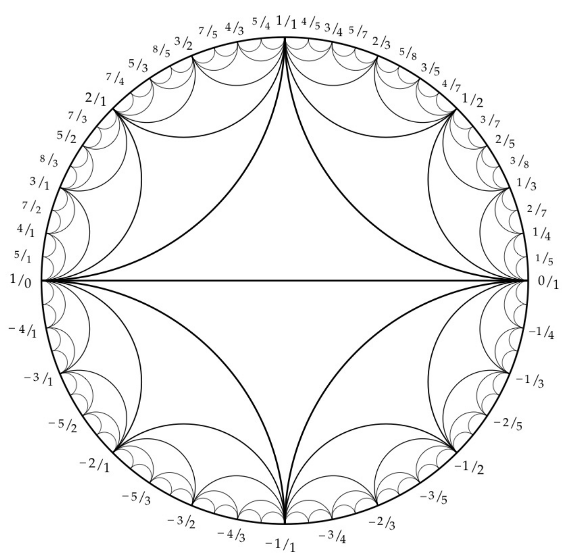
When they were first released, Chicken McNuggets were sold in packs of 6, 9, and 20. The Chicken McNugget Monoid is the set of numbers of Chicken McNuggets that can be purchased (Chicken McNugget numbers), i.e. numbers of the form $6a + 9b + 20c$. Notice that this expansion is not unique: $18 = 6 \cdot 3$ and $18 = 9 \cdot 2$. This is a form of nonunique factorization!

We will analyze this problem, and related problems in numerical monoids, through the lens of nonunique factorization.

Prerequisites: None.

Farey Tales. (J-Lo)

The goal of this course is to understand this picture:



Along the way we will encounter Farey series, the Euclidean algorithm, and a geometric interpretation of continued fractions. But really it's all about the picture.

Prerequisites: Basic theory of 2×2 matrices (multiplication, inverses, determinant, how they act on vectors).

Finite-ness. (Susan)

What does it mean for a set to be finite? One possible definition is that the set can be put into bijective correspondence with a natural number. But is that really the definition we want? It seems kind of inelegant to just take all the finite sets we already know and say, "That's all, folks!" Here are a couple possible definitions of finite-ness that don't reference the natural numbers:

- **Dedekind's Definition:** A set S is finite if every injection $f : S \to S$ is a surjection.
- **Tarski's Definition:** A set S is finite if every total-order on S is a well-order.
- **Kuratowski's Definition:** A set S is finite if and only if it is an element of the smallest class of sets containing $\emptyset$ and all $\{s\}$ for $s \in S$ which is closed under unions.

All of these definitions are equivalent to the natural numbers definition... as long as you believe the Axiom of Choice. Otherwise, only one of them is. And also the proofs are surprisingly subtle. Wait... is finiteness hard?

Prerequisites: None.

From High School Arithmetic to Group Cohomology. (Apurva)

Much of mathematics is invented by taking something ordinary and reinterpreting it in a new way – a change of perspective.

We all know how to add 2 digit numbers and can do it instinctively. But what if the digits came from non-abelian groups? Does it still make sense to talk about addition and "carrying" if the addition is non-abelian?

By understanding why the carrying process works and axiomatizing it, we will be able to define the (second) group cohomology of a group and Ext groups. The notion of group cohomology then shows up in unexpected places like field theories in physics and algebraic K-theory.

This will be an IBL class. Instead of a regular lecture, we'll learn things by solving problem sheets.

Prerequisites: Basic group theory - group homomorphism, first isomorphism theorem, cyclic groups, products of groups, kernel and image of group homomorphisms.

Functions of a Complex Variable. (Mark)

Spectacular (and unexpected) things happen in calculus when you allow the variable (now to be called $z = x + iy$ instead of x) to take on complex values. For example, functions that are "differentiable" on a disk in the complex plane now automatically have power series (Taylor) expansions. If you know what the values of such a function are everywhere along a closed curve, then you can deduce its value anywhere inside the curve! Not only is this quite beautiful math, it also has important applications, both inside and outside math. For example, functions of a complex variable were used by Dirichlet to prove his famous theorem about primes in arithmetic progressions, which states that if a and b are positive integers with $\gcd(a, b) = 1$, then the sequence $a, a + b, a + 2b, a + 3b, \ldots$ contains infinitely many primes. This was probably the first major result in analytic number theory, the branch of number theory that uses complex analysis as a fundamental tool and that includes such key questions as the Riemann Hypothesis. Meanwhile, in an entirely different direction, complex variables can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow. Dirichlet's theorem is certainly beyond the scope of this class, and heat conduction probably is too, but we should see a proof of the so-called "Fundamental Theorem of Algebra, which states that any nonconstant polynomial (with real or even complex coefficients) has a root in the complex numbers. We should also see how to compute some impossible-looking improper integrals by leaving the real axis that we're supposed to integrate over and boldly venturing forth into the complex plane! This class runs for two weeks, but it should be worth it. (If you can take only the first week, you'll still get to see a good bit of interesting material, including one or two of the things mentioned above.)

Prerequisites: Multivariable calculus, including Green's Theorem; if the week 1 crash course doesn't get to Green's Theorem, it will be covered near the beginning of this class.

Fundamental Groups. (Kayla)

In this class, we develop an algebraic tool that helps us distinguish topological spaces up to homotopy. The fundamental group is a group that we associate to a homotopy class of topological spaces that helps us detect "holes" in our space. This class will be a great introduction to algebraic topology and will give you a great sense of how group theory and topology intersect.

Prerequisites: Group Theory and Topology

Galois Correspondence of Covering Spaces. (Apurva)

Galois theory first arose in the study of roots of polynomials. Its main theorem establishes a mirrored structure, called the Galois correspondence, between the algebraic extensions of fields and certain automorphism groups.

A parallel theory exists in topology with field extensions replaced by covering spaces and the absolute Galois group replaced by the fundamental group. This parallel theory existed independently for a while, and was unified by Grothendieck under a broader umbrella of algebraic geometry.

In this class, we'll see the topological side of the story: we will define covering spaces, deck transformations, and prove the correspondence between sub-covering spaces and subgroups of deck transformations. As an application, we will prove the NielsenSchreier theorem which states that *every subgroup of a free group is free.*

We won't need any topology, as we'll mostly be working with graphs and surfaces. The intrepid student is very welcome to generalize to higher dimensions.

Disclaimer: We won't be talking about the Galois correspondence for fields at all. This is a purely geometry/topology class.

Prerequisites: You should be very comfortable with group theory stuff like normal subgroups, group actions, and the orbit-stabilizer theorem.

Galois Theory Crash Course. (Mark)

In 1832, the twenty-year-old mathematician and radical (in the political sense) Galois died tragically, as the result of a wound he sustained in a duel. The night before Galois was shot, he hurriedly scribbled a letter to a friend, sketching out mathematical ideas that he correctly suspected he might not live to work out more carefully. Among Galois' ideas (accounts differ as to just which of them were actually in that famous letter) are those that led to what is now called Galois theory, a deep connection between field extensions on the one hand and groups of automorphisms on the other (even though what we now consider the general definitions of "group" and "field" were not given until fifty years or so later). If this class happens, I expect to be rather hurriedly (but not tragically) scribbling as we try to cover as much of this material as reasonably possible. If all goes well, we might conceivably be able to get through an outline of the proof that it is impossible to solve general polynomial equations by radicals once the degree of the polynomial is greater than 4. (This depends on the simplicity of the alternating group, which we won't have time to show in this class but which *may* be shown in a separate week 5 class.) Even if we don't get that far, the so-called Galois correspondence (which we should be able to get to, and prove) is well worth seeing!

Prerequisites: Group theory; linear algebra; some familiarity with fields and with polynomial rings.

Game Theory. (Kayla)

This class will take an economical approach to the study of game theory. We will explore ideas including optimal strategies, iterated deletion of dominated strategies, Nash equilibrium, etc. The aim of the course is to play some classical games and formalize the idea of strategizing to win a game!

Prerequisites: None.

Generating-Function Magic. (Bill)

"A generating function is a clothesline on which we hang up a sequence of numbers for display."
– Herbert Wilf

The *generating function* of a number sequence $a_0, a_1, \ldots$ is the formal power series $\sum_{i=1}^{\infty} a_i x^i$. By encapsulating the number sequence within a single algebraic object, generating functions allow for us to derive results about the original number sequence in what are often highly unexpected ways.

In this class, we'll use generating functions to prove two of my favorite results:

- The number of integer partitions with odd-sized parts is equal to the number of integer partitions with distinct-sized parts. (We'll also see a beautiful bijective proof of this result!)
- The n^{th} Catalan number C_n, which counts the number of valid configurations of n pairs of parentheses, satisfies $C_n = \binom{2n}{n}/(n+1)$.

Prerequisites: Students who have seen an example of generating functions used in the past will have an easier time, but I will make sure this is not necessary.

Going in Cycles. (Misha)

A *knight* is a chess piece that jumps from a square to any other square exactly $\sqrt{5}$ units away. Put one of these in the corner of an 8×8 chessboard. Can it visit every other square of the board exactly once, then come back to the start?

This is an instance of the Hamiltonian cycle problem. In general, it's very hard to solve. We will talk about some ways we can guarantee a solution exists—or quickly demonstrate that it doesn't.

Prerequisites: It will help to be familiar with some graph-theoretic terminology.

Graph Coloring and Containment. (Pesto)

A k-coloring of a graph is a way to assign each of its vertices one of k colors such that no pair of adjacent vertices shares a color.

The nicest statement we could hope for in graph coloring is

(1) If a graph doesn't *contain* a graph with k vertices all adjacent to each other, then it can be colored with at most $k - 1$ colors.

That statement is false for your first guess of the definition of "contain", but we can change the definition of "contain" in one way ("minors") to get one of the biggest open conjectures in graph theory, and we can change the definition of "contain" in another way ("induced subgraphs") and tweak the statement slightly to get another of the biggest open conjectures in graph theory.

We'll learn about both definitions of "containment," understand both of those conjectures, prove some special cases of them, and see some other nice results in graph coloring.

Prerequisites: None.

Group Theory & Rubik's Cubes. (Gabrielle)

Group theory has many practical applications! We will not study any of those in this class. Instead, we will describe the set of moves one can perform on a Rubik's cube as a group, and we will use group theory (some of which you may learn in this class, such as group actions) to analyze this group, in order to characterize all valid configurations of the Rubik's cube and determine some solution methods. This class is an active learning class, which means that you will play with Rubik's cubes and discover the answer to these questions yourself (with some guidance).

Prerequisites: Group theory (definition of groups, subgroups, cosets, homomorphisms; know what a symmetric group is; know what is meant by "even" and "odd" permutations); talk to me if you're interested but lack some of the prerequisites!

Harmonic Analysis on Finite Abelian Groups. (*Mike Orrison*)

In this course, we will focus on how and why you might want to rewrite a complex-valued function defined on a finite abelian group as a linear combination of simple functions called characters. As we will see, doing so quickly leads to discussions of far-reaching algorithms and ideas in mathematics such as discrete Fourier transforms (DFTs), fast Fourier transforms (FFTs), random walks, and the Uncertainty Principle.

Prerequisites: Complex numbers (basic arithmetic and geometric interpretations), linear algebra (bases, invertible matrices, eigenvalues, complex inner products, orthogonality), and group theory (examples of finite abelian groups and their subgroups, cosets, homomorphisms).

Hedetniemi's Conjecture. (*Yuval Wigderson*)

Two months ago, the world of graph theory was shocked when Yaroslav Shitov, a relatively unknown Russian mathematician, published a three-page paper disproving Hedetniemi's conjecture, a famous open problem that's been intensively studied for more than fifty years. In this class, we'll go through Shitov's beautiful and subtle argument to disprove this conjecture.

Hedetniemi's conjecture concerns proper colorings of graphs, which are a central topic in graph theory (for instance, the famous Four-Color Theorem says that every graph that can be drawn in the plane can be properly colored with at most four colors). More specifically, Hedetniemi's conjecture deals with proper colorings of the *tensor product* of two graphs, which is a simple construction that generates a new graph $G \times H$ from two graphs G, H. As we'll see, the definition of the tensor product implies that if G or H can be properly colored with k colors, then $G \times H$ can also be colored with k colors. Thus, if $\chi(G)$ denotes the minimum number of colors needed to properly color G, then this implies that

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}.$$

Hedetniemi conjectured that this inequality is actually always an equality, and this is what Shitov disproved, by finding graphs G, H with $\chi(G \times H) < \min\{\chi(G), \chi(H)\}$.

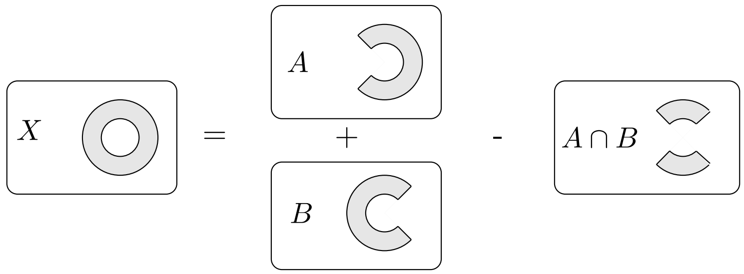
You might wonder why it took fifty years to disprove this conjecture, given that a counterexample amounts to writing down two graphs satisfying a simple property. One reason is that the graphs Shitov constructs are absolutely enormous—to my knowledge, the smallest known G has about 2,000,000 vertices, while the smallest known H has about $10^{4{,}000{,}000}$ vertices; for comparison, the number of particles in the universe is about 10^{80}, so H is incomprehensibly huge. So if you're interested in thinking about truly enormous graphs, or interested in seeing what real-life math research looks like, this class is probably for you.

Prerequisites: None, but the class will be easier to follow if you've seen a little bit of graph theory before.

Homological Algebra: The Art of Gluing. (*Jeff Hicks*)

"Decomposition and gluing" is a general mathematical technique which is used to understand an object in terms of its pieces. When people make that analogy of comparing mathematics to jigsaw puzzles and assembling the small things you know until you see the whole picture, this is the what they are talking about. As a concrete example, think of the way we understand polytopes like cubes and tetrahedra by gluing them together from their vertices, edges and faces.

The goal of this class is to understand the theory of decomposition and gluing. As a simple example, let's suppose that we are trying to compute the number of elements in a set X. We will decompose X into two subsets A and B so that $X = A \cup B$. We can compute the number of elements in X by the principle of *inclusion-exclusion,* which states $|X| = |A| + |B| - |A \cap B|$.



Our framework for generalizing these kinds of arguments is called homological algebra. Instead of using numbers to record our properties, we'll use linear algebra to construct our properties and glue. This framework will allow us to generate insight into the general theory of decomposition and gluing, showing that problems amenable to attack by gluing have subtleties beyond the original questions we were trying to answer. We will mostly be looking at topological spaces—spheres, tori, etc— as our objects of interest to motivate the theory.

As a warmup: you might want to think about why trying to compute the number of connected regions in the shape X above by gluing will magically tell you that X has a hole in the middle (come talk to me about your thoughts, or let me know if you have any questions!)

Prerequisites: We will be using linear algebra in this class. You should know what the dimension of an abstract vector space is, and the rank-nullity theorem. If you're taking linear algebra this week, the first day or two will be a bit rocky.

Infinite Graphs. (*Mia Smith*)

In traditional graph theory, a graph is defined to be a finite set of vertices which are connected by edges. However, the requirement that the vertex set be finite feels a little … stifling. In this class, we will eschew this requirement and allow *infinitely* many of vertices. By drawing on tools from set theory and graph theory, we will explore what happens when we allow graphs to have an infinitely large set of vertices.

To start, some classical results from graph theory turn out to be far trickier to prove on infinite graphs. For instance, simply proving that every connected graph has a spanning tree will necessitate a dive into the world of posets. However, the admission of infinite graphs opens up possibilities. We can define universal graphs and ask things like, is there a graph that contains every other countable graph as a subgraph? What about a planar graph that contains every other planar graph?

Prerequisites: Some familiarity with graph theory and sets is recommended.

Infinitely Many Proofs of Infinitely Many Primes. (Eric)

I'll fit as many proofs as I can into one class of the fact that there are infinitely many prime numbers. Some will be straightforward, others will connect us to great open problems (the Riemann zeta function will appear!), others will use topology (??), and more!

Prerequisites: Some proofs may require a small amount of knowledge about topology or rings, but nothing beyond the basic definitions.

Infinite-Ness. (Susan)

What happens when you take an infinite set and add an element to it? This looks like a silly question if we're thinking in terms of cardinality. The answer is: nothing happens. Adding a single element to an infinite set has no effect on its cardinality. But if we're thinking of an *ordered* infinite set, the question becomes much more interesting. The answer is: it depends on where you put the extra element.

In this class we'll be discussing two different perspectives on infinite sets—cardinality and ordinality. We'll consider the relationships between these two notions of infinite-ness, and how to find interpretations for the operations of addition, multiplication, and exponentiation in either context.

We'll finish off by discussing examples of sets that are really, really big. We'll discuss inaccessible cardinals, and prove that they are so big they prove their own independence.

Prerequisites: Some familiarity with cardinality could be helpful—knowing how to show that the rationals and the naturals are the same size, but that the real numbers are bigger should suffice. A quick conversation with Susan (or really any mentor of your choice) should bring you up to speed.

Infinite Trees. (Susan)

König's infinity lemma states that a tree of infinite height with finite levels has an infinite branch. So let's ask the obvious[3] followup question: what happens when you have a tree of uncountable height with countable levels? Does this result in an uncountable branch?

Surprisingly[4], no, it doesn't—in this class we'll construct a tree of uncountable height in which every level and every branch is countable! And this isn't the weirdest thing we'll see in this class. We'll see trees that may not even exist—we have to go outside of Zermelo-Fraenkel set theory to find them. Come to this class if you want to find out what it means for a tree to be uncountably tall, delve into the mysteries of the Diamond Axiom, and learn how to pronounce the word "Aronszajn."

Prerequisites: None.

Introduction to Algebraic Number Theory. (J-Lo)

Here are two helpful perspectives that can be helpful when solving problems:

(1) Sometimes the easiest way to answer a question about numbers is to change what "number" means.
(2) Factoring can become a whole lot nicer if you do it with infinitely many numbers at once.

In this course, we will develop these perspectives through a guided series of exercises that introduce you to algebraic numbers, algebraic integers, and ideals.

Prerequisites: Dividing polynomials with remainder, adding and multiplying mod n.

Introduction to Gerrymandering. (Assaf)

Gerrymandering is the art of winning elections that you really had no business winning by drawing districts behind secret doors in smokey rooms. In this class, we'll embark on a mathematical dive into the world of gerrymandering with shapes, political metrics, data, and graphs. By the end, we'll talk about the current state-of-the-art arguments that appeared in the recent supreme court cases.

This class will have a non-mandatory coding component for those of you who want to learn a bit of coding.

Let's break politics!

Prerequisites: None.

Introduction to Group Theory. (Shiyue)

What do the integers and the set of permutations of n things have in common? They are both groups! You might have seen many many groups before, but what are they? In this class, we will learn the foundational theory of groups, maps between them, and a plethora of examples that will show you the fascinating world of modern algebra.

Prerequisites: None.

Introduction to Number Theory. (Gabrielle)

Number theory is the study of the natural numbers, which we thought we finished understanding completely in elementary school. It turns out that there is a lot we do not know! We will prove some results that you may have taken for granted in the past (the division algorithm and unique factorization of integers into primes, for example). By the end of this class, you will know much more, such as: What is the relationship between the greatest common divisor and least common multiple of two integers? When does a polynomial with integer coefficients have solutions modulo a positive integer m? When is a number a a square modulo a prime p? (If you don't know what we mean by greatest common divisor, least common divisor, or modulo an integer, do not fear–you will learn all of this, too!)

Prerequisites: None.

Introduction to Ring Theory. (Will)

Rings are abstract algebraic structures (like groups and vector spaces) which capture the idea of a system with addition, subtraction, and multiplication, but not necessarily division. They show up in any mathematical subject that involves algebra.

In this class, we'll develop the axioms and basic properties of rings: homomorphisms, ideals, quotients, and localization: the practice of strategically adding multiplicative inverses of certain elements. We'll pay particular attention to the rings of integers and polynomials, which share many intriguing parallels.

Prerequisites: Some examples may reference basic facts about groups, matrices, and vector spaces, but you won't need to be familiar with these things to take the course.

Knot Theory. (Kayla)

Why knot learn more about knots? Especially when you canknot find a punnier subject! In this class, we will explore the world of knots and try to figure (8) out how we can tell them apart. Knot theory is still a dynamic branch of mathematics, so in our quest to distinguish all knots, some of our attempts may be trefoiled. But nonetheless, we will valiantly try to distinguish knots using things like Reidemeister moves, as well as numerical, color, and polynomial invariants!

Prerequisites: None.

Linear Algebra. (Apurva)

Linear algebra is a fundamental language used in almost every area of theoretical and applied math, from abstract algebraic geometry to applied data science. If you can add things together you are probably using linear algebra. Linear algebra is set theory with a vengeance. `https://youtu.be/A05n32Bl0aY`.

In this class, we'll learn about vector spaces, bases, matrices, linear transformations, rank-nullity theorem, change of basis isomorphisms, and take a peek at some other advanced stuff like determinant and eigenvectors. We'll also be doing a lot of proofs.

This class is mostly IBL, i.e. instead of a regular lecture we'll learn by solving problem sheets.

Prerequisites: None

Logic and Arithmetic. (*Steve Schweber*)

A lot of mathematics can be boiled down to arithmetic. Or, perhaps we should say that a lot of *the practice of* mathematics can be boiled down to arithmetic: regardless of what mathematical objects we're studying, what we're actually *doing* is writing down definitions, theorems, and proofs, and these are just strings of symbols. (Once we precisely define all those terms, of course. But hey, how hard could that be?) And arithmetic is great for talking about strings of symbols.

This suggests a great bit of silliness: can we prove things about what mathematics is/does/can be by just thinking about the natural numbers? It turns out that we can, and there's a lot to say here. This class will focus on understanding what this perspective shift can tell us about the limits (in a non-calculus-y sense) of mathematics, both in terms of what mathematical languages can express and what mathematical systems can prove.

Prerequisites: None.

Long Live Determinants! (Will)

The determinant of a square matrix is a polynomial in its entries which contains a lot of important information. It's sometimes fiddly to work with: when Sheldon Axler wrote his highly-regarded textbook *Linear Algebra Done Right*, he adopted the motto of "Down with determinants!", arguing that they are computationally impractical and pedagogically unhelpful.

On the other hand, determinants are really cool by themselves! In this class, we'll look at the several different definitions of determinant, and see that, in spite of its arbitrary-seeming formulas, it is an important and natural feature of a matrix. Then in the rest of the class, we'll look at some remarkable identities involving the determinant: as time allows, we'll count things, prove magic formulas, and/or index lines in projective space.

Prerequisites: Basic familiarity with linear algebra (linear maps, manipulating matrices). Abstract vector spaces and eigenstuff would help. You do not need to know what a determinant is.

Magic. (*Don Laackman*)

It turns out that you can give a quite precise meta-mathematical definition of what a magic trick is. For magic tricks involving deception or misdirection, there isn't much mathematical content, but there is a huge world of magic tricks where not knowing what you don't know about math can quite literally generate magic. In this class, we'll take a look at a couple of my favorite mathematical magic tricks, consider what they have in common, and even come up with some tricks of our own!

Prerequisites: None.

Markov Triples and Continued Fractions. (Vronique and Kayla)

Markov numbers are integers that appear in the solution triples of the Diophantine equation, $x^2+y^2+z^2=3xyz$. A conjecture, unproven for over 100 years, states that a Markov triple $\{x,y,z\}$ is uniquely determined by its maximum. In this class, we prove a statement related to this conjecture to determine an ordering on subsets of the Markov numbers based on their corresponding rational. Indeed, there is a natural map from the rational numbers between zero and one to the Markov numbers. The proof uses the cluster algebra of the torus with one puncture and a resulting reformulation of the conjectures in terms of continued fractions.

Prerequisites: None, but cluster algebra is an advantage.

Martin's Axiom. (Susan)

Induction is awesome! You know what else is awesome? Posets! In fact, my one problem with induction is that I only ever get to do it on well-ordered sets. First I prove my result for zero, then for one, then for two, etc. If I'm lucky, maybe I also get to prove my result for infinite ordinals. That's kinda cool. But nowhere near as cool as it could be—I wanna do induction on posets!

Enter Martin's Axiom: an extra-set-theoretic axiom that allows us to do just that. I take the poset of all partially-proved results, and then glue them haphazardly into a giant, glorious Franken-Sign of a proof that gives us an uncountable result.

Wanna see how it works? Come to this class!

Prerequisites: None.

Matching Bears With Campers. (Rice)

Hooray! Mathcamp just received a shipment of 128 adorable teddy bears, one for each camper! Each camper writes down a complete ranking of teddy bears, ordered by how much they like each bear.

Unbeknownst to the campers, the teddy bears are sentient creatures, and they are hungry. Each bear writes down a complete ranking of campers, ordered by how much they want to eat each camper.

The Mathcamp staff collect these rankings and match the bears up with the campers. Their goal is to create a *stable* matching: a matching M where there is no bear b and camper c such that b and c prefer each other over their match in M. After all, if the matching is unstable, b and c can just run off with each other, abandoning their matches in M, and that would be sad.

Day 1: Can the staff succeed, no matter what rankings the bears and campers submit? (And more!)

Day 2: Some stable matchings are better for campers, others for bears. Can we make a stable matching that's good for both bears and campers, or is the bears' and campers' welfare fundamentally at odds?

Day 3: Oops! The staff accidentally bought 127 bears, just one short. What happens now? (The answer is really surprising!)

Days 4 and 5: The staff realize that the bears want to eat the campers, use their staff powers to fight off the bears, and this time order 128 bears that are *actually* just teddy bears. Now that only campers have preferences, what are some fair ways for the bears to be matched with the campers?

Note: days 1 and 2 of the class will be IBL, i.e. you'll derive the results yourselves in small groups!

Prerequisites: None!

Mathcamp Crash Course. (Kevin)

There are two fundamental parts to doing mathematics: the toolbox of notation and techniques that go into proofs, and the ability to communicate your ideas through writing and presentation. Most math books, papers, and classes (including at Mathcamp!) take these things for granted; this is the class designed to introduce and reinforce these fundamentals. We'll cover basic logic, basic set theory, notation, and some proof techniques, and we'll focus on writing and presenting your proofs. If you are new to advanced mathematics, or just want to make sure that you have a firm foundation for the rest of your Mathcamp courses, then this class is *highly* recommended. If you want to build up confidence in working with others mathematically, from simply asking questions in class to writing proofs for others to read to presenting at a blackboard, this class may also be right for you.

Here are some problems to test your knowledge of this fundamental toolbox:

(1) Negate the following sentence without using any negative words ("no", "not", etc.): *"If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."*

(2) Given two sets of real numbers A and B, we say that A *dominates* B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two disjoint, nonempty sets A and B such that A dominates B and B dominates A.

(3) Prove that there are infinitely many prime numbers.

(4) Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)

(5) Define rigorously what it means for a function to be increasing.

(6) Prove that addition modulo 2013 is well-defined.

(7) What is wrong with the following argument (aside from the fact that the claim is false)?

  Claim: On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

  Proof: We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k+1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. QED.

(8) Explain what it means to say that the real numbers are uncountable. Then prove it.

If you would not be comfortable writing down proofs or presenting your solutions to these problems, then you can probably benefit from this crash course. If you found this list of questions intimidating or didn't know how to begin thinking about some of them, then you should *definitely* take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Prerequisites: None.

Mathcamplandia. (*Luke Joyner*)

The powers that be (THE POWERS THAT BE!!!) have noticed that Mathcamp moves around year to year, and more generally that mathematicians often work in spaces and places that may not be well suited to their needs. They've asked you to build a city of math (CITY OF MATH!!!) that will house a permanent home for Mathcamp, any alumni and friends who might want to live in such a place, other mathematicians, and roughly 100000–400000 other people. They want you to find a real location for this city, and design the whole thing... in a week. (You're welcome to adapt an existing place, or create a new place from scratch, if you can find a site that allows it.) Along the way, we'll dive into the fascinating world of cities and urban design... there'll be a little topology and geometry involved, and you're welcome to bring math to your project in whatever creative ways you see fit, but this class mostly focuses on mapping, design and making.

Prerequisites: Very basic understanding of Graph Theory, an open mind, willingness to develop ideas through stages of editing (Note: drawing skills are not required, but drawing by hand or on the computer will be an important part of our final project; I will lead an optional seminar on drawing techniques for anyone interested on Wednesday).

Note: homework for this class will be a mix of math, reading, group activities, and mostly progress toward designing a city in small groups. This is a one-chili class for the low level of math knowledge required, but it is not exactly an *easy* class, as it is not so easy to design a city well in a week. It's a "you get what you put in" kind of class.

Matroids and Greed. (Will)

Kruskal's algorithm is a way of choosing, in a graph with weighted edges, a spanning tree of minimal weight. It's a very simple greedy algorithm: just keep adding the least-weight edge you can, subject to the condition that you don't form a cycle.

A matroid is a mathematical structure that captures the idea of "independent subsets of a set." Key motivating examples include linearly independent subsets of a finite set of vectors, as well as forests and trees in a graph.

And in fact, just as in the case of graphs, we can use a greedy algorithm to find minimum-weight independent subsets of any matroid. But more is true: matroids are actually precisely characterized by the fact that a greedy algorithm works in this way. In this class, we'll look at a few equivalent ways of defining matroids, and then see why the greedy algorithm property is one of them.

Prerequisites: Know what "linear independence" is. Know what a tree (in the context of graph theory) is.

Morse Theory. (Kayla)

Topologists like donuts. Suppose that we want to cover our donut in chocolate sauce when it is oriented vertically (so that the hole of the donut is perpendicular to the floor). What would the chocolate sauce function look like? Would this function have critical points? What does it mean to be a critical point of a chocolate sauce function?

Morse theory is a branch of topology that enables us to analyze the topology of a manifold by studying differentiable functions on that manifold. Moreover, some differentiable functions can tell us information about how to build a CW complex structure for our space!

Prerequisites: Topology and calculus.

Multi-Coefficient Solving of Problems. (Pesto)

Polynomials are a frequent topic of olympiad-style competitions, since there are many and interesting problems using them. For instance,

(1) If $a_1, \ldots, a_n$ are distinct real numbers, find a closed-form expression for

$$\sum_{1\le i\le n} \prod_{1\le j\le n, j\ne i} \frac{a_i + a_j}{a_i - a_j}.$$

(2) Find the product of the lengths of all the sides and diagonals of a regular n-gon of diameter 2.

Don't see the polynomials? Come to class and find them.

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to olympiad-style problems you will have solved as homework the previous day.

Prerequisites: Linear algebra: understand the statement "Two-variable polynomials of degree at most 2 are a vector space of dimension 6."

Multivariable Calculus Crash Course. (Mark)

In real life, interesting quantities usually depend on several variables (such as the coordinates of a point, the time, the temperature, the number of campers in the room, the real and imaginary parts of a complex number, ...). Because of this, "ordinary" (single-variable) calculus often isn't enough to solve practical problems. In this class, we'll quickly go through the basics of calculus for functions of several variables. As time permits, we'll look at some cool applications, such as: If you're in the desert and you want to cool off as quickly as possible, how do you decide what direction to go in? What is the total area under a bell curve? What force fields are consistent with conservation of energy?

Prerequisites: Knowledge of single-variable calculus (differentiation and integration)

Musical Lattices. (J-Lo)

Ignoring the difference between black and white keys, a piano is essentially just a set of equally spaced points on a line: press a point, get a note. Each point plays a note $2^{1/12}$ times the frequency of the previous one. But where did this particular spacing come from? Why 12? More importantly, why should all the music we want to create be built out of a single interval in the first place?

[5] `http://stanford.edu/~jonlove/3dpiano.cdf`

What if, instead, our piano went in *two directions*: say, steps along the x-axis go up perfect fifths, and steps along the y-axis go up octaves? And if we wanted to play major thirds too, we could make a *three-dimensional piano*![5]

But why should we even stop there — we can make *n-dimensional pianos*.

This class will explore the properties of n-dimensional pianos (more commonly known as "lattices"). Through a combination of lecture and guided exercises, we will study Diophantine approximation, Minkowski's Theorem, simultaneous approximation of real numbers, and more, in order to discover facts about musical scales.

Prerequisites: None. (The concepts may seem more familiar to you if you've taken linear algebra or group theory before, but everything we need will be developed from scratch).

Non-Euclidean Geometries. (Véronique)

This class is an introduction to non-Euclidean geometries. We will look at Euclid's postulates, and, more precisely, theorems proved with and without the fifth postulate. Then, we will study some non-Euclidean geometries such as the spherical geometry, the projective geometry, and the inversive geometry.

Prerequisites: Group theory

Not-Proofs of Fermat's Last Theorem. (Gabrielle)

In this class, we're going to learn many ways that Fermat's Last Theorem was not proved. This will be a history class in part, but we are going to look at the proofs themselves–a proof when $n = 4$ (Fermat), a failed-but-fixed proof for $n = 3$ (Euler), one of the first proofs for infinitely many primes (Germain), and an idea which would have worked to prove the statement for all p, had it not been for nonunique factorization (Kummer).

Prerequisites: None.

Not Your Grandparents' Algorithms Class. (*Sam Gutekunst*)

This class will study algorithms for making progressively more and more complicated decisions: we'll start with finding an "optimal" diet[6] and end with finding optimal routes for Beyoncé's next concert tour. Unlike a traditional algorithms class, we'll spend approximately zero time focusing on running time beyond vague notions about whether or not an algorithm is efficient in a formal sense. Instead, we'll emphasize the ideas behind a whole bunch of beautiful algorithms! We'll cover: Fourier–Motzkin elimination and the ellipsoid algorithm for linear programming, branch and bound for discrete optimization, cool graph theory algorithms that work stupidly well, and state-of-the-art approximation algorithms for the Traveling Salesman Problem.

Prerequisites: None.

Packing Permutation Patterns. (Misha)

Prepare by picking a permutation π and a pattern P. Probabilistically pick $|P|$ pieces of π: perhaps putting them together produces P? Let $\rho_P(\pi)$ be the probability of producing P.

To *pack* P in π, puff up this probability, making P as plentiful as possible. We will ponder the packing problem for $P = 132$ (and plenty of its pals) using a progression of powerful problem-solving procedures.

(For returning campers: this material is a subset of my class on flag algebras last year.)

Prerequisites: None.

PARTIAL DIFFERENTIAL EQUATIONS. (Ben and Assaf)

PAIN IS WEAKNESS LEAVING THE BODY! PARTIAL DIFFERENTIAL EQUATIONS? MORE LIKE PAINFULLY DIFFICULT EXERCISE! WANT FOURIER TRANSFORMS? WANT INFINITE DIMENSIONAL VECTOR SPACES? WANT INTEGRATION BY PARTS? HAVE YOU EVER TAKEN THE EXPONENT OF A DERIVATIVE? SOLVE ALL OF QUANTUM MECHANICS, SOLVE HEAT AND MOTION AND WAVES AND MAKE LAPLACE'S EQUATION LOOK LIKE ADDITION! YOU WILL TRANSCEND TIME ITSELF. IN THIS CLASS, YOU DO THE WORK, YOU GET RESULTS! CALL US AT 1888-MCSP-PDE FOR YOUR FREE TRIAL TODAY! Limited time offer, some restrictions apply, misuse of this product may cause blindness and hair loss, not available in Alaska or Hawaii.

Prerequisites: Topology, multivariable calculus, complex analysis, linear algebra, nonlinear algebra, measure theory, and an undergraduate degree in physics.

Perfect Numbers. (Mark)

Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called *perfect*, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes – a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.

Prerequisites: None

Permutation Combinatorics. (Bill)

In this class, we'll study the underlying combinatorial structures of permutations. We'll find that many simple questions about permutations (e.g. how many permutations contain no decreasing subsequence of length three?) have startlingly beautiful answers (e.g., The n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$). We'll also find that other questions about permutations have quite *surprising* answers. For one example of this, consider the following riddle:

Suppose 100 players $1, 2, \ldots, 100$ each walk into identical rooms. In each room, there are 100 boxes in a row, containing the numbers $1, 2, \ldots, 100$ in a random order (though the order is the same in all 100 rooms). Each player opens 75 boxes, and wins a cheap plastic trophy if one of the boxes that they open contains their number. If, however, all 100 players win trophies, then all 100 players also win $\$1,000,000$.

At face value, the cash prize seems nearly impossible to win. Surprisingly, there's a simple strategy that guarantees a $> 70\%$ chance of winning it. (Wait, what!?) Perhaps even more surprisingly, the motivation and analysis for the strategy comes straight from studying the cycle structure of permutations.

Prerequisites: Know what a permutation is.

Perspectives on Cohomology. (Apurva, Assaf, J-Lo, and Eric)

Cohomology is a useful mathematical tool that comes in a variety of flavours, and many of your staff friends use cohomology in their research. In this class we'll have a different staff every day talk about their favourite flavour of cohomology, what is special about it, what it is useful for, and how they think about it. The topics, in order:

- Cellular Cohomology 🌶🌶🌶 (Apurva)
- Čech Cohomology 🌶🌶🌶🌶 (Assaf)
- The Chow Ring 🌶🌶🌶 (J-Lo)
- Galois Cohomology 🌶🌶🌶 (Eric)

Prerequisites: Each lecture will aim to be mostly self-contained, but some familiarity with linear algebra will be useful for all of the lectures.

Polytopes (Higher Dimensional Polygons). (*Angélica Osorno*)

In this series we will explore how to generalize the definition of a polygon to higher dimensions. We will learn about intricacies and subtleties of this higher dimensional geometry. We will also explore applications to optimization theory (e.g., the traveling salesman problem). If time permits, we will also talk about some open research problems in this area.

Prerequisites: None.

Probabilistic Models and (a little bit of) Machine Learning. (Mira)

Machine learning can be approached in two ways. You can put your data in a black box, try a bunch of standard techniques, find one that does a decent job on your task, and consider yourself done. If all you want is to train your machine to do a certain task, these "black-box" techniques can be very powerful. A prime example of this approach is neural networks (which we will NOT discuss in this class).

But the black-box approach doesn't give you a deeper understanding of how learning actually happens and what intelligence really means. An alternative approach is to posit that a learner begins with some model of the world – which has to be a *probabilistic* model, since the world is full of uncertainty and we only observe it incompletely, through the lens of our data. Then learning proceeds by updating this model based on observed data according to the laws of probability (also known as Bayesian inference). This is the approach that we will pursue in this class.

In addition to learning the theory, we will actually implement some learning algorithms in simple probabilistic graphical models, using the free web-based probabilistic programming language WebPPL. The homework for this class is essential, and you will be expected to devote at least an hour to it every day.

Prerequisites: Solid knowledge of calculus. It will help if you have done a little bit of programming before, but WebPPL is quite easy to use, so beginners are welcome too.

Problem-Solving Cornucopia. (Mark)

Come and work together (or separately, if you prefer) on a daily variety of interesting problems, which are not arranged by topic - so part of the challenge is to figure out what techniques might be helpful for any particular problem! Although the difficulty of the problems will vary, most should be in the 2- to 3-chili range. This is not a competition-oriented class, although you might well pick up some ideas and/or "tricks" that will be useful to you in the context of competitions. Although there might be an occasional "break" when a point of general interest comes up, most of the time you'll be actively thinking about problems, with hints provided on request.

Prerequisites: None, although some problems will require calculus.

Problem Solving: Induction. (Misha)

Some of you maybe first saw induction in the context of proving a result like

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}.$$

Such a proof is fairly straightforward, and maybe your main worry was "Can my last sentence just be 'by induction, we're done' or do I need something fancier?"

In this class, we'll see how these proofs can get much more complicated. Our induction will start out strong, and on each day of class it will get stronger than all the previous days combined. You'll see examples of crazy induction in algebra, analysis, graph theory, set theory, number theory, and other theories. You'll learn how to use induction (and how *not* to use it) to solve problems of your own, olympiad and otherwise.

In class, we will solve problems together; I will focus less on answering the question "why is this claim true?" and more on answering the question "why would we think of solving a problem this way?" There will be plenty of problems left for homework, and you will not get much out of a problem-solving class unless you spend time solving those problems.

Prerequisites: None.

Procrastination. (Rice)

If you're like me, you've had the experience of putting off all your homework until the last minute, despite knowing the whole time that pacing yourself would make your life way easier. Come learn about an elegant mathematical model of procrastination, and about how simply realizing that you're going to procrastinate will (usually) cause you to make much wiser decisions.

Prerequisites: None.

Quantum Mechanics. (*Nic Ford*)

Quantum mechanics is one of the great triumphs of twentieth-century physics, and a lot of people seem to think they know a lot about it. You might have heard stories about the uncertainty principle, measurements changing the results of experiments, things being in two places at once, entangled particles communicating instantaneously over large distances, and a cat that's neither alive nor dead until you look at it. In this class, we'll talk about how quantum systems actually behave; I hope to convince you that while these popular explanations sound designed to weird you out, in a sense they're usually not strange *enough* to describe how the universe actually works!

We'll start in the first half from an abstract, axiomatic perspective to understand how quantum states and observables work mathematically; the "actual physics," where we'll examine how a couple realistic systems evolve in time, will wait until the last couple of days. My goal is to show you the mathematical description of many of the quantum phenomena you might have already heard about, like quantum measurements, entanglement, the Heisenberg uncertainty principle, and interference, but no prior knowledge of any of this is necessary.

Prerequisites: You'll definitely need linear algebra to follow the class, including vector spaces, bases, linear independence and dimension. Some prior exposure to inner products and eigenvalues will be helpful, but we'll also be covering them in class. Calculus will also show up, but mostly only for the second half. It will also be helpful to know a bit about how momentum and energy behave in Newtonian physics, but this isn't a hard requirement.

Quiver Representations. (Will)

One of the central ideas of linear algebra is that, while we often express maps between vector spaces by matrices, they aren't exactly the same thing. Indeed, if we change the bases of the vector spaces involved, the matrix expression for our map changes as well. By choosing coordinates strategically, we can express a map by a particularly nice matrix, which depends only on the rank of the map. This can reveal structure that wasn't visible in the original matrix.

If we consider maps from a vector space to itself, thus only allowing one change of basis, the situation gets more complicated. But it turns out we can still pick a basis that puts our matrix in a nice canonical form, known as Jordan normal form.

What if we have *two* linear maps between a pair of vector spaces? Or a chain of maps $U \to V \to W$? Can we still pick the bases of the spaces involved such that all the maps are simultaneously described by nice-looking matrices? The subject of quiver representations provides a unifying framework for questions like these. In this class, we'll see how considering networks of vector spaces and maps between them can tell us when these problems have satisfying answers. And while there won't be a direct connection to cluster algebras, some actors from Véronique's week 1 class will make an unexpected reappearance.

Prerequisites: Familiarity with linear algebra. In particular, you should know what effect change of basis has on a matrix. Knowing what it means to diagonalize a matrix would also help.

Randomized Algorithms. (Bill)

The only thing I love more than algorithms is *randomized* algorithms. In this class we'll see examples where a little bit of randomization, used in just the right way, can let us solve algorithmic problems that are otherwise unapproachable (or very difficult). Examples of questions we'll answer include:

- Given a binary sequence x and another shorter binary sequence p, how quickly can I determine whether p appears as a contiguous subsequence of x?
- Given a graph $G = (V, E)$, how do I efficiently split the vertices into two parts V_1, V_2 that minimize the number of edges between V_1 and V_2?
- How do I route messages on a computer network in order to minimize network congestion?

In the last day of the class, we'll analyze a special case of the so-called constructive Lovász Local Lemma. In doing so, we'll see one of the weirdest and coolest algorithm analyses I know, in which we argue an algorithm's correctness using the fact that random sequences of zeros and ones are difficult to compress.

Prerequisites: Familiarity with probability. Familiarity with Big-O notation would be useful but not necessary. No knowledge of Chernoff Bounds or advanced probability is required.

Real Analysis Week 1: Limits. (Véronique)

How can we prove that $\lim_{n\to\infty} \frac{n}{n+1} = 1$? Because, for large n, $\frac{n}{n+1}$ is almost 1? Sure, but how can we prove this formally? And what is a continuous function, other than a function that you can draw without lifting your pen? And finally, how do we compute $\lim_{x\to 0} \frac{\sin(x)}{x}$? We answer these questions in this class, among others.. We will learn about a little bit of topology and a lot of analysis: we'll discuss numerical sequences and functions (limits and continuity).

Prerequisites: A bit of calculus

Real Analysis Week 2: Measures. (Ben)

We like integrating things, and we like taking limits. Unfortunately, the Riemann integral runs into technical problems when dealing with taking limits of things. When trying to solve problems by methods such as Fourier analysis or approximation, this bad behavior with limits makes life a lot more difficult.

One way to solve this problem is to introduce the idea of a "measure." This lets us define a more general idea of integration that is more well-behaved with respect to taking limits! Moreover, measure theory is a gateway to a lot of useful areas of math, letting us do a great deal of analysis with more rigor and care.

Prerequisites: Know what limit points are (of a sequence, of a set).

Reciprocity Laws in Algebraic Number Theory. (Eric)

If I give you an integer polynomial $f(x)$, it may factor mod p for some primes p but not for others. How can we determine for which primes a given polynomial factors? What sorts of answers can we expect?

This turns out to be super hard to answer in general. One of the goals of modern number theory is to give good general answers to this and related questions. I hope to give you a tour of this landscape of algebraic numbers, zeta functions, and Galois groups.

We'll explore this landscape starting in the foothills with quadratic reciprocity, testing our mettle on the slopes of class field theory, and we'll try to gain a glimpse of the peaks of the Langlands mountains by the end.

Prerequisites: Algebraic number theory to the level of knowing that every prime number factors into a product of prime ideals in the ring of integers of a number field. You should be familiar with the statement of quadratic reciprocity. I will introduce a few pieces of Galois theory as necessary.

Representation Theory of Associative Algebras. (Véronique)

Representation theory is a branch of mathematics that studies abstract algebraic structures (in our case, algebras) by representing their elements as linear transformations of vector spaces, and studies modules over these abstract algebraic structures. In this class, we will define modules and algebras, in particular, path algebras. Then, we will prove how modules on a path algebra are equivalent to quiver representations, studied in Wil's class. The class on quiver representation is not an absolute prerequisite to this class, but you should definitely know what a quiver representation is. If it is not the case, ask me or Will!

Prerequisites: Course on quiver representations (just need to know the definition of a quiver representation)

Rescuing Divergent Series. (Mark)

Consider the infinite series $1-1+1-1+1-1+\dots$. What is its sum? Maybe $(1-1)+(1-1)+\cdots=0$, maybe $1-(1-1)-(1-1)\cdots=1$. At one time mathematicians were quite perplexed by this, and one even thought the issue had theological significance. Now presumably its nonsense to think that the "real" answer is $\frac{1}{2}$, just because the answers 0 and 1 seem equally good, right? After all, how could the sum of a series of integers be anything other than an integer?

Prerequisites: A bit of experience with the idea of convergence.

Riemann Surfaces. (Apurva)

Riemann surfaces were invented by Riemann to understand inverses of complex functions. In this class, we'll explore Riemann's ideas of gluing pieces of the complex plane together and the theory of ramified coverings. We'll prove the Riemann-Hurwitz formula and as an application prove Fermat's last conjecture for polynomials in one variable, namely that

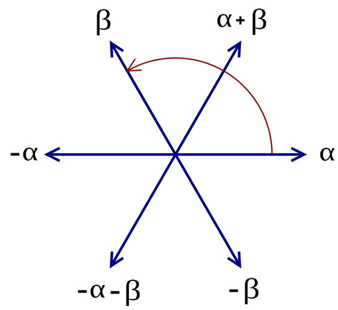
$$(x(t))^p + (y(t))^p = (z(t))^p$$

has no non-trivial solutions if $p > 2$.

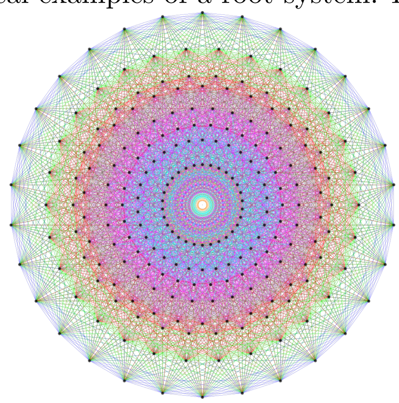
Prerequisites: Basic knowledge of complex analysis – complex analytic functions, open mapping theorem, Laurent and Taylor series. We'll also need basic notions from topology like continuity and compactness.

Root Systems. (Kevin)

Root systems are arrangements of vectors in some Euclidean space that satisfy certain basic reflection and projection properties. Here's an example:



This is A_2, one of the prototypical examples of a root system. Here's another example:



This is (a two dimensional projection of) the infamous E_8 root system, one of the few exceptions to an otherwise elegant classification of root systems.

Root systems are an important tool to understand Lie algebras and Lie groups, but in this class, we'll focus on their combinatorial properties and their beautiful representation by Dynkin diagrams[7].

Prerequisites: None.

Sperner, Monsky, and Brouwer. (*Laura Pierson*)

Is it possible to divide a square into an odd number of triangles of equal area? If you stir your coffee, must there always be a point that doesn't move? If you have a big triangle divided into small triangles, and you color all the vertices so that the vertices of the big triangle are all different colors and each edge of the big triangle has only two different colors, must there be a small triangle all of whose vertices are different colors? In this class, we'll answer all these questions and see how they're related to each other. Along the way, we'll also encounter the game of hex and the 2-adic numbers!

Prerequisites: None.

Sporadic Groups and Where to Find Them. (*Theo Johnson-Freyd*)

We will survey the finite simple groups, focusing on the exceptional, aka "sporadic," ones. Most groups are "muggles": they live in repeating families. The sporadic groups are sorted into "generations" according to which exceptional object they act on — the magical landscape inhabited by the exceptional beasts. The first generation acts on Golay's error correcting code, and we will spend some time learning to compute in that code. Then we will discuss how a "gauged quantization" procedure convert Golay's code into Leech's lattice, producing the second generation, and then a second "gauged quantization" procedure converts the Leech lattice into Monster Moonshine and the third generation.

Prerequisites: Know what the following words mean: finite group, (normal) subgroup, alternating group, vector space, complex numbers, arithmetic mod p.

Studying Betting Games with Other Betting Games. (Bill)

Sometimes the best way to study a betting game A is to define a *different* game B on top of it. By analyzing game B, we can then extract information about the original game.

This cool combinatorial technique can be used to prove all sorts of surprising facts. Examples of where we'll use it include:

- Analyzing the expected time for a random walk to leave an interval.
- Analyzing a famous game involving pennies, known as Penney's Game (and named, somewhat confusingly, after its inventor Walter Penney).
- Analyzing the probability of a gambler winning big when the bets are stacked against them.

Prerequisites: Comfortable with probability and expected values.

Super Mario Bros. is NP-hard. (Tim!)

One of the first video games I ever played was Super Mario Bros. on my friend's 8-bit Nintendo Entertainment System. Some of the levels were really hard! But now I can some feel some vindication, because in 2015, researchers mathematically proved that Super Mario Bros. is hard. Specifically, it's impossible for a computer to solve[8] Super Mario Bros. any faster than some of the most notorious computationally hard problems, like the Traveling Salesman Problem and boolean 3-satisfiability (3-SAT).

In this class, we'll talk about what it means for a math problem to be computationally hard, and then we'll show that Super Mario Bros. (and if we have time, other classic Nintendo games like Donkey Kong Country and Legend of Zelda) are NP-hard.

Prerequisites: None.

Systems of Differential Equations. (Mark)

Many models have been devised to try to capture the essential features of phenomena in economics, ecology, and other fields using systems of differential equations. One classic example is given by the Volterra-Lotka equations from the 1920s:

$$\frac{dx}{dt} = -k_1 x + k_2 xy, \frac{dy}{dt} = k_3 y - k_4 xy,$$

in which x, y are the sizes of a predator and a prey population, respectively, at time t, and k_1 through k_4 are constants. There are two obvious problems with such models. Often, the equations are too hard to solve (except, perhaps, numerically); more importantly, they are not actually correct (they can only hope to approximate what really goes on). On the other hand, if we're approximating anyway and we have a system

$$\frac{dx}{dt} = f(x, y), \frac{dy}{dt} = g(x, y),$$

why not approximate it by a linear system such as

$$\frac{dx}{dt} = px + qy, \frac{dy}{dt} = rx + sy?$$

Systems of that form can be solved using eigenvalues and eigenvectors, and usually (but not always) the general behavior of the solutions is a good indication of what actually happens for the original (nonlinear) system if you look near the right point(s). If this sounds interesting, come find out about concepts like trajectories, stationary points, nodes, saddle points, spiral points, and maybe Lyapunov functions. Expect plenty of pictures and probably an opportunity for some computer exploration using *Mathematica* or equivalent. (If you don't want to get involved with computers, that's OK too; most homework will be doable by hand.)

Prerequisites: Linear algebra (eigenvectors and eigenvalues), calculus, a little bit of multivariable calculus (equation of tangent plane).

Take it to the Limit. (Ben)

The normal idea of a limit has a lot of nice and wonderful properties. We'll start this class by stating some of these properties, as well as another not-so-wonderful property: it is not always defined.

In this course, we will give up some of the other wonderful properties of the limit in order to define limits of more sequences. This will include some normal methods—such as the Cesàro limits—and some less normal ones—such as ultrafilters.

Prerequisites: None.

The Centuries-Old English Tradition of Publicly Performing Hamiltonian Cycles in Cayley Graphs of Symmetric Groups (Change Ringing). (Eric + Tim!)

In 16th century England, people figured out that they could make church bells louder by hanging them on wheels. Eventually they got bored of just ringing their bells back and forth and wanted to play music. But too bad, they can't. The bells just physically can't do it.[9] What they could do, though, is play all the bells, and then play them all again in a slightly different order. Their goal was to play all the possible permutations of the bells without repetition (an *extent*). This exercise continues to this day; there are over 5000 bell towers in England that are equipped for "ringing the changes."

With only 6 bells, the number of ways to ring all permutations is 6!!![10] But in reality, people perform only a fraction of these extents, developed over time through the constraints they faced:

- Due to physical limitations, consecutive permutations must be similar to each other in a very specific sense.
- Sheet music and other written notes are not allowed; everything must be performed from memory.
- People frequently make mistakes, so there should be enough built-in structure for the conductor to be able to sort out confusion on the fly.

Over the centuries, people developed ways of building up extents from smaller, memorizable, somewhat-recursive building blocks. With enough painstaking effort, we could check change by change that these are valid extents (i. e., that we haven't repeated any orders of the bells). But could we find some underlying structures that would save us work in verifying that we have a valid extent? Can we use this structure to help in building new extents, or in actually ringing ones we come up with? Can we figure out the math that's been secretly underlying this tradition for centuries, and redevelop several hundred years of change ringing in a week using the power of group theory?[11] The answers to some of these questions are yes!

Prerequisites: None

The Class Number. (Gabrielle)

Kummer was able to prove Fermat's Last Theorem in the case where n is a regular prime. In this class, we're going to figure out what that means and why that is useful.

Prerequisites: Intro to Algebraic Number Theory.

The Combinatorics of Boarding an Airplane. (Bill)

Suppose 100 people board an airplane in a random order. When each person gets to their assigned seat, they pause for one unit of time to put their luggage in the overhead bin, possibly holding up everyone behind them. How long will it take, on average, before everyone gets to sit down?

This topic has led to actual academic papers in math and theoretical computer science. In this class, we'll be considering the simplest version of the question. Along the way, we'll see several beautiful ideas from probability and combinatorics, including a famous theorem of Erdős and Szekeres that proves the existence of long monotone subsequences within any permutation.

Prerequisites: None.

The Good and Bad Mathematics of Pennsylvania's Gerrymandering Lawsuit. (Mira)

Partisan gerrymandering has never been successfully challenged in US federal courts, and with the Supreme Court's most recent decision, pretty much all hope is gone.

On the other hand, in 2018, the Pennsylvania Supreme Court declared that Pennsylvanias gerrymandered map violates the Pennsylvania Constitution's guarantee of "free and fair elections". Since many other states have similar guarantees, the focus of anti-gerrymandering activism looks like it's going to switch to state courts.

That all sounds promising, except for one thing. What the PA Court called the "most compelling" argument in the case was based on completely bogus statistics. There was also a really good statistical argument presented to the court, but the judges didn't seem to understand it. (The good argument, made by mathematicians, was complicated; the bad one, made by political scientists, had nice pictures.)

In this class, we'll look at some of the original expert witness testimony presented before the court, the original research papers that both arguments are based on, the opposing side's attempted rebuttal, and the judges' reactions. You will get a sense of both how important and how difficult it can be to communicate mathematical concepts to non-mathematicians. You will also be in a better position to evaluate similar anti-gerrymandering arguments going forward; there are likely to be many in the next few years, and the popular press is likely to make a complete mess out of them.

Prerequisites: None. (In particular, you can take this whether or not you took Assaf's Week 1 class.)

The Hoffman–Singleton Theorem. (Pesto)

There are exactly three known graphs in which

(1) every vertex has the same degree,
(2) there are no cycles of length 3 or 4, and
(3) every vertex is at distance at most 2 from every other vertex.

It's not known whether a fourth such graph exists, but if it does, then its vertices have degree 57![12] We'll prove so by a beautiful application of linear algebra to graph theory.

Prerequisites: Linear algebra: Understand the statement (not necessarily a proof) that "an $n \times n$ symmetric matrix has n eigenvectors".

The Hopf–Poincaré Index Formula. (Assaf)

I bet you've heard of hairy balls, and how the spheri-ness of the sphere prevents one from combing it. That's really weird when you think about it in those terms. How could the global structure of the sphere be related to combability?

This class will discuss surfaces, Euler characteristics, and vector fields, and the truly surprising relation between them. Along the way, we'll discuss curves on surfaces, homotopy groups, winding numbers, and the classification of surfaces.

Prerequisites: Some calculus and topology in $\mathbb{R}^n$ would be helpful, but is not necessary.

THE Intuitive Proof of the Hairy Ball Theorem. (Assaf)

The usual proof of the Hairy Ball Theorem is some magic wand nonsense involving homology, degree maps, and wishy-washy hand waving. In this class, we'll get rid of everything except for the wishy-washy hand waving to prove the Hairy Ball Theorem using only an intuitive definition of continuity.

Prerequisites: A good grasp on the intuitive meaning behind continuity

The Mathematical ABC's. (Susan)

Ever noticed how no one ever uses "d" as a variable in a calculus class? Or how "i," "j," and "k" are perfectly fine natural numbers if you're doing combinatorics, but not if you're studying noncommutative rings? In this class we'll go over the alphabet from A to Z, and talk about how to use (or not to use) these letters we know and love. If we have time, maybe we'll learn Mathematical Greek!

Prerequisites: None.

The Mathematics of Fairness. (Mira)

This class will be a hodgepodge of independent topics related in some way to fairness. Days 1 and 2 will be at least partially active learning; Days 3 and 4 will be interactive lecture. Come for as many days as you like!

Days 1 and 2: Apportionment. The US Constitution mandates that "representatives ... shall be apportioned among the several states ... according to their respective numbers." This is usually taken to mean that the number of representatives in each state should be proportional to its population. But exact proportionality is not possible: for example, California cannot have 54.37 representatives. (Europeans face the same problem when they have to figure out how many representatives each party should have in Parliament.)

This is the problem of apportionment – and it is a lot trickier and more interesting than might appear at first glance. Over the course of US history, Congress went through five different apportionment methods, always accompanied by fierce political debates and sometimes leading to paradoxes that no one expected. The method that we currently use was proposed by a Harvard mathematician named Huntington in 1921 and adopted by Congress on the recommendation of the US Academy of Sciences! The math behind it is surprisingly non-trivial.

Day 3: An ancient fair division problem. Here is a passage from the Mishnah, the 2nd century codex of Jewish law:

> *A man has three wives; he dies owing one of them 100 [silver pieces], one of them 200, and one of them 300.*
>
> *If his total estate is 100, they split it equally.*
>
> *If the estate is 200, then the first wife gets 50 and the other two get 75 each.*
>
> *If the estate is 300, then the first wife gets 50, the second one 100, and the third one 150.*
>
> *Similarly, any joint investment with three unequal initial contributions should be divided up in the same way.*

For 1800 years, this passage had baffled scholars: what could possibly be the logic behind the Mishnah's totally different ways of distributing the estate in the three cases? Then, in 1985, a pair of mathematical economists produced a beautifully simple explanation based on ideas from game theory. They showed that for any number of creditors and for any estate size, there is a unique distribution that satisfies certain criteria, and it turns out to be exactly the distribution proposed in the Mishnah. The proof is very cool, based on an analogy with a simple physical system.

Day 4: TBD. I have a bunch of ideas, and will also take student preferences into account.

Prerequisites: None.

The Monotone Sequence Game. (Misha)

If all campers this year are standing in a row, in any order, we can pick out 12 of them lined up in increasing, or decreasing, order of height.

This is only guaranteed this year: last year, we only had 120 campers, and you need at least 121 for the trick to work.

All of the above is just motivation for the questions I'm really interested in, which are the following:

(1) How many campers do you have to compare to find such a sequence of 12?
(2) Why is the process in question 1 called an "online edge-coloring game"?
(3) What happens if you have to choose all the campers you compare in advance?

Prerequisites: None.

The Polish Attack on the Enigma. (Ben)

In the early 1930s, the German Enigma machine seemed to offer a great deal of security, due to the sheer number of possible setups for the machine. However, Polish intelligence realized that the code could be broken, using group theory.

We'll see how the Polish attack on the early Enigma worked, which is both a historically interesting topic and a testament to the power of group theory.

Prerequisites: Group theory.

The Probabilistic Method. (Bill)

The probabilistic method is the art of proving combinatorial results using elegant probabilistic arguments. Often, the result being proven will seem to have *nothing to do with probability.* For example, this theorem from combinatorial number theory:

Theorem: Call a set $A \subseteq \mathbb{N}$ *sum-free* if for all distinct $x, y \in S$, the quantity $x + y \notin S$. Then every set $B = \{b_1, \ldots, b_n\}$ of n nonzero integers contains a sum-free subset A of size $|A| > \frac{1}{3}n$.

In this class we'll see some of the most beautiful (and mind-blowing!) applications of the probabilistic method. Along the way we'll prove famous results from Ramsey theory, combinatorial number theory, graph theory, and even real analysis. Often the proofs that we will encounter will be an order of magnitude shorter than any other known proofs (and in some cases, it will turn out that there *are no other known proofs*!).

Prerequisites: Familiarity with probability. Familiarity with linearity of expectation would be helpful but will also be reviewed during the class.

The Riemann Zeta Function. (Mark)

Many highly qualified people believe that the most important open question in pure mathematics is the Riemann hypothesis, a conjecture about the zeros of the Riemann zeta function. Having been stated in 1859, the conjecture has outlived not only Riemann and his contemporaries, but a few generations of mathematicians beyond, and not for lack of effort! So whats the zeta function, and whats the conjecture? By the end of this class you should have a pretty good idea. Youll also have seen a variety of related cool things, such as the probability that a "random" positive integer is not divisible by a perfect square (beyond 1) and the reason that $-691/2730$ is a useful and interesting number.

Prerequisites: Single-variable calculus, including infinite series. Previous experience with functions of a complex variable may help a bit, but is *not* required.

The Sound of Proof. (Eric)

Can you hear what a proof sounds like? I'll present five proofs from Euclid's Elements, and then play (recordings of) five pieces of music written to capture each proof in sound. You'll get to try and work out which piece of music lines up with which proof, and then we'll dissect how a couple of the compositions "sonify" the proofs. All of the material I'm drawing on is from an art piece entitled "The Sound of Proof" by mathematician Marcus du Sautoy and composer Jamie Perera at the Royal Northern College of Music in Manchester.

Prerequisites: None.

The Weierstrass $\wp$ Function. (Assaf)

In kindergarten, we learned about the trigonometric functions and their roles in parametrizing the circle. By grade 1, after mastering calculus, we also learned about the definitions of the inverses of these functions in terms of integrals, and showed that they are derivatives of each other.

In this class, we will go up a dimension and discuss the complex analogue of a trigonometric function, called the Weierstrass $\wp$ function. This doubly-periodic meromorphic function behaves like a trigonometric function in parametrizing complex tori in $\mathbb{C}P^2$.

Along the way, we'll pass through series of meromorphic functions, the complex projective space, and hopefully prove Abel's theorem, and start looking at Riemann surfaces in $\mathbb{C}P^2$.

Prerequisites: Complex analysis

Thinking of Images as Mathematical Objects. (*Olivia Walch*)

This class will explore different ways of talking about images and drawings mathematically. Topics covered will include: how photo editors work, how to draw a nice-looking line, vector graphics, style transfer, and generative art.

Prerequisites: None.

Topology. (Kayla)

Want to understand why mathematicians joke that coffee mugs and donuts are actually the same? Come learn about this in topology! In the class, we will be introducing many basic concepts in point-set topology. We will be discussing when we think of two topological spaces as the same, maps between topological spaces, basic properties of topological spaces and how they play with maps between the spaces, the product topology and the quotient topology (how to formalize the idea of "gluing" things together)!

Prerequisites: None.

Traffic. (Assaf)

As the saying goes, "you're not in traffic, you are traffic." Traffic is a game that almost[13] everyone plays. Everyone wants to be rational, but sometimes this rationality comes back and bites the collective. In this class, we'll explore scenarios where this effect happens. We'll look at Braess' Paradox, the Bus Motivation Problem, and spend some time discussing the formalism of congestion games.

Prerequisites: None.

Two Games and a Code. (Mira)

Question: What do the following two games have in common?

(1) You think of a number between 0 and 15. I ask you seven yes/no questions about it. You are allowed to tell at most one lie (or, if you prefer, to answer truthfully throughout). At the end, I'll tell you if and when you lied, and then I'll guess your number! (We'll actually play this in class.)

(2) You and six friends are playing a cooperative game. You are each given a black or white hat at random. As usual, each person can see the color of everyone else's hats and has to guess the color of his or her own. Each of you writes your guess — either "BLACK" or "WHITE" — on a piece of paper, without showing it to anyone else. If a player doesn't want to guess, they also have the option of writing "PASS". Then everyone holds up their papers simultaneously. Your team wins if, among you, you have *at least one correct guess and no incorrect guesses.* ("Passes" don't make you lose, but they don't help you win either.) You are allowed to agree on a strategy before the hats are passed out, but no communication is allowed afterwards. What strategy will maximize your probability of winning, and what is the best probability you can achieve? (We'll play this in class if we have time.)

Answer: The same strategy, based on a perfect code!

Come and find out what that means.

Prerequisites: None.

Tychonoff's Theorem. (Ben)

The topological idea of "compactness" is, roughly speaking, a finiteness condition. However, while the infinite product of finite sets is not finite, it turns out that the infinite product of compact spaces is still compact. This surprising result, called "Tychonoff's Theorem," has applications not only in topology but also in analysis, logic, and graph theory.

This course will prove Tychonoff's Theorem and then show some of its applications to various areas of mathematics.

Prerequisites: Point-set topology (know what it means for a topological space to be compact).

Unique and Nonunique Factorization. (Gabrielle)

In this class, we are going to be studying the nice properties of unique factorization domains, ways to determine if a ring has unique factorization or not, and cool results that we can use unique factorization to prove. Then, because we are not afraid of challenges, we are going to confront some examples of domains where there is nonunique factorization and learn about the tools (elasticity and catenary degree) we can use to study them.

Prerequisites: Some comfort with rings, integral domains, ideals (i.e. know and be comfortable with the definitions)

Units in Algebraic Number Theory. (Kevin)

J-Lo began his Intro to Number Theory class by presenting an instance of Pell's equation: $x^2 - 2y^2 = 1$. By factoring this using $\sqrt{2}$ and then raising the resulting factors to powers, we can generate an infinite family of solutions over the integers.

Pell's equation translates directly into the fact that $x + y\sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$. The units in these sorts of "integers" have a rich algebraic structure described fully by a famous result called Dirichlet's Unit Theorem. In this class, we'll see the surprising way that beautiful geometric ideas contribute to proving something that looks so purely algebraic.

Prerequisites: It is helpful to have some familiarity with the basics of many topics (algebraic number theory, complex numbers, group theory, linear algebra, and ring theory), but none are strictly required. There will be sections on the homework to catch you up on everything you need before it's relevant in class.

Wedderburn's Theorem. (Mark)

You may well have seen the quaternions, which form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$ with basis $1, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j.$$

Have you seen any examples of *finite* division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Prerequisites: Some group theory; knowing what the words "ring" and "field" mean. Familiarity with complex roots of unity would help.

Why the Millennium Problems? (J-Lo)

With such a wide variety of unsolved problems out there, why did seven in particular stand out enough to counted as Millennium Prize Problems? In this class I will give a brief overview of the history behind each problem, and even though I won't be able to state all the problems precisely, I hope at least to convey a sense of the mathematical significance of each, and why people care about them.

Prerequisites: None.

Why We Like Complex Projective Space. (Will)

Parallel lines in Euclidean space don't intersect. This might seem self-evident, but it's kind of annoying if we're trying to figure out how two geometric figures intersect. After all, most of the time two lines will intersect in a unique point, and it's a hassle to check an extra condition just in case they don't. Similar inconsistencies show up when we intersect algebraic curves and surfaces: for example, if we intersect two conic sections, we might get anywhere from 0 to 4 points.

It would be nice to work in a world where these intersection problems have consistent answers. In fact, we can accomplish this by extending to the complex numbers and adding points to our space "at infinity" in a systematic way. The resulting setting, called complex projective space, is trickier to visualize, but it ends up being the most natural place to solve many geometric problems.

In this class, we'll define complex projective space, get a sense for its shape and importance, and look at some classic results on intersections of lines and curves that clean up nicely in this new space. If you're interested in taking a first look at algebraic geometry, seeing a classic example of mathematicians adjusting definitions to make nicer theorems, or dividing by zero with staff supervision, you might enjoy this class.

Prerequisites: Familiarity with complex numbers. A bit of linear algebra (manipulating matrices and using them to describe transformations of space) may come in handy, but it won't be necessary.

You Can't Solve the Quintic. (Eric)

We'll prove that there's no general formula in radicals that works to produce the roots of a degree 5 polynomial. This particular proof due to Arnold is a beautiful combination of combinatorial and topological arguments. The only things you'll need to know going in are what complex numbers, the quadratic formula, and permutations are; but we'll make tons of connections with classes on group theory, fundamental groups, Riemann surfaces and more from the first 4 weeks of camp.

Prerequisites: Be comfortable with the complex plane, know the quadratic formula, know cycle notation for permutations.

Young Tableaux and Combinatorics. (Shiyue)

Young tableaux are rich combinatorial objects that inspire many combinatorial techniques and theory. Counting standard Young Tableaux (SYT) is already combinatorially interesting. Moreover, given a Young diagram, this count is also the dimension of the irreducible representation of symmetric group associated with the Young diagram. To study this, we will derive the beautiful bijection between SYT and shapes/permutations, namely the celebrated RobinsonSchenstedKnuth correspondence, Hillman-Grassel combinatorial algorithm, and SYT's connections to Schur functions (another ubiquitous gadget in algebraic combinatorics and study of symmetric groups). We will eventually derive the hook length formula that counts the number of standard Young Tableaux, using all these crucial ingredients that we have picked up on the way.

Prerequisites: None.

Young Tableaux and Enumerative Geometry. (Shiyue)

Enumerative geometry is a subfield of algebraic geometry where mathematicians are interested in asking "How many A are there such that B?" where A is usually algebraic varieties and B is some geometric conditions. In the 19th century, Hermann Schubert raised a series of such questions like "how many lines in $\mathbb{P}^3$ can pass through 4 generic lines?" If you know the answer to this question (the answer is two), how about proving it? Hilbert asked in his 15th problem to systemize our knowledge of such questions, and hence the subfield of algebraic geometry and combinatorics – Schubert Calculus. To get there, we will start from projective geometry, basic facts about intersections of projective varieties, basic intersection-theoretic objects such as Grassmannians, and Schubert varieties. The meat of this course will be seeing how Littlewood-Richardson rule combined with counting semistandard Young tableaux will give us a proof of the question above, and a few other examples.

Prerequisites: Linear algebra.

Young Tableaux and Probability. (Shiyue and Andrew)

Guess who is super excited about Young tableaux in MC2019? In this course, we will give a different proof for the Hook Length formula, using probability. Maybe in Week 2, you have seen that the Hook Length Formula gives us a way of counting the number of standard Young tableaux, which turns out to be the dimension of irreducible representations of the symmetric group S_n. In Week 3, you might have seen a complicated proof of the Hook Length Formula which required building up several identities involving generating functions. In this course, we will use probability to think about hook walks and the recursive relations between standard Young tableaux and their sub-standard Young tableaux (which is a very natural idea when one tries to build up a standard Young tableaux). Eventually, we will give an elegant derivation of the Hook Length Formula.

Prerequisites: Induction, some knowledge about probability will be helpful but not necessary.

Young Tableaux and Representation Theory. (Shiyue)

If you run into any representation theorist and ask them what rep theory is, they will probably tell you that it is a way of studying your algebraic structures as maps into vector spaces (think: understanding group theory (hard) using linear algebra (easy)). This can still be quite scary. But believe it or not, representation theory of symmetric groups can be highly combinatorial and not scary at all!

In this course, we will see a fascinating and ubiquitous interplay between algebra and combinatorics. We will play around with a fun combinatorial object called Young tableaux (think: stacking boxes and throwing balls in the boxes), learn representation theory, and see how Young tableaux give us simple and elegant ways of understanding symmetric groups and their representations. More particularly, we will construct irreducible representations of symmetric groups using Young tableaux, and prove that the dimensions of these irreducible representations can be understood combinatorially.

Prerequisites: Linear algebra, group theory

Zeta Functions. (*Sachi Hashimoto*)

Many problems in mathematics, like Fermat's last theorem, which asks which integer nth powers are the sums of two nth powers, are deep statements about zeta functions. The most famous zeta function is the Riemann Zeta Function, whose values at special integers can be computed; Euler showed that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Special values of zeta appear in another millennium problem, the BSD conjecture, which relates the zeta value to the number of points on that curve. These statements about zeta functions are truly mysterious. We will meet a few zeta functions and contemplate their mystery. In concrete terms, zeta functions count the number of points of a geometric object. Throughout the class, we will make use of the open source programming language Sage, a Python-based language, but no programming experience is necessary.

Prerequisites: Group theory; quadratic reciprocity.

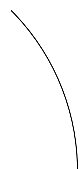
Colloquia

Degrees of Unsolvability. (*Steve Schweber*)

There are some mathematical problems – even some *interesting* ones – which, in a precise sense, cannot be solved by a computer (even in principle). For example, there is no way to write a computer program which will determine whether a given computer program will ever stuck in an endless loop. Perhaps surprisingly, in a precise sense some unsolvable problems are *more unsolvable* than others! There is a rich structure of "degrees of unsolvability" – also called "Turing degrees" – and after saying a bit about why unsolvable problems exist in the first place, I'll discuss some of its features. In particular, we'll see why there are "incomparably unsolvable" problems (that is, being able to solve one wouldn't help you solve the other).

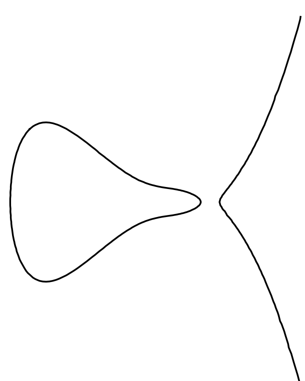
Dictionary Shapes. (J-Lo)

Have you ever wondered what it would be like to see every right triangle at once? Well here they are!



And here's every possible grid in the plane!

Is it possible to find a pair of triangles – one right, one isosceles – with integer side lengths, which have the same area and the same perimeter? Well, any such pair can be found somewhere on here:



Exceptional Mathematics: from Egyptian Fractions to Heterotic Strings. (*Theo Johnson-Freyd*)

Most of the mathematical universe is regular and repeating, but every once in a while there is an exception, and it leads to all sort of interesting and irregular phenomena. I will explain how the exceptional solutions to a simple problem from antiquity — find all integer solutions to $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1$ — lead to 20th and 21st century highlights: topological phases of matter, heterotic string theory, and E_8, the most exceptional Lie group.

Good Practice: Teaching, Learning & Beauty in Math and Beyond. (*Luke Joyner*)

"In mathematics, the art of asking questions is more valuable than solving problems." – Georg Cantor

Mathematicians often speak of "elegance" as a desired quality in a proof, and more generally, of "pure math" as something to hold dear... but as soon as math gets applied – which can mean a lot of things – notions of beauty and truth get a lot more complicated, to the extent they matter at all. Meanwhile, here at Mathcamp, a premium is put on math that is fun and satisfying, which often means either creative problem-solving, a series of proofs that lead to important results, or math wearing unexpected clothes. In this colloquium, we'll consider a range of attitudes toward teaching and learning, both in math, and in other fields that pursue beauty. In particular, I'll share examples from my own work in mapmaking, architecture & urban design, fields that must negotiate the messy social tangle of people and the world, without the luxuries of pure abstraction and validating proof.

Knowledge Puzzles. (*Don Laackman*)

I know what this colloquium is about.

You don't know what this colloquium is about.

I know you don't know what this colloquium is about.

You know I know what this colloquium is about.

You know I know you don't know what this colloquium is about.

Now, you know what this colloquium is about.

Knowledge puzzles are a type of logical problem in which everyone involved has imperfect information about their universe; and yet, by only sharing information about how much they know, they can often discover everything. The theory of knowledge puzzles has only barely been explored, but I will talk about some of the coolest problems and theorems the field has to offer!

Should We Vote on How We Vote? (*Mike Orrison*)

Voting is something we do in a variety of settings, but how we vote is seldom questioned. In this talk, we'll explore a few different voting procedures from a mathematical perspective as we try to make sense of the paradoxical results that can occur when we vote in more than one way.

Stably Surrounding Spheres with Smallest String Size. (*Zach Abel*)

Once upon some tricky casework, while I pondered how much lacework,

I would need to tie some netting tightly 'round my unit orb,

While the sphere I thus tried trapping, suddenly a bound came, capping

How much string is needed wrapping, wrapping round my precious orb.
Merely 3π is required to accomplish such a chore!
Only this? Well, just eps more.
Ah, distinctly I remember, how I then wished to dismember,
Each and every strand that emb(e)raced that rare and radiant orb.
Seeking to dislodge my prize, I quickly saw, to my surprise, by
Choosing k strands to incise, my sphere from net could not be torn!
How much–how much length is needed? Tell me–tell me, I implore!
$(2k + 3)\pi$, and more.

Superpermutations, Traveling Salesmen, and The Melancholy of Haruhi Suzumiya. (*Ari Nieh*)

Suppose that, not content with linear time, you want to watch your favorite TV series with the episodes in every possible order. That is, you want every possible arrangement of all n episodes to appear as a contiguous block. How many total episodes would you have to watch? Obviously it will require some repetition, but how much? For example, if your favorite show has two episodes, you can watch them in the order 1-2-1, which will give you all 2! possible orderings.

Until 2014, it was conjectured that this problem's answer was given by a very straightforward formula. In this talk, we'll see why that formula is wrong, and what this has to do with the traveling salesman problem, 4chan, and an anime called *The Melancholy of Haruhi Suzumiya.*

The Most Depressing Theorem I Know. (Mira)

Say there is a human trait that (e.g. probability of being in a car accident) on which two groups (e.g. men and women) differ.

Say also that there is a cost or benefit associated with this trait. In our example, if your probability of being in an accident is high, then you insurance goes up – that's the cost.

The theorem says that any algorithm for predicting such a trait will either be either biased or unfair. I'll tell you precisely what I mean by "biased" and "unfair" in the talk, but the upshot is: there *are* no fair algorithms for prediction. And it's not because the people who write these algorithms are racist or sexist or careless (though that may be true too), but because, in many cases, perfect fairness is just mathematically impossible.

Tournaments Having the Most Cycles. (*John Mackey*)

A round robin tournament is a set of matches in which each contestant plays all other contestants. Just how confusing can the outcome of a round robin tournament be? Directed cycles are an impediment in ranking the participants in a tournament, so let's try to figure out which tournament outcomes have the most directed cycles.

This problem is about three-fourths solved. Open problems and connections with the zeta function will be discussed.

When Bare Hands Fail. (*Po-Shen Loh*)

Many people associate Combinatorics with elementary, ad-hoc arguments that are understandable by the general public. Indeed, that is one of the aspects which makes Combinatorics fascinating: that a solution composed of a chain of (albeit fantastically clever) bare-hands steps can be so difficult to find, yet is easy to follow once it is found. However, this is not universally true, and indeed, many modern combinatorial problems are solved by importing theorems and techniques from other branches of mathematics, such as algebra, probability, or even topology. In this talk, we'll demonstrate this using a natural graph theoretical statement as motivation.

Why Should We Care About Category Theory? (*Angélica Osorno*)

One of the first mathematical concepts we learn as children is counting, and when we do so, we think of counting the number of elements in a specific set. Soon after, we forget about sets and we just consider the abstract numbers themselves. This abstraction simplifies many things, but it also makes us forget about some structure that we had when we were thinking about sets. That structure can be encoded by a category. In this talk, we will describe certain concepts in category theory, and you will realize that in most of your mathematics classes, you have been working with categories – you just didn't know about it. There will be plenty of examples that will show that category theory provides a unifying language for mathematics, and that many constructions are more naturally understood when they are seen through the categorical lens.

Your Clock on Your Phone: Tracking Your Body's Internal Time with Math. (*Olivia Walch*)

The body's circadian clock regulates timing for almost all major physiological functions, with the clock itself primarily regulated by light exposure. Measuring circadian time experimentally can be time-consuming and expensive, but mathematical modeling can help overcome this challenge. By providing a model of the circadian clock with a person's recent light exposure, the model can generate a prediction of that person's internal time. In this talk, I'll discuss one such model of the circadian clock, as well as our experiences putting that model into mobile apps used around the world. I'll also talk about connections between this work and sleep, jet lag, and optimal control.