CLASS DESCRIPTIONS—MATHCAMP 2018

Classes

Algebraic Number Theory. (Shiyue)

Every integer can be written uniquely as a product of primes. We all take this for granted, but there are lots of other number systems where unique factorization does not hold. For several decades in the 19th century, mathematicians wrongly thought they had proved Fermat's Last Theorem until they realized this problem. Then Richard Dedekind invented the concept of "ideal". Ideals are not exactly numbers, but in the "ideal" world, unique factorization always works. Unfortunately, this didn't help prove Fermat's Last Theorem, but it generated a huge amount of interesting number theory in its own right.

In this course, we will follow this journey, first finding number systems that don't have unique factorization, then introducing some ring theory and learning to work with ideals, and finally proving Dedekind's great theorem on unique factorization. This is only the first step in the fascinating field of algebraic number theory.

Prerequisites: None; taking group theory at the same time would be useful, but not required.

A Mathematician's Perspective on the World. (*Po-Shen Loh*)

"When will we ever use this?" Over the past few years, the speaker has been working to bridge the gap between mathematics and real life, to boost public interest in mathematics. In this talk, he will share some of his mathematical insights on the real world, accumulated from over a year of mathematical introspection on topics of common interest. Some have appeared in features of the New York Times, Wall Street Journal, and FiveThirtyEight (part of ESPN). He will also share his experience in connecting with the general public.

Prerequisites: None.

An Introduction to q-Analogues. (*Maria Gillespie*)

A q-analog of an expression P is a polynomial or rational expression in the variable q for which setting $q = 1$ gives P. The theory of q-analogs comes up in combinatorics, algebra, geometry over finite fields (including the popular game SET), and quantum mechanics, and we'll discuss all of these aspects but focus mostly on the applications to combinatorics.

Classes will be lecture-based but with class participation; I may pause occasionally and let the students try an example before resuming lecture. It will be a one-hour class for each of three days.

Prerequisites: Comfort with multiplying polynomials together is essential. Students should also be familiar with basic combinatorial formulas involving binomial coefficients, factorials, and Pascal's triangle. A bit of familiarity with generating functions may be helpful but is not required.

ARML Power Rounds. (Misha)

For those not in the know: ARML is a team-based high school math competition in the US. In one part of the competition, called the Power Round, the entire team of 15 people works together on a bunch of questions on the same topic.

In this class, we'll work on some past ARML Power Rounds together, and talk about the math behind the topics.

Prerequisites: None.

A Slice of PIE. (Brian, *camper teaching project*)

If you've ever wondered about the sizes of things that aren't infinite, you might have run into the Principle of Inclusion-Exclusion:

$$\left|\bigcup_{A\in\mathcal{F}} A\right| = \sum_{S\subseteq\mathcal{F}} (-1)^{|S|-1} \left|\bigcap_{A\in S} A\right|$$

This is a formula which tells you the size of a union of a bunch of sets based on the sizes of their intersections. But... we use a lot of information about the sets, and end up with one very small piece of information. Shouldn't we be able to figure out more stuff if we know the sizes of ALL possible intersections?

In this class, we'll be taking a look at how we can modify inclusion-exclusion to be more general. First, we'll see how we can find the size of something called the "symmetric difference" of all of the sets. We'll also see how this is like cutting the union in half, and find out what happens when we cut into more than two pieces. Along the way, we'll see some generating functions and complex numbers, and get some insight into why Inclusion-Exclusion is so useful.

Prerequisites: You should be familiar enough with Inclusion-Exclusion to know how it applies to this problem: How many 3-digit numbers have an even first digit or a last digit divisible by 3 (or both)?

Axiomatic Geometry. (Misha)

In 1899, Hilbert came up with a list of axioms for Euclidean geometry.

We'll talk about why we need them, and about the horrible, no-good, actually-kind-of-fun things that happen when we leave out some of them.

Prerequisites: None.

Axiomatic Music Theory. (J-Lo)

What makes some combinations of notes more pleasant to listen to than others? Why does the chromatic scale have 12 notes? And what's up with those same four chords being used in every pop song?

At first glance, music might seem like a collection of completely arbitrary facts that just coincidentally combine in consonant ways. But on the contrary, many of these complicated musical constructions can be derived as corollaries from just a couple of basic conditions that we think music ought to satisfy. The rich, complex intricacy? That's actually number theory under cover.

Prerequisites: None (knowing some group theory may help, but this can also be picked up as you go along).

Axiomatic Music Theory part 2: Rhythm. (J-Lo)

Want more music theory? We talked a lot about pitch and intervals and triads and chord progressions in week 3, but very little about another crucial component of any musical piece: *rhythm.* Join a small group and explore how the Euclidean algorithm can be used to derive many traditional rhythmic patterns, find unexpected ways to transform rhythms in ways that preserve the set of time differences between pairs of beats (using a result known as the "hexachordal theorem"), and discover implications of the fact that pitch and rhythm are really the same thing.

Prerequisites: None (in particular, Axiomatic Music Theory is not required).

Big Numbers. (Linus & Pesto)

You have 60 seconds to write down the biggest number you can. What do you do? This class is about different ways to get very large numbers:

- From addition to multiplication to exponentiation to power towers and beyond: the method of "Do "Do "Do it again" again" again"
- Structure in randomness: Ramsey Theory & numbers like the famous Graham's Number
- Why can't I write "The largest number definable in less than a thousand letters, plus one?"
- Busy Beaver: one way you *can* write basically that.

Prerequisites: None.

Calculus on Graphs. (J-Lo)

What would happen if you took calculus (the study of how functions change over time) but took out all the continuous stuff? What if instead of the real numbers, you were to input some discrete quantity, like the integers, or the vertices of a graph? Though the computations often become simpler than in regular calculus, we end up in a world with many analogous properties and similar-looking theorems!

In this class, small groups will work together to solve problems that explore this world of functions defined on graphs, and eventually discover what these functions can tell us about the shape of the graph itself.

Prerequisites: None (in particular, calculus is not required).

Calculus without Calculus. (Tim!)

If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Jessica is 5 cubits tall and Larsen is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Jessica's head to the top of Larsen's head that touches the ground in the middle. What is the shortest length of string you can use?
- Ania rides a bike around an ellipical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 20 meters way along the shoreline throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- Where in a movie theater should you sit so that the screen takes up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Prerequisites: Some calculus will be useful for context, but we won't actually use calculus (that's the point).

Chaotic? Good! (Ben)

Say we take a number, double it a few hundred times, and want to predict the first digit after the decimal place when we've done this. How hard can that be? It can get quite hard, it turns out, because we'd be trying to predict the behavior of a chaotic dynamical system!

Starting from the goal of predicting how certain dynamical systems evolve, we'll see how "chaos" in the colloquial sense sneaks into the picture. Then, we'll discuss a more precise definition of chaos and how to see that a system is chaotic. As we go, we'll run into a couple kinds of bifurcations and meet some Cantor sets. We'll also introduce the method of symbolic dynamics, which will greatly simplify our analysis of some dynamical systems.

Prerequisites: Know what fixed points and periodic points are, and how to classify fixed points as attracting or repelling. Talk to me if you're interested in taking this and haven't seen these!

Cohomology via Sheaves. (Apurva)

Why could all the king's horses and all the king's men,
Not put Humpty Dumpty back together again?

Because Humpty Dumpty lacked sheaf datum. A sheaf is a mathematical tool that allows us to glue local mathematical data together. In this class, we'll learn how to use the *locally constant sheaf* to compute topological invariants (cohomology) of spaces, which in turn enable us to use algebraic techniques to study topology.

This will be an IBL class. This is NOT a class on sheaves, this is a class on cohomology of spaces.

Prerequisites: You should be able (and willing) to compute the rank and nullity of linear transformations. You should be familiar with the notions of "connected components" and "continuous functions".

Combinatorial Designs. (Ania)

You probably know Set and Dobble games. Hopefully you think they are cool and "mathy". Maybe you are even curious about that math behind them? If so, this class is for you! It turns out that even though at the first glance Set and Dobble seem to be completely different they do have a common denominator. Both of them are examples of a combinatorial design, which is a structure of finite sets that satisfies generalized concepts of balance and/or symmetry.

During the class we will explore the elementary theory describing those objects, prove basic relationships between their parameters (including Fisher's inequality and maybe theorem about symmetric designs) and also analyze some specific designs (Hadamard design, Steiner Triple System, Kirkman system). There will be many pretty pictures and fun problems (including solving a bit modified sudoku and drawing fancy graphs!) and we will also look for the other examples of designs in everyday life/math.

Prerequisites: Basics of finite combinatorics (inclusion-exclusion, binomial coefficients, etc.).

Combinatorial Game Theory. (Laura)

We'll look at some examples of two player impartial games and the winning strategies, and learn about how to figure out the winning positions for a game and assign nimbers to positions to figure out how to win!

Prerequisites: None.

Combinatorial Poetry. (*Matt Stamps*)

Have you ever noticed how various creative writing techniques in literature are used to express different modes of thinking and feeling? Poetry, for instance, makes use of linguistic attributes, such as rhyme or rhythm, to elevate the aesthetic of a piece, but also to convey additional meaning that leads to a deeper appreciation for its subject. Is there an analogous notion in mathematics? Certainly there are proofs that are clearer and more concise than others — mathematicians like to call these elegant — but are there elegant proofs that elevate the reader's understanding and insight so substantially they fall into the category of poetry? In this class, we will explore some mathematical "poems" that not only establish the truth of their respective propositions, but transform the way we think about a given subject. Class meetings will consist of team problem solving and practice writing mathematical poetry about familiar combinatorial objects, including the Fibonacci numbers, Pascal's Triangle, and (if time permits) the Catalan numbers. Particular emphasis will be placed on finding different ways to enumerate and construct one-to-one correspondences between finite sets so as to reveal their underlying combinatorial structure.

Prerequisites: None (though it would be helpful if you have seen some basic discrete math and/or enumeration before).

Combinatorial Topology. (Jeff)

So, you want to be a topologist because you love drawing pictures. But, you've never taken point-set topology[1] . How much can you prove about topology?
Turns out, quite a bit. In this class, we'll be developing simplicial complexes, which give combinatorial representations of topological spaces. Then, we'll look at discrete Morse theory, a combinatorial representation of a construction from differential topology. Along the way, we'll draw lots of pictures and diagrams, and get a feel for what topology should do, without messing around with all of those icky open sets.

Prerequisites: None!

Commutative Algebra and Algebraic Geometry. (Mark)

In its classical form, algebraic geometry is the study of sets in n-dimensional space that can be described by polynomial equations (in n variables). This is both a very old and a quite active branch of mathematics, and for over a century now it has relied heavily on commutative algebra — that is, on the properties of commutative rings and related objects. We'll start by looking at some of those, including prime and maximal ideals and a review of quotient rings, and we'll see how the algebra can be used to give us information about the geometric sets. For instance, we'll show that if a set can be given by polynomial equations, then a finite number of such equations will do. We may also see how to translate the idea of dimension into the language of algebra. There may well be cameo appearances by the axiom of choice (in the guise of Zorn's lemma) and a bit of point-set topology (on a space whose points are ideals!), but you don't need to know any of those things going in. In the second week, I hope to prove Hilbert's famous Nullstellensatz ("Theorem of the Zeros"), arguably the starting point for modern algebraic geometry, at least for the case of two variables.

Prerequisites: Familiarity with polynomial rings, ideals, and quotient rings.

Conflict-Free Graph Coloring. (Pesto)

You put some cell towers on a graph, each broadcasting at some frequency. Every vertex in the graph is a cell phone that needs to be able to listen at at least one frequency, but can't listen at a frequency if two towers adjacent to it are both trying to broadcast at that frequency. How many cell towers and how many distinct frequencies do you need?

This problem defines a version of graph coloring called "conflict-free" graph coloring. For this new version of graph coloring, we'll prove an analogue of the most important unsolved problem in graph theory, generalize the four-color map theorem, and prove that we (probably) can't solve it efficiently in general.

Prerequisites: None.

Continued Fraction Expansions and e. (Susan)

The continued fraction expansion of e is

[1] Disclaimer: I have never taken point set topology

$$1+\cfrac{1}{0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{\ddots}}}}}}}}}}}}}}$$

Okay, but seriously, though, why?!?! Turns out we can find a simple, beautiful answer if we're willing to do a little integration. Or maybe a bit more than a little? No previous experience with continued fractions necessary. Come ready to get your hands dirty—it's gonna be a good time!

Prerequisites: None.

Convergence Issues; or: Monsters in Real Analysis. (Ben)

During the 1800s, infinite series became increasingly important in analysis, especially series of functions, such as Fourier series. These were useful for solving differential equations, but some mathematicians working with them began to notice problems. Abel discovered a Fourier series which was discontinuous at some points, despite being a sum of continuous trigonometric functions. Dirichlet found that he could change the value of one series by rearranging its terms, a result that Riemann vastly generalized. Finally, in 1872 Weierstrass presented a function that was continuous everywhere, but differentiable *nowhere*. These struck one prominent mathematician, Poincaré, as being about as far from "honest functions" as you could possibly get: he called them "monsters."

Well, this course is about the monsters. We'll start with Riemann's theorem on rearranging series (the one referenced above) and then move on to working with sequences and series of functions. In this work, we'll see a few common features of our monsters, and learn about the helpful concept of uniform convergence. Then, once we learn a few of the standard results about uniform convergence, we'll be ready to talk about the biggest monster I mentioned above: Weierstrass's Nowhere Differentiable Function.

Prerequisites: This course focuses on ideas from calculus (limits, series, derivatives, integrals), so you should know what all of these are.

Crash Course. (Lara)

Mathematicians prove things! So in this class, you'll write lots of proofs. We'll spend a bit of time going over notation you'll need in all your other Mathcamp classes. However for the majority of this class, we'll just answer fun problems and learn how to make our answers water tight, using powerful techniques such as induction and contradiction.

Here are a few problems we're likely to think about together: (1) Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable. (2) Using the Fermat Numbers $F_n = 2^{2^n} + 1$, prove that there are infinitely many primes. (3) Let a be a nonzero integer. Show that there is no integer n which satisfies the equation $(n-a)^3 + n^3 = (n+a)^3$.

Prerequisites: None.

Crash Course in Complex Dynamics. (*Scott Kaschner*)

This course begins with repeatedly composing functions and, after switching to complex numbers, works its way through fractal geometry and chaos. The internet is awash in pictures of "fractals" that purportedly have something to do with mathematics. We will spend time investigating some of the algorithms used to generate certain classes of these images. Is it mathematics, artwork, or both? While we can appreciate any of those options, it would be good to know for certain when the pictures are mathematically accurate. Also, the Mandelbrot set will be involved somehow; this is a complex dynamics course after all.

Prerequisites: Calculus (including series).

Curse of Dimensionality. (Dyusha & Michelle, *camper teaching project*)

As those of you who went to Po's lecture already know, oranges can be deceiving.

Since our very birth, all of us, and all of our progeny unto eternity, have lived and will forever live in a three dimensional world. For humans (even Po), an orange is just a healthy snack, but for n-dimensional beings (where n is large), life is not nearly as simple. Even essentials such as peeling an orange to reveal juicy flesh are no longer viable. For instance, training a neural network to discern good poetry from bad is simple in three or four dimensions, but optimizing thousands, if not millions, of factors, in a high-dimensional space is quite the challenge. Similarly, clever strategies to find the k nearest neighbors from a given point in 2 dimensions quickly fall apart when dimensions grow larger.

This funky behavior in large dimensions is known as the *Curse of Dimensionality.* In this class, we'll dive into several examples of things that work just fine in low dimensions, discuss how the curse arises, and propose some strategies to combat it.

Prerequisites: If you've heard about neural networks and vector spaces, that would be helpful but not at all required.

Cycles in permutations. (Ania, Jessica)

Let's take a random permutation. What is the average number of fixed points in it? What is the average number of cycles? What is the probability that is has exactly one cycle? What is the probability that elements 1 and 2 are in the same cycle? Come to the class to find out and solve riddles about prisoners and planes!

Prerequisites: None.

Cyclotomic Polynomials and Migotti's Theorem. (Mark)

The cyclotomic polynomials form an interesting family of polynomials with integer coefficients, whose roots are complex roots of unity. Looking at the first few of these polynomials leads to a natural conjecture about their coefficients. However, after the first hundred or so cases keep confirming the conjectured pattern, eventually it breaks down. In this class we'll prove a theorem due to Migotti, which sheds some light on what is going on, and in particular on why the conjecture finally fails just when it does.

Prerequisites: Some experience with complex numbers, preferably including complex roots of unity; some experience with polynomials.

de Bruijn Sequences. (Pesto)

de Bruijn sequences are sequences of 0s and 1s containing all possible subsequences of a given length exactly once: for instance, 0001110100 contains every possible sequence of 3 0s and 1s.

For what lengths do such sequences exist, and how easily can we find them if they do?

Bonus homework: how were they used as an early error-correction code in Sanskrit poetry, 2500 years before error-correction codes even existed?

Prerequisites: None.

Differential Topology. (Kevin)

It is often said that topologists can't tell the difference between a donut and a mug. But if the mug has some parts that aren't so smooth, a differential topologist sure can distinguish them!

You may have heard of the Klein bottle, a surface (also known as a two-dimensional manifold) that can be embedded in three-dimensional space, but with a self-intersection. It can, however, live perfectly happily in four dimensions. We'll use the power of differential topology to show that, in general, we can fit any manifold of dimension d in $2d$-dimensional space.

Prerequisites: Knowledge of derivatives. We'll talk about linear maps and continuous maps in $\mathbb{R}^n$, but there will be some first day homework questions to catch you up if you haven't seen them before.

Dirichlet. (Nicholas, *camper teaching project*)

Prime numbers have been among the central objects of study in number theory since the time of the ancient Greeks. While Euclid's proof of the infinitude of primes is well known, his proof tells us little about how dense the primes actually are in the natural numbers. As it turns out, the answer to this question is intimately related to the zeros of the Riemann zeta function. In fact, the Riemann Hypothesis is equivalent to the statement

$$\pi(x) = \frac{x}{\log x} + O(x^{\frac{1}{2}} \log x),$$

where $\pi(x)$ is the number of primes less than or equal to x.

In this class we will take an introductory look at this function and others like it, which are known as Dirichlet Series. Find out how functions of a complex variable are able to encode information related to number theory, and, if time permits, see the role of Dirichlet Series in proving a weaker result which states that $\pi(x)$ and $\frac{x}{\log x}$ are asymptotically equal.

Prerequisites: Comfort with infinite series and some experience with functions of a complex variable.

Dirichlet's Theorem on Primes a mod b. (Kevin)

Did you know there are infinitely many primes? Did you know there are infinitely many primes congruent to a mod b, as long as a and b are relatively prime? We will study the world of L-functions, which generalize the Riemann zeta function, and use them to prove this fact.

Prerequisites: Familiarity with complex numbers. Some knowledge of groups and basic complex analysis (such as the word "pole") is helpful.

Divisor Theory of Graphs. (*David Perkinson*)

This is a course on the divisor theory of graphs and the abelian sandpile model. Topics would include: chip-firing and the "dollar game" on a graph, the discrete Laplacian, the Jacobian and Picard groups of a graph (the cokernel of the Laplacian — computing their structure gives an excuse for appreciating the Smith normal form of an integer matrix), tree-bijections, the matrix-tree theorem, orientations of graphs, and the Riemann-Roch theorem for graphs due to Baker and Norine. If there is time, I would also talk about the "dual" version of this theory — the abelian sandpile model.

The material is described in detail in my text with Scott Corry (to appear this year), which can be found at

`http://people.reed.edu/~davidp/divisors_and_sandpiles/`

An outline of a course I taught on this subject at Reed can be found at

`http://people.reed.edu/~davidp/374/`

Classes will be 1.5 hours long.

Prerequisites: Some linear algebra, including determinants.

DIY Hyperbolic Geometry. (*Katie Mann*)

If you've ever done geometry on the surface of a sphere, you'll know that it's a wacky place: what should be "parallel" lines always eventually intersect, the area of a disc is less than πr^2, and when you try to flatten a piece of sphere (perhaps you've tried this with a piece of orange peel), you end up tearing it.

In this class we'll explore an even wilder place to do geometry, the hyperbolic plane. This space is the opposite of the sphere: you can draw both parallel and non-parallel lines that never intersect, the area of a disc is way larger than you expect, and when you try to flatten a piece of it, you are forced to wrinkle it up. You might think you've never seen such a place before, but *actually* hyperbolic is the most common kind of geometry.

This class is called "DIY" for a reason: I'll do some talking, but you should also expect to make and experiment with paper models, attempt to build a hyperbolic soccer ball, and discover the laws of the hyperbolic universe for yourself.

Prerequisites: None.

Elliptic functions. (Mark)

Complex analysis, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k),$$

where $\sigma_i(k)$ is the sum of the i-th powers of the divisors of k. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1^3 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1],$$

which you are welcome to check if you run out of things to do.)

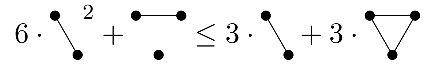
Prerequisites: Functions of a complex variable, in particular Liouville's Theorem.

Flag Algebras. (Misha)

We prove graph theory results from 1907 by methods developed in 2007. (Later, we'll solve some more recent problems, as well.)

Flag algebras are a tool in graph theory introduced by Razborov to talk about how densities of small subgraphs in large graphs relate to each other. We do some graph theory, then we do some algebra to automate our graph theory, then we do some more algebra to automate our algebra.

Selling point: we get to write inequalities where our variables are graphs. Here's an example:



Prerequisites: You will be lost without graph theory. Some ideas from abstract algebra and linear algebra will show up.

Galois Theory. (Viv)

You may know a formula for solving quadratic equations, but what about polynomials of higher degree? For many years, the question of solving quintic equations plagued the mathematical community. Ultimately, it was answered by a hotblooded young punk named Evariste Galois, who described a startlingly beautiful connection between field theory and group theory shortly before being killed in a duel. We'll learn about this connection and how it applies to quintics.

Prerequisites: Group theory.

Galois Theory and Number Theory. (Aaron & Viv)

Can you find an example of a polynomial which is irreducible over the integers but reducible mod every prime?

In this class, we'll see what this question has to do with Galois theory. We'll also investigate many other relations between Galois theory and number theory.

Possible further topics include:

(1) Galois groups of finite fields
(2) Cyclotomic extensions
(3) Kummer Extensions
(4) Extensions of local fields (and the infamous secret society of p-adics)
(5) Inseparable extensions
(6) Purely inseparable extensions
(7) Perfect fields

Prerequisites: Galois Theory, some background in algebraic number theory may be helpful but it's not necessary that you took the week 1 class on algebraic number theory.

Game Theory (The Economic Variety). (Ben)

In economics, political science, and even biology, game theory is a useful tool for making predictions. This class will discuss a few of the ideas of game theory, such as: what is a Nash Equilibrium, why do we care about them, and how do we find them? We will also talk about how to formalize our intuition for analyzing games similar to chess or go, where players alternate moves, and what kinds of predictions we end up with here. We'll continue by looking at "repeated games," where players play the same game with each other over and over. Finally, we'll discuss the Folk Theorem, which demonstrates a fascinating difference between analyzing finitely repeated games and infinitely repeated ones.

Although this course will be mostly lecture-based, there will also be several opportunities for discussion, to give a chance to cooperatively work with the concepts of game theory.

Prerequisites: None.

Generating Functions, Catalan Numbers, and Partitions. (Mark)

Generating functions provide a powerful technique, used by Euler and many later mathematicians, to analyze sequences of numbers; often, they also provide the pleasure of working with infinite series without having to worry about convergence.

The sequence of Catalan numbers, which starts off $1, 2, 5, 14, 42, \ldots$, comes up in the solution of many counting problems, involving, among other things, voting, lattice paths, and polygon dissection. We'll use a generating function to come up with an explicit formula for the Catalan numbers.

A *partition* of a positive integer n is a way to write n as a sum of one or more positive integers, say in nonincreasing order; for example, the seven partitions of 5 are

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, \quad \text{and} \quad 1+1+1+1+1.$$

The number of such partitions is given by the partition function $p(n)$; for example, $p(5) = 7$. Although an "explicit" formula for $p(n)$ is known and we may even look at it, it's quite complicated. In our class, we'll combine generating functions and a famous combinatorial argument due to Franklin to find a beautiful recurrence relation for the (rapidly growing) partition function. This formula was used by MacMahon to make a table of values for $p(n)$ through $p(200) = 3972999029388$, well before the advent of computers!

Prerequisites: Summation notation; geometric series. Some experience with more general power series may help, but is not really needed. A bit of calculus may come in handy, but you should be able to get by without.

Graph Minors. (Pesto)

The graphs K_5 and $K_{3,3}$ are nonplanar. In fact, *Wagner's Theorem* says that a graph is nonplanar if and only if it contains one of those two graphs as a "minor". We'll define minors, prove that theorem, and talk about why K_5 is hardly necessary in that statement. Also, we'll state the most famous unsolved problem in graph theory, a generalization of the Four-Color Theorem, and prove it for some special cases.

Prerequisites: Graph theory: Understand the first sentence of the blurb.

Group Theory. (Mira)

Abstract algebra studies how mathematical objects interact and combine to form new objects. For example, numbers combine via addition or multiplication (among other things); functions combine via composition; individual moves on a Rubik's cube combine into more complicated patterns; knots combine by intertwining. Abstract algebra is what happens when you don't care about the objects themselves, but only about the structure of their interaction.

Groups theory lies at the heart of abstract algebra. It examines a type of interaction that occurs over and over in many mathematical contexts: a binary, associative operation with an identity and inverses. (Don't worry if you don't know what that means – we'll explain.) The general results you prove in group theory can be applied to geometry, number theory, combinatorics, topology, physics – basically everywhere! That's why this class is a prerequisite for so many other classes at Mathcamp.

In this course we will cover the basic definitions of group theory, Lagrange's Theorem, homomorphisms, quotient groups, the First Isomorphism Theorem, and a little bit on symmetries and permutation groups. If you've seen most of these topics before, no need to take this class. If you haven't, come join us for a first foray into the beautiful realm of abstract algebra.

Prerequisites: None.

Guess Who? (Tim!)

I've become a bit obsessed with the board game *Guess Who?*. It's a simple children's guessing game — you ask yes/no questions to try to determine your opponent's secret character from among 24 possibilities, and whoever guesses correctly with fewer questions wins.

A YouTube star (and former NASA scientist) claimed to find the "best strategy," and you can probably think of this strategy too. But actually this strategy is not best. It can be totally destroyed, as we will see.

But then, what really is the "best strategy." There's an answer to this, but this is where things start getting weird. We'll explore it all together. Expect to see terms like "decision trees," "entropy," "matrix games," "convex optimization," and numbers like $\frac{127}{255}$. This class will also involve a lot of *relaxation*, but in this case that's a technical term. . . .

Prerequisites: None.

Hardness. (Linus)

What does it mean for a math problem to be hard? Is solving a Sudoku puzzle hard? Is Portal 2 hard?

I know the answers aren't "yes" and "no" respectively. I know this because I can make a Portal 2 level that makes the player enter a Sudoku solution and only lets them beat the level if it works.

So, Portal 2 is at-least-as-hard as Sudoku. This class is all about at-least-as-hard. It's complexity theory, the same kind of math as the P versus NP problem.

Prerequisites: Be comfortable with me saying the word "algorithm" a lot.

Haskell. (Larsen)

This is not a math class. (Not on the surface, anyway.) We will learn how to code with a programming language called Haskell, which was named after a mathematician. Haskell is "lazy" and "purely functional" — in other words, it is structurally very different from the programming languages you're probably familiar with. In this class, we'll type some code that looks like this, and learn why we do it that way:

```
data NaturalNumber = Zero | S NaturalNumber
instance Eq NaturalNumber where
  Zero == Zero = True
  Zero == S y  = False
  S x  == Zero = False
  S x  == S y  = x == y
```

Prerequisites: None.

Hat Problems ft. Hamming Codes. (Agustin)

There may come a time in your life in which you are imprisoned with only one way out: play a game that involves you and some inmates wearing hats of two possible colors, and have at most one inmate guess the color of their own hat wrong. And that's if the prison guards are nice.

This class is meant to prepare you for the various cruel hat problems your prison guard will inevitably force you to solve. We'll start with a few puzzle-y hat problems, and then spend some of the time attacking a trickier hat problem. To solve it we'll have to talk about Hamming codes, which we can use to transmit messages accurately, even if an error is introduced! After all, you know what they say– come for the hats, stay for the linear error-correcting codes.

Prerequisites: None.

Higher Homotopy Groups. (Larsen)

In week 4 we will have a class on the fundamental group, also denoted π_1. But if there is a π_1, surely there must be a π_2, and π_{17}! The "higher homotopy groups" are the natural generalization of the π_1, measuring the higher-dimensional complexity of a space. They are notoriously difficult to work with, but have some strange properties, for example they are all abelian.

Prerequisites: The Fundamental Group.

Hilbert's Paper, Scissors, and Rock. (Steve)

Consider the property "has the same area as." For complicated shapes this probably has something to do with integrals; for polygons, however, your instinct is probably that we're just talking about *cutting and rearranging* — that two polygons have the same area if and only if we can slice one into a bunch of pieces and rearrange them to get the other. Now, some of you may have heard about the Banach-Tarski paradox and related results and be worried, but if we keep everything nice and simple it turns out we don't get into trouble: if A and B are polygons, then A and B have the same area if and only if we can cut A into finitely many pieces via straight line cuts and reassemble those pieces to get a copy of B. That is, two polygons have the same area if and only if they are **scissors congruent**. Note that scissors congruence, unlike the integral approach, doesn't actually require us to ever measure an area, but rather defines "same-area-ness" entirely on its own.

In 1900, Hilbert published a list of 23 problems. The third problem — the first to be solved, within a year and by Hilbert's own student Dehn — asked whether scissors congruence can define *volume* correctly, in addition to area. That is, given polyhedra A, B of equal volume, can we always cut A into finitely many pieces via straight plane cuts and reassemble those pieces to get a copy of B?

In this class we'll see Dehn's answer to Hilbert's playful question, "Scissors beats paper; does it also beat rock?"

Prerequisites: Proof by induction.

History of Math. (Sam)

Do you remember a time when variables didn't exist? When we had to communicate equations with sentences like:

> Take absolute number on the side opposite to that on which the square and simple unknown are. To the absolute number multiplied by four times the [coefficient] of the square, add the square of the [coefficient of the] unknown; the square root of the same, less the [coefficient of the] unknown, being divided by twice the [coefficient of the] square is the [value of the] unknown [Quoted from Katz, from around 700CE in India]

If so, you're probably about 400 years old. Think about that: for the overwhelming majority of mathematical history, symbolic notation simply did not exist. This class is about moments like that transition: moments when the way we do mathematics fundamentally changed. Topics may include:

- The emergence of mathematics
- Greek mathematics and proof
- The development of symbolic notation
- Analytic geometry, and the recognition that curves and equations captured **the same** thing
- Cauchy and a rigorous foothold for calculus
- The emergence of probability in the 17th century
- The St. Petersburg paradox, and a fundamental challenge to mathematical tools

Prerequisites: None.

How Curved is a Potato? (Apurva)

Ans: 4π.

Every potato has total Gaussian curvature of 4π and so does the surface of the Earth. What is even more interesting is the fact that Gaussian curvature is entirely *intrinsic* to the surface and does not depend upon how it is embedded within an ambient space. Gauss called this the Theorema Egregium. To truly understand this phenomenon mathematicians had to invent manifolds and the field of Riemannian geometry.

In this class, we'll learn how linear algebra and calculus come together to make differential geometry, define principal, mean, and Gaussian curvatures and understand their geometric significance. By the end of this class, you'll have a much deeper appreciation of linear algebra.

Prerequisites: The only things we'll need from calculus are partial derivatives and vector valued functions. You should be familiar with or be willing to learn on the fly - Taylor series expansion in 2 variables, eigenvalues and eigenvectors of 2x2 matrices.

How to Get Away from Dead Guys. (Pesto)

To get his PhD at MIT, Adam gave a talk on how get away from dead guys if you are in some area A and can run at rate 1, and they have to stay out of A but can run at a rate $z > 1$; you get away if you get to the edge of A with no dead guy at the same spot, but the dead guys try to stop you and be at the same spot on the edge of A. I'll tell you how to get away for some z, how the dead guys can stop you for some z, and why it's hard to say who wins for some more z.

Also, like a dead guy who can't talk well, I'll stay in the game of four.

Prerequisites: If you don't know what the statement "The problem of finding a set of k vertices in a graph such that every edge contains at least one is NP hard" means, ask before class.

How to Juggle. (Viv)

In this class, you will learn to juggle...in theory.

We'll discuss juggling sequences, a mathematical model for juggling that revolutionized the juggling world!

Prerequisites: None!

Infections, contractions, and tumors, Oh My! Agent Based Modeling of Biological Systems. (*Angela Gallegos & Kamila Larripa*)

What do tumors, infections, and the uterus all have in common? Actually quite a bit more than you might think! All three have dynamics that can be described using mathematics, and patterns that can be explored when you look at interactions between different individuals—whether those individuals be tumor cells, humans, or muscle cells. In this course we will use NetLogo to explore how we can computationally model these types of systems and you will get to explore your own research questions in our week together!

Note that classes may run over time so that you will have time to work in the computer lab on your homework and projects. Homework: Necessary! You will work on modeling projects during the week and present your results in class on the last day.

Prerequisites: Specific courses are less important, than comfort with mathematical abstraction (word problems, for example) and computer programming is helpful.

Infinitesimal Calculus. (Tim!)

If you've learned the definition of *continuous function*, you may have learned that a function f is continuous if an infinitely small change in x results in an infinitely small change in $f(x)$. This is a pretty good definition: it's short, and you can picture it on a graph, and you can see the connection to more geometric descriptions ("a function is continuous if you can draw its graph without lifting your pen"). It's also how many of the pioneers of calculus thought about the subject.

But if you take a proof-based calculus class, you might see this definition instead: *A function f is continuous at c if for all $\epsilon > 0$, there is a $\delta > 0$ such that for all x with $|x - c| < \delta$, we have that $|f(x) - f(c)| < \epsilon$.* What an ugly definition! To be sure, it's correct, and is often useful, but nevertheless it's clunky and counterintuitive. Why would any class use it instead of the "infinitely small" definition? The problem is that there is *no such thing* as an infinitely small (or *infinitesimal*) real number.

Most proof-based calculus classes usually throw in the towel on infinitesimals at this point and haul out ϵ and δ instead. But not us. We'll just add some infinitesimal numbers to the real numbers to get the *hyperreal numbers*. And we'll get to have nice definitions like the one at the start of this blurb. We'll go through all the highlights of a calculus class, with proofs that are correct and often much simpler than the standard ones, but which are simultaneously alien and bizarre.

You see, when you start playing with the fundamental building blocks of reality, things can start going totally bananapants. And perhaps we'll come to understand why most calculus classes shy away from infinitesimals.

Prerequisites: Some calculus.

Infinitesimaller Calculus: Sequences, Integrals, and Democracy. (Tim!)

In Infinitesimal Calculus, we constructed the hyperreal numbers, and started to use them to construct pretty proofs about continuous functions and derivatives, ending with the Extreme Value Theorem. But there is so much more calculus to prettify!

We'll handle sequences and their limits with ease, we'll define integrals the way you've *always* wanted to, and we'll bask in the Fundamental Theorem of Calculus.

And hey, since we've been talking about hyperreal numbers in terms of voting, we're actually pretty well prepped to do some voting theory! We'll use ultrafilters to prove Arrow's Impossibility Theorem, which says that the only fair voting system is a dictatorship. But instead of hyperreal numbers, we'll use hyper-something else!

Prerequisites: Infinitesimal Calculus.

Information Theory and the Redundancy of English. (Mira)

NWSFLSH: NGLSH S RDNDNT!! (BT DN'T TLL YR NGLSH TCHR SD THT)

The redundancy of English (or any other language) is what allows you to decipher the above sentence. It's also what allows you to decipher bad handwriting or to have a conversation in a crowded room. The redundancy is a kind of error-correcting code: even if you miss part of what was said, you can recover the rest.

How redundant is English? There are two ways to interpret this question:

- How much information is conveyed by a single letter of English text, relative to how much could theoretically be conveyed? (But what is information? How do you measure it?)
- How much can we compress English text? If we encode it using a really clever encoding scheme, can we reduce the length of the message by a factor of 2? 10? 100? (But how will we ever know if our encoding is the cleverest possible one?)

Fortunately, the two interpretations are related. In this class, we will first derive a mathematical definition of information, based on our intuitive notions of what this word should mean. Then we'll prove the Noiseless Coding Theorem: the degree to which a piece of text (or any other data stream) can be compressed is governed by the actual amount of information that it contains. We'll also talk about Huffman codes: the optimal way of compressing data if you know enough about its source. (That's a big "if", but it's still a very cool method.)

Finally, we'll answer our original question — how redundant is English? — in the way that Claude Shannon, the father of information theory, originally answered it: by playing a game I call Shannon's Hangman and using it as a way of communicating with our imaginary identical clones!

The class is 4 days long, but you can skip some of the days and still come to the others. Here's how it works:

Day 1: Introduction and definition of information. (Required for the rest of the class.)
Days 2, 3: Noiseless coding and Huffman codes. (The mathematical heart of the class, where we'll prove the Noiseless Coding Theorem.)
Day 4: Shannon's Hangman and the redundancy of English. (You can come to this class even if you don't come on Days 2 and 3 — you just need the material from Day 1.)

Prerequisites: None.

Intersecting Curves. (J-Lo)

The following statement is extremely concise and elegant: A line and a parabola intersect at exactly two points.

Unfortunately it also appears to be wrong. We could just declare it to be wrong and leave it at that, or we could start changing our assumptions until we can force it to be true.

This course takes the second approach. Working through a collection of examples in groups, you will determine the hypotheses you need in order to force the above statement (and many others like it) to be true, and explore how these elegant (and now true) statements have algebraic implications, such as to Pythagorean triples and elliptic curves.

Prerequisites: None.

Intersecting polynomials. (Tim!)

You might think that everything there is to know about one-variable real polynomials has been known for hundreds of years. Except, in 2009, while bored at a faculty meeting, Kontsevich scribbled down a brand new fact about polynomials. You'll discover it.

Prerequisites: None.

Intro Graph Theory. (Mia)

A new island has been discovered in the Arctic Ocean! While the geographers are arguing over how to divide the island, the cartographers begin to wonder about the map: how many colors are needed to color the countries so that any two countries that share a border get different colors? The Four Color Theorem says just four. However, it took over 100 years and a computer program that checked 1,936 different cases to prove this theorem. In this class, we will won't prove the Four Color Theorem, but we will prove the Five Color Theorem.

Now, let's make this harder. Suppose the countries decide that they have *non-negotiable* color preferences. For instance, the country Zudral demands to be eggplant or magenta. And the country Scaecia refuses to be anything but light blue, sky blue, or cornflower blue. Given that each country now has a list of allowable colors, how does this change the cartographers' ability to color the map?

Rather surprisingly, the answer is barely. So long as each country has a list of at least 5 colors, the map can be properly colored! That means an analog of the Five Color Theorem still holds. But does the Four Color Theorem still hold? In this class, we will prove some fascinating results in graph theory and eventually answer this question.

Prerequisites: None.

Intro Number Theory. (Mark)

How do you find $\gcd(a, b)$ for two large integers a and b without having to factor them? Which integers are the sum of two (or the sum of three, or the sum of four) perfect squares? What postages can you get (and not get) if you have only 8 cent and 17 cent stamps available? How does the RSA algorithm (used for such things as sending confidential information, such as your credit card number, over the Internet) work? Besides the answers to such questions, number theory offers insight into many beautiful and subtle properties of our old friends, the integers. For thousands of years professional and amateur mathematicians have been fascinated by the subject (by the way, some of the amateurs, such as the 17th century lawyer Fermat and the modern-day theoretical physicist Dyson, are not to be underestimated!) and chances are that you, too, will enjoy it quite a bit.

Prerequisites: Modular arithmetic (which I can catch you up on, if necessary).

Intro Ring Theory. (Lara)

As my awesome colleague and friend Susan so aptly put it: to study rings, 'we cut ourselves loose from our usual number systems—the complexes, the reals, the rationals, the integers, and just work with. . . stuff. Stuff that you can add. And multiply.'

Rings naturally arise in many areas of math and serve as a powerful tool for problem-solving, allowing us to understand familiar structures from a fresh perspective.

We'll give lots of examples of rings to get a sense of how ubiquitous they are, then we'll play with rings to our hearts' content! We'll learn how to tell which rings are 'the same' and how to map between rings, as well as the many wonderful things special subsets called 'ideals' can help us do.

Prerequisites: None.

Irrationality Proofs. (Susan)

I'm sure you can prove that $\sqrt{2}$ is irrational. Let's do some weirder things! In this class, we'll prove the irrationality of $\sqrt{2}+\sqrt{3}$, $\log_5(7)$, e, and π!

Prerequisites: None.

King Chicken Theorems. (Marisa)

Chickens are incredibly cruel creatures. Whenever you put a bunch of them together, they will form a pecking order. Perhaps "order" is an exaggeration: the chickens will go around pecking whichever chickens they deem to be weaker than themselves. Imagine you're a farmer, and you're mapping out the behavior of your chickens. You would like to assign blame to the meanest chicken. Is it always possible to identify the meanest chicken? Can there be two equally mean chickens? Are there pecking orders in which all the chickens are equally mean?

Prerequisites: None.

Knot Theory. (Jeff)

In the 1860s, Lord Kelvin developed the following theory of matter: atoms, the indivisible particles that composed the universe, were actually tiny whirlwind vortices in the ether. He beleived that shape of these vortices were tiny knots, and you could make compounds out of these knots by linking them together. Inspired by the quest to classify atoms, a mathematician named Tait made a list of all knots up to 10 crossings (no small feat, considering that there are around 250 of them.)

Kelvin's theory turned out to be bunk (as both the idea of ether and tiny vortices were too crazy), but mathematicians kept on thinking about knots. It took mathematicians nearly a hundred years to realize that Tait's list was wrong, and we still have a lot to learn about knots. We now study knots not because they represent atoms, but because they are some of the simplest objects a topologist can study: maps from the circle to 3-dimensional space. And despite these objects being so fundamental, a classification of knots eludes mathematicians to this very day.

Prerequisites: None. In this class, we'll take the first step to classifying knots, by describing invariants of knots and giving a procedure to (non-uniquely) describe every knot.

Latin Squares and Finite Geometries. (Marisa)

In 1782, Euler conjectured an answer to the following yes/no question: is it possible to arrange six *regiments* consisting of six officers each of different *ranks* in a 6×6 square so that no rank or regiment will be repeated in any row or column? We know the answer now, but surprisingly, the question remained open until 1901. In this class, we'll be exploring combinatorial design questions like this one through the lenses of Latin Squares (like Sudoku puzzles) and Finite Geometries (like the Fano plane).

Prerequisites: None.

Linear Algebra. (Mark)

You may have heard that linear algebra involves computations with matrices and vectors - and there is some truth to that. But this point of view makes it seem much less interesting than the subject really is; what's exciting about linear algebra is not those computations themselves, but ...

(1) the conceptual ideas behind them, which are elegant and which crop up throughout mathematics, and
(2) the many applications, both inside and outside mathematics.

In this class we'll deal with questions such as: What is the real reason for the addition formulas for sin and cos? What happens to geometric concepts (such as lengths and angles) if you're not in the plane or 3-space, but in higher dimensions? What does "dimension" even mean, and if you're inside a space, how can you tell what its dimension is? What does rotating a vector, say around the origin, have in common with taking the derivative of a function? What happens to areas (in the plane), volumes (in 3-space), etc. when we carry out a linear change of coordinates? If after a sunny day the next day has an 80% probability of being sunny and a 20% probability of being rainy, while after a rainy day the next day has a 60% probability of being sunny and a 40% probability of being rainy, and if today is sunny, how can you (without taking 365 increasingly painful steps of computation) find the probability that it will be sunny exactly one year from now? If you are given the equation $8x^2 + 6xy + y^2 = 19$, how can you quickly tell whether this represents an ellipse, a hyperbola, or a parabola, and how can you then (without technology) get an accurate sketch of the curve? How do astronomers know the chemical composition of distant stars? (We may not get to all these things in the two weeks, but we should cover most of them at least to some extent.)

Prerequisites: Although the blurb refers to taking a derivative, you'll be able to get by if you don't know what that means. If you have no previous exposure to abstract concepts, you should at least take the Mathcamp Crash Course at the same time.

Linear Programs & Convex Optimization. (Linus)

What do these problems have in common?

- You play rock-paper-scissors, but winning with rock is worth 2 points instead of 1. What is the Nash equilibrium?
- Is there an English anagram between only vegetable words and only fruit words? "Parsnip lima" → "Apple raisin" almost works, but it changes an 'm' to an 'a' and "lima" isn't really a vegetable.
- Can you place overlapping L-shaped triominoes into a 5-by-7 square grid so that every square is covered the same number of times?

The answer: they can each be expressed as a system of linear inequalities. These systems are a rare intersection between (a) super useful and (b) super beautiful theory. Probable topics:

– Every system of linear inequalities has a "dual" system ("LP duality")
– How a computer can solve systems of linear inequalities quickly
– Applications of LP duality in combinatorics (such as Hall's Marriage Lemma)

Prerequisites: Ability to program *not* needed; this class is not about writing computer code. Be comfortable working with vectors in n-dimensional space. Basic linear algebra recommended but not required.

Low-Dimensional Zoology. (Larsen)

A *manifold* is a shape that looks like n-dimensional Euclidean space, if you only look at small parts of the shape. For example, a sphere is a 2-dimensional manifold — just ask the many people throughout history who thought the Earth was flat. A torus is another good example, but a pair of intersecting planes is not, since the intersection doesn't look anything like a single plane. In other words, manifolds are the most natural kind of shape from a topological point of view, since they only differ from each other on the "global" scale.

But what manifolds are out there? How can we characterize them and tell them apart? Depending on how you count, there is only one 1-dimensional manifold: a circle. In higher dimensions, there are progressively more distinct manifolds, so to keep things simple we'll focus on "low" dimensions, in this case meaning dimensions up to 3. We will define various constructions and distinguishing properties that will help us completely classify 2-dimensional manifolds, and we'll do (some of) the same for 3-dimensional manifolds.

Prerequisites: Knowledge of point-set topology could help, but isn't necessary.

Machine Geometry; or, Area and Coarea. (Misha)

This class is inspired by, but not strictly about, computer algorithms for solving geometry problems. We avoid ugly coordinates, but also remain skeptical of proofs that rely on cleverly spotting the right similar triangle or cyclic quadrilateral.

We will begin with the area method, and use it to prove theorems in affine geometry: geometry where we can't measure distances or angles.

We like distances and angles, though. To be able to handle those, we will define a notion of coarea, which relates to area as cosine relates to sine.

Prerequisites: None.

Machine Learning (No Neural Nets). (Linus)

Machine learning is about getting examples of a function and guessing what that function is.

For example, let's say you want to classify emails as spam or not spam. You have a large supply of example emails which are already classified. You guess that the correct truth is some majority function of words — such as "If the email mentions at least three of medicine, cheap, rich, campaign, virus, then it's spam." If your guess is correct, then how can you figure out which words to use? What if instead of a perfect truth, there's a 1% chance of error? Can you figure out words that give a 99% success rate?

Okay, here's a more complicated question: let's say we have a large collection of Mathcamper board game ratings. Not everyone has rated every game. If I haven't played Spirit Island yet, then what's the best way to guess how much I would enjoy it? How confident should I be of my guess?

Prerequisites: Basic linear algebra; be willing to think about vectors in high-dimensional space. Programming skill *not* needed.

Mathematical Art History. (Viv)

Lucia Pacioli, one of Leonardo da Vinci's contemporaries, is quoted as saying

"Without mathematics, there is no art."

Now, it's certainly easy for many of us to agree with that, but there are also many examples throughout history of times when the art world was obsessed, wittingly or unwittingly, with mathematical ideas. We'll talk about some of these times, including topics like perspective, the golden ratio, proto-Cubism, and fractals.

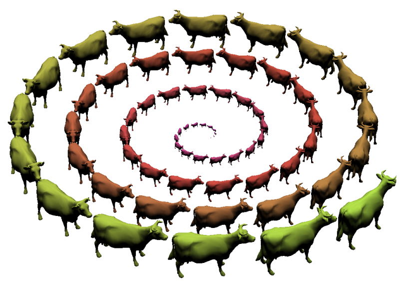
Prerequisites: None!

Mathematica Workshop. (Misha)

All campers get a free 1-year license for Mathematica from Wolfram.[2] But do you know how to use it?

In this class, you will learn how to do things in Mathematica, including:

- Drawing pictures such as the one below:



- Solving crossword puzzles.
- Various tricks I've learned for dealing with hard combinatorial problems.
- Using the incredibly extensive help function.

There will be time for you to work on doing your own cool things in Mathematica, which I will try to help you with.

Prerequisites: None.

Mathematics of Democracy. (Mira)

Everyone knows that elections involve choices, but it turns out that the most important choice is one that most voters don't even think about. Before anyone can vote, you have to choose a voting *system*. For instance, do we split the country into small districts each of which elects a single Congressional representative, or do we use larger districts each of which elects several representatives? Do the voters get to list only their first choice of candidate, or do they get to rank or rate all the candidates? And of course, once the votes are in, what algorithm do we use to select the winner(s)? The choice of voting system can have a huge effect on the outcome of an election, so this topic is obviously important from a political point of view. But it also turns out to be really interesting mathematically.

During the first two days, we will focus on systems for electing a single person (e.g. a president). We will prove some depressing theorems showing that no voting system can have all the nice properties you want it to have, and that all voting systems are vulnerable to strategic voting. Then we will

[2]Also, if you're going off to college, many universities provide free Mathematica licenses for their students.

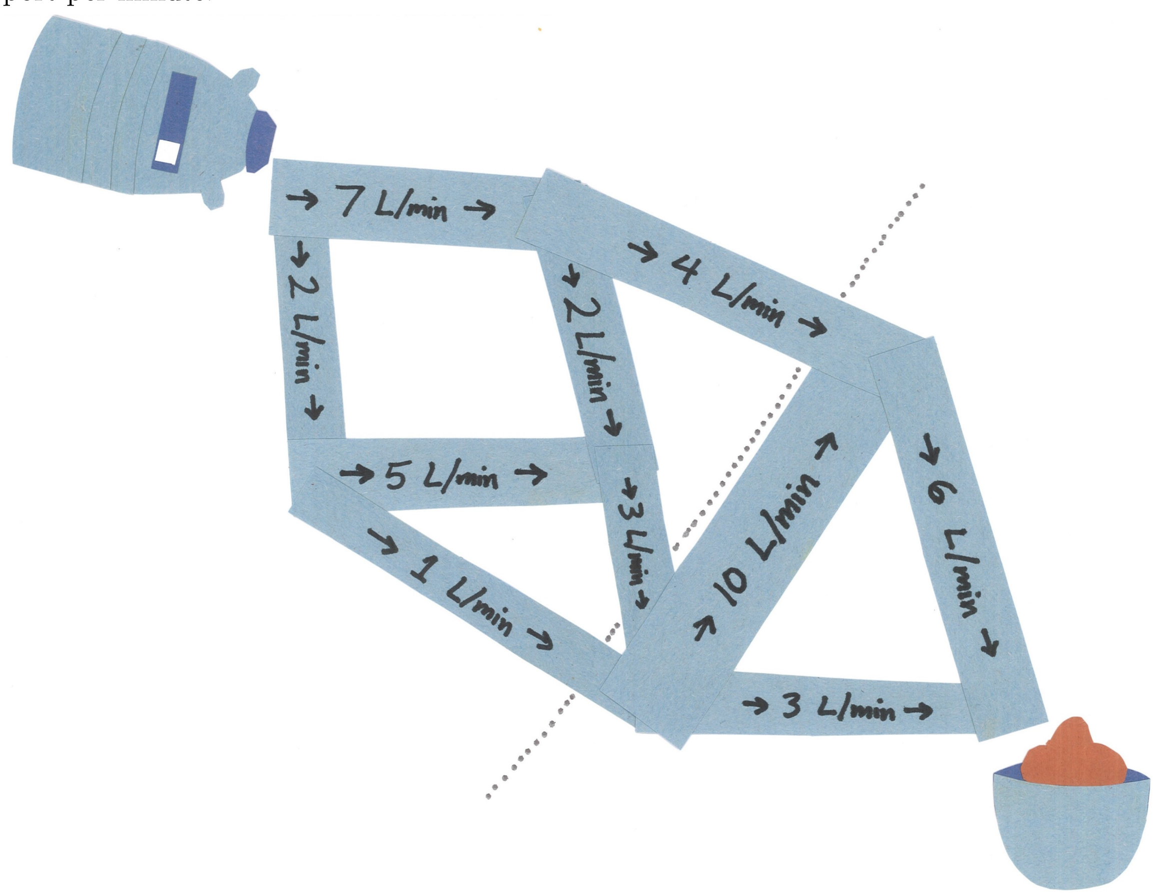
move on to multi-winner systems (systems for electing a Congress or parliament), and talk about apportionment, gerrymandering, and various methods of proportional representation.

If you've taken a voting theory class outside of Mathcamp, or at Mathcamp $2000 + N$ for $N < 17$, you may know some of the material in the first day or two, but probably not the last three days. (This stuff is rarely taught in standard voting theory courses.) If you took Mira and Ari's voting theory superclass at MC2017, the first three days will be mostly review, but the last two will be things we didn't cover last summer. Talk to Mira for more details.

Prerequisites: None.

Max-Flow Min-Cut. (Tim!)

The JCs have built a network of pipes to carry liquid nitrogen, so that they can make more ice cream! Each pipe in the diagram below is labeled with its capacity, the number of liters of nitrogen it can transport per minute.



How fast can this system transport nitrogen from the dewar to the bowl? It can't be more than 8 liters per minute, because there's a bottle-neck: at most 8 liters per minute can flow across the dotted line. In fact, there's a way to route nitrogen through the pipes that achieves *exactly* that rate (such a routing is called a *flow*). Can you find that maximum flow?

More generally, is there an efficient algorithm to find the maximum flow for any network? And, is there always a "bottle-neck" (usually called a *cut*) that matches the maximum flow? We'll find the answer to these questions (the answer is yes to both!). Then, we'll use the algorithm to help solve other seemingly unrelated problems!

Prerequisites: Graph theory.

MCMC. (Mira)

MCMC does not stand for "Mathcamp Mathcamp", but it stands for something almost as exciting: "Markov Chain Monte Carlo". It is a technique for sampling from very complicated probability distributions. That might not sound particularly glamorous, but in fact, MCMC lies at the heart of the Bayesian revolution in statistics that began in the 1990's and is still going strong. The revolution is changing the way scientists use mathematical models in their work. There is literally no branch of the natural or social sciences where researchers aren't currently using MCMC.

If the above sounds intriguing but you have no idea what any of the words mean (Markov chain? Monte Carlo? Probability distribution? Bayesian?), that's totally OK – we're going to start from scratch. We'll define exactly what MCMC does and why this is so important. We'll analyze it mathematically and see how and why it works. We'll talk about it in the broader context of the Bayesian revolution. And we'll give lots of examples from different branches of science, including real data sets for you to play around with. For those who want to go more in depth, I will also be offering a couple of MCMC-based projects later in camp.

Prerequisites: No hard prerequisites, but if you have never done any statistics, the Week 1 class on Statistical Modeling will give you some much-needed context and help you hit the ground running.

Metric Space Topology. (Jeff)

One of the oldest notions in mathematics is the idea of distances. I think before we even had a notion of numbers, angles or lines, we had an idea of what it meant for two things to have a distance between them.

- We have real-world notions of distance: "San Francisco and Los Angeles are 559 km apart."
- We have relational ideas of distance: "The Penguin is a distant relative of the velociraptor", or " I think that my friends from elementary school have grown apart since then".
- We have correlative notions of distance: "The English King James Bible and American King Bible are closer than the New American Bible", or "the Dow Jones Industrial Average tracks closely with Standard and Poor's 500 index".

A metric space is a framework for looking at all of these ideas together, allowing us to develop a general theory which attacks many different kinds of problems. We'll also get an introduction to topology — the study of the shape of space when you're allowed to deform your geometry — through the lens of metrics.

Prerequisites: None.

Minus Choice, Still Paradoxes. (J-Lo)

The Banach-Tarski paradox says that a ball can be broken into finitely many pieces, and using only translations and rotations, can be rearranged into two balls, each the same volume as the original. The "Axiom of Choice" is a crucial part of this construction, and historically some people have used Banach-Tarski as an argument against this axiom.

But it turns out that paradoxical decompositions (breaking something into finitely many pieces and rearranging them into two copies of the original) exist even without the Axiom of Choice! This class will discuss some of these constructions, which are all based in group theory, and see what specific role Choice plays in the Banach-Tarski case.

Prerequisites: Group Theory.

Modular Forms. (Shiyue)

A modular form is a function on the complex upper half plane satisfying certain symmetries. The theory of modular forms is really part of complex analysis, but has ubiquitous applications in number theory. It became hugely popular in the 1990s, when it was used to prove Fermat's Last Theorem.

We won't get to the proof of FLT in this course – we'll only have time to scratch of the surface of the theory, but it will give you a sense of how complex analysis (calculus over complex numbers) and number theory interact. We will talk about Eisenstein series, the Ramanujan cusp form, the valence formula (a super cool geometric result), the structure theorem for level 1 modular forms (which was totally amazing to me when I first saw it), the Miller basis, and hopefully touch on Hecke operators.

Prerequisites: Linear algebra; calculus; a very solid familiarity with complex numbers; the basic definitions of group theory.

Nonzero-sum games. (Pesto)

We'll play and talk about combinations of (Iterated) (Community) (Hidden) Prisoner's Dilemmas (with punishment). Ask me at TAU about their relationship to theories of ethics.

Prerequisites: None.

One-Day Complex Analysis. (Larsen)

If you poke a hole in the complex numbers, they all stay in one piece—unlike the real line. Thanks to this and other facts, calculus with complex numbers works a lot better than regular calculus. Did you know that if a complex function is differentiable, it is infinitely differentiable? And if it is also bounded, it must be constant? After this one-day crash course, you will see why calculus is what makes complex numbers special.

Prerequisites: Calculus.

One-Day Complex Analysis. (Larsen)

In week 4 we will have a class on the fundamental group, also denoted π_1. But if there is a π_1, surely there must be a π_2, and π_{17}! The "higher homotopy groups" are the natural generalization of the π_1, measuring the higher-dimensional complexity of a space. They are notoriously difficult to work with, but have some strange properties, for example they are all abelian.

Prerequisites: None.

Oversize Airline Luggage. (Tim!)

Most airlines have the same funny rule for how large your luggage can be: The total length + width + height can be at most a specified value. You show up to the airport with a bag that's too large. But then you have an idea: You decide to put your bag inside another bag, but at an odd angle so that the outer bag has a smaller length + width + height than the inner bag. Can you do it?

Prerequisites: Some calculus.

Partially Ordered Sets. (*Mike Orrison*)

Relations on and between sets are ubiquitous in mathematics, and one of the most useful kinds of relations is that of a partial order. Familiar examples include the less-than-or-equal-to relation on the set of real numbers, the divides relation on the set of positive integers, and the is-a-subset-of relation on the set of all subsets of a set.

This course will introduce some foundational ideas associated with partially ordered sets. We will then use this material to begin working with one of my favorite ideas in mathematics, Möbius inversion, which is an eye-opening generalization of inclusion-exclusion and a discrete analog of the Fundamental Theorem of Calculus.

Prerequisites: Matrix algebra (adding and multiplying matrices) and the ability to create and work with unions, intersections, and direct products of sets.

Probability & Paradoxes. (Larsen)

"This is a very *good* conjecture. It just happens to be false." –G.H.S.

Quick, think of a random number! But what does "random" actually mean? A subtle mathematical theory of *probability spaces* underlies concepts of randomness that may be familiar to you, such as correlation and independence. We'll define these concepts in rigorous ways and use them to prove some conjectures — but some of our conjectures might be false. In probability, many seemingly-rigorous lines of reasoning can lead to contradictory results, or "paradoxes". The Monty Hall problem is a famous example of such a paradox, since an incorrect answer can be very convincing. Fortunately, flawed logic can sometimes be just as interesting as correct logic, so in this class we will study both.

Prerequisites: None.

Problem Solving: Tetrahedra. (Misha)

In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

(This is a repeat of the class I taught in 2016.)

Prerequisites: None.

Problem-Solving: The Just Do It method. (Linus, at least)

You are navigating a 50-by-50 maze on your computer when suddenly your monitor shuts off. You don't remember where the maze's walls are. You only remember that you are in the top-left corner, the exit is in the bottom-right corner, and there is some path from you to the exit.

Is there some sequence of arrow key presses you can do that guarantees that you exit the maze at some point?

Learn the "Just Do It" method, a problem-solving technique in combinatorics that solves problems like this one by revealing that they are actually trivial.

Prerequisites: None.

PS: Inequalities. (Pesto)

High-school olympiads usually try to choose problems relying on as little prior knowledge as possible. In inequalities problems, they usually fail completely; training is necessary to solve most and sufficient to solve many of them. We'll go over the common olympiad-style inequalities, and solve problems like the following:

(1) Prove that if a, b, and c are positive and $ab+bc+cd+da=1$, then $\frac{a^3}{b+c+d}+\frac{b^3}{a+c+d}+\frac{c^3}{a+b+d}+\frac{d^3}{a+b+c}\geq\frac{1}{3}$.
(2) [USAMO 2004] Prove that if a, b, and c are positive, then $(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3)\geq(a+b+c)^3$

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to olympiad-style problems you'll have solved as homework the previous day.

Prerequisites: None.

Public-Key Cryptography. (J-Lo)

The e-commerce site parana.com has a problem: thousands of customers want to provide their credit card info, but anything sent over the internet can be intercepted by pirates!

So parana.com produces a scrambling function, which customers can use to hide their sensitive info. But here too there is a problem: for customers to be able to use it, this function must be public! So what's stopping the swashbucklers from just computing the inverse of this function and unscrambling all the messages?

"Public Key Cryptography" is a search for the best possible scrambling technique. Some common candidates include RSA, Diffie-Hellman, and Elliptic Curve Cryptography; in addition to discussing the pros and cons of each, we'll see how all of these are actually special cases of the same deeper problem.

Prerequisites: None.

Quadratic reciprocity. (Mark)

Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?
(1) "Is q a square modulo p?"
(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Prerequisites: Some basic number theory (if you know Fermat's Little Theorem, you should be OK).

Ramsey Theory. (Misha)

To a first approximation, Ramsey theory is about proving theorems that say "If we color all the whatsits of a sufficiently large thingy with yea many colors, then we will be able to find a monochromatic doodad."

We'll follow a meandering path between some results of this kind and results of a few other kinds. Topics of interest include upper and lower bounds, clever constructions that everyone should see at least once, and connections to number theory and geometry.

Prerequisites: None.

Random Matchings in Cubes. (*Fedya Manin*)

Congratulations! You've been commissioned to bake a cubical raisin pudding for Marisa and Alfonso's wedding. Unfortunately, after it's done, you find out that Alfonso is very particular about raisins, and wants them to be arranged in a perfectly rectilinear grid throughout the pudding. Fortunately, your pet mole Alfred is willing to burrow through the pudding and put every raisin in the right spot. Assuming the original placement of the raisins was totally random, how long will it usually take Alfred to do this? How does this time depend on the total number of raisins? The answer depends on the dimension of the pudding (and Alfred); dimension 2 is the hardest.

Prerequisites: Some calculus.

Rational Points on Elliptic Curves. (Shiyue)

Diophantine equations are equations in n variables with integer coefficients where you are looking for integer solutions. For instance, Fermat's Last Theorem says that the Diophantine equation $x^n + y^n = z^n$ for $n > 2$ has no nontrivial solutions.

The higher the degree of the equation, the harder it is to analyze. For equation of degree 2, we will show how to find all solutions using geometry of conic sections (ellipses, hyperbolae, and parabolae). Most of the class will focus on equations of degree 3, which correspond to a family of curves called elliptic curves. The structure of the solutions in this case is given by Mordell's Theorem, which describes the group structure of rational points on an elliptic curve. We won't prove the theorem in its most general form, but only focus on points of finite order. But even these special cases will help you get an idea of the tools needed for the full proof. More broadly, this course serves as an introduction to the field of arithmetic geometry, in which insights from algebraic geometry are applied to questions in number theory.

Prerequisites: Group Theory (cyclic groups, direct products, quotient groups; the Week 1 introductory class will be sufficient).

Representation Theory (1/2). (Aaron)

Representation theory is a field of math aiming to describe the symmetries of your favorite shapes. It can be pithily summarized as group theory meets linear algebra.

In the first two days, we'll introduce the basic notions from representation theory and describe the amazing coincidences satisfied by the representations of a group. These will be proved in week 2.

In days 3 and 4, you'll work on problems in class, applying the aforementioned coincidences to compute all the representations of symmetry groups platonic solids, and many other fun problems.

Prerequisites: Linear algebra and group theory. From group theory, familiarity with group actions is essential. One should also be familiar with symmetric groups, conjugacy classes, commutators, and quotient groups, among other things. From linear algebra, one should be familiar with linear transformations, eigenvalues, trace, and direct sum, among other things.

Representation Theory (2/2). (Aaron)

This class is a followup to the first week of representation theory, where we applied various theorems to classify all representations of various symmetry groups, such as those of platonic solids. Suitably motivated, this week, we'll explain why those theorems are true.

Prerequisites: Linear algebra, group theory.

Riemann's Explicit Formula. (Kevin)

Riemann's explicit formula is

$$\sum_{n=1}^{\infty} \frac{\pi_0(x^{1/n})}{n} = \int_0^x \frac{dt}{\log t} + \sum_\rho \int_0^{x^\rho} \frac{dt}{\log t} - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t}$$

where π_0 is the normalized prime counting function and ρ ranges over all nontrivial zeros of the Riemann zeta function.

We'll explore how a crazy formula like this could possibly arise, and we'll see how the Riemann zeta function's zeros end up so intricately connected to the prime numbers.

Prerequisites: Calculus.

Shuffling and Card Tricks. (*David Roe*)

Have you ever wondered how many times you need to shuffle a deck of cards until it is completely randomized? What does that even mean? How many card tricks do you know that rely on math rather than sleight of hand? Come to this class if you want to become mathematical card shark!

Prerequisites: None!

Simple Models of Computation. (Pesto)

Almost all programming languages are equally powerful—anything one of them can do, they all can.

[3] `http://www.xkcd.com/208`

We'll talk about less powerful models of computation—ones that can't even, say, tell whether two numbers are equal. They'll nevertheless save the day if you have to search through 200MB of emails looking for something formatted like an address.[3]

This is a math class, not a programming one—we'll talk about clever proofs for what those models of computation can and can't do.

Prerequisites: None.

Special Relativity. (Nic Ford)

Around the beginning of the twentieth century, physics was undergoing some drastic changes. The brand-new theory of electromagnetism made very accurate predictions, but it forced physicists to come to grips with a strange new truth: there is no such thing as absolute space, and there is no such thing as absolute time. Depending on their relative velocities, different observers can disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

In this class, we'll talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics has to be rebuilt to accommodate these observations. We will see how, as Minkowski said, "Space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." At the end, we'll also briefly look at how to revise the classical definitions of momentum and energy and see why we should believe that $E = mc^2$.

Prerequisites: None.

Spectral Graph Theory. (Laura)

Spectral graph theory is a way of turning problems about graphs into linear algebra by associating a matrix to a graph (called the adjacency matrix) and studying its eigenvalues. We'll look at a classic application of this technique to Moore graphs of girth 5, which can be defined as regular graphs (all vertices have the same number of neighbors) where any two non-adjacent vertices have a unique common neighbor. Surprisingly, the only possible degrees such a graph can have are 2, 3, 7, or 57 (and it's not even known whether one of degree 57 exists)! The proof is an elegant application of linear algebra to studying graphs!

Prerequisites: Basic linear algebra. Familiarity with bases, eigenvalues, eigenvectors, and trace.

Statistical Modeling. (Sam)

This is a weeklong crash course in statistical modelling. We'll cover a general framework for statistical analysis and get our hands dirty analyzing data. We'll mostly focus on linear regression and its extensions, which cover a surprisingly general class of statistical models (including what are often the number 1 and 2 methods on "top ten algorithms for machine learning" lists!). We'll also answer the following questions:

- What was going on with those lines-of-best fit I had to draw in 7th grade science?
- What even is a p-value? Can statisticians agree?
- Can I be more right by agreeing to be wrong on average?
- Does LSD improve your ability to do math (spoiler alert: the answer to this is a resounding no. But we'll work with data from a surreal experiment in the 60s where someone asked this question!)

Fair warning: this is a bona-fide useful class. During homework and lab sessions you'll learn R, a powerful language for statistical analysis. We'll see some statistical theory along the way, but the main goal of this class is for you to leave comfortable with some fundamental and ubiquitous techniques for analyzing data.

Prerequisites: Basic probability: you should know what it means for random variables to be normally distributed and independent, or talk to me!

Street Fighting Mathematics. (*Sanjoy Mahajan*)

Street Fighting Mathematics is the art of guessing results and solving problems without doing a proof or an exact calculation. Tools for this include extreme-cases reasoning, dimensional analysis, successive approximation, lumping, and pictorial analysis. And applications include mental calculation, solid geometry, musical intervals, infinite series, and fluid mechanics.

Prerequisites: None.

Stupid Games on Uncountable Sets. (Susan)

Let's play a game. I'll pick a countable ordinal number, then you pick a bigger one, and then I'll pick one that's even bigger. We'll continue this for infinitely many turns, and when we're done we'll check to see who's won. Sound like fun?

As it turns out, these "games" can be a powerful tool for studying important ideas in set theory. In this class, we will learn about the ordinal numbers, clubs, and stationary sets. We will prove the existence of a stationary set which is also co-stationary, and see how this results in a game which has a clear winner and loser, but no winning strategy.

Prerequisites: None.

Symmetries and Polynomials. (Aaron & Apurva)

In this course, you'll discover how to solve the cubic and quartic equations. You'll then work out how solving the cubic and quartic equation relates to symmetries of platonic solids and take a peek at Galois theory.

The class will be IBL, specifically, we'll work on problems in class, instead of a traditional lecture format. The basic problem set is 2 chili, but there will be enough optional problems for those who want to make it (much) harder.

Prerequisites: Familiarity with Group theory will be useful, but not necessary.

Systems and Signals Analysis. (Jeff)

One of the simplest ways to model a real world phenomenon is to model it with a function of a single variable. For instance, if I want to understand a piece of music, I could use the function

$$\rho(t) := \text{the air-pressure density of the sound wave that passes through my ear at time } t.$$

Or if I wanted to give you an understanding of a lighthouse in the distance, we could study the electromagnetic field strength as a function of time. The list goes on. We'll call all of these types of functions *signals.*

While this provides a simple description of things we see and hear in the real world, this does not give a good description of how humans observe the world. With most phenomenon, people perceive the frequency content of the signal instead of the time content: when we listen to music, we perceive pitches instead of air density. When we watch a movie, we perceive colors at different intensities instead of the electromagnetic wave.

One of the mathematical tools we'll develop in this class is the *Fourier transform*, which takes a signal and produces the frequency data of that signal. We'll then talk about *systems,* which describe models which modify signals — think like how a mute on a trumpet changes the tones the trumpet produces. Finally, we'll classify a wide variety of systems from simple measurements.

Prerequisites: You should know how to integrate and differentiate trig functions, as well as understand and give a solution to the differential equation $\frac{df}{dt} = f$. Complex numbers and Euler's formula are useful, but not strictly necessary.

Teaching Computers to Read. (*Greg Burnham*)

Human language is tantalizingly close to a formal system. We *feel* like there is a clear relationship between the words we express and the information these words convey. At worst, we just need to be a little more verbose and explicit. If this intuition is true, then we should be able to write computer programs to perform linguistic tasks – like reading a document and answering questions about it. But we've been trying to write such programs for 50 years, and the results are mixed at best.

This class will be a quick survey of some interesting topics in the (very broad) field of computational natural language understanding. We'll try to motivate why it's so difficult to write computer programs capable of performing linguistic tasks and then describe what tasks computers *can* currently perform, focusing on how recent algorithmic and technological progress has allowed for improved performance. We will conclude by noting that the big problems remain unsolved and speculating on what might be necessary for the next steps forward.

As a teaser, here is a simple example illustrating why computational language understanding is hard. Consider the following two sentences, which differ only in the last word:

"The cat caught the mouse because it was clever."

"The cat caught the mouse because it was careless."

What does the pronoun "it" refer to in each sentence? Humans share a clear intuition about the right answer to this question. And yet, consider what it would take to write a computer program with this same capability. That's the problem in a nutshell.

Prerequisites: None!

The Art of Live-TeXing. (Aaron)

Have you wanted to type math notes in real time but were too intimidated by the prospect? If so, this is the class for you. I'll describe some tips and tricks to make live-TeXing a feasible, and even enjoyable, experience. Students are encouraged to bring their laptops to class.

Prerequisites: Minimal prior experience with latex would be helpful.

The Class Number. (Kevin)

A special case of Fermat's Last Theorem says that there are no nonzero integers such that $x^3+y^3 = z^3$. If ζ_3 is a third root of unity, we can factor this as $(x + y)(x + y\zeta_3)(x + y\zeta_3^2) = z^3$. If we know that factorization is unique once ζ_3 gets involved, we can make a lot of progress towards proving this case!

If we play this game with ζ_{23}, we don't have unique factorization. But by studying the *class number*, a finer measure of the structure of rings of integers, we can overcome this obstacle!

Can we, in fact, prove all of Fermat's Last Theorem this way? We're still doomed with ζ_{37}, but at least we tried.

This class will pick up right where Shiyue's Algebraic Number Theory class stops. If you're not sure what that means, come talk to me after Week 1 ends.

Prerequisites: Week 1 Algebraic Number Theory or equivalent.

The Continuum Hypothesis. (Susan)

In 1874, Georg Cantor proved that there are more real numbers than natural numbers, essentially proving for the first time that there are different sizes of infinity. This result suggests an obvious follow-up question: is there an infinity between the size of the naturals and the size of the reals, or are these infinities in some sense right next to each other? The statement that there are no intermediate infinities became known as the continuum hypothesis.

For almost a hundred years, mathematicians struggled to either prove or disprove the continuum hypothesis. Finally, in 1963 Paul Cohen proved that the continuum hypothesis was independent of Zermelo-Fraenkel set theory. It could neither be proved nor disproved from our standard set theory axioms.

But how do you prove that you can't prove something? This proof involves a bizarre technique known as forcing, in which you take a tiny toy universe, add some objects, and take the closure under being-a-set-theoretic-universe. In this class we'll build the machinery required for forcing, and use it to prove the independence of the continuum hypothesis.

Prerequisites: Some knowledge of the ordinal numbers, particularly ω_1. Stupid Games on Uncountable Sets could function as a prereq.

The Erdős Distance Problem. (Ben)

Grab a piece of paper, and mark a bunch of points on it. Now, grab a ruler, and check how many distances you can make by measuring the distance between two of your points. The Erdős Distance Conjecture, very roughly, says that the number of these distances will grow almost linearly in the number of points, no matter how carefully you try to keep the number of distances small. The conjecture was first formulated in 1946 and was established to be true only in 2011, with quite a few intermediate results.

In this course, we'll explore a few of these intermediate results, starting with Erdős's result from the 1946 paper formulating the conjecture and going up to a result from the 1990s. This later result will involve a few tools from graph theory and many useful analytic techniques. Although we won't be able to get to the full proof of the conjecture, we will see how a wide variety of mathematical ideas can be brought to bear on one particular problem. We will also see some of the evolution of methods in this problem, with later proofs relying on ways of attacking the problem that the first authors hadn't realized yet.

Prerequisites: None; we will use some graph theory but I'll cover what we need.

The Fundamental Group. (Larsen)

What do a circle, a square, and the Republic of South Africa all have in common? They all have a hole in the middle! We may be able to give a name to the hole (e.g. "Lesotho") but the hole isn't a part of the original shape itself, despite still somehow being an intrinsic feature.

If X is topological space (a shape, with continuity properties), then there is is a special group called the *fundamental group of* X or $\pi_1(X)$, with algebraic properties depending on the topological characteristics of X. If X doesn't have any holes in it (e.g. if X is a line), then $\pi_1(X)$ is the trivial group, but if X is a circle (or South Africa), then $\pi_1(X)$ is the group of integers $\mathbb{Z}$. The fundamental group is very useful as an invariant, and we will also use it to prove interesting facts, for example: If you have a map of Colorado (of any size, possibly warped or folded) and the map is in Colorado, then there is a point on the map representing its own exact location.

The fundamental group is the first part of a wider field called Algebraic Topology.

Prerequisites: Group Theory.

The Maths of Peppa Pig. (Linus)

Let's examine some of the advanced mathematical topics featured in the short film series Peppa Pig. This class will be a disjointed, ill-structured compilation of random mathematical thoughts I have while watching Peppa Pig next week.

Prerequisites: Get ready for algorithms and maybe topology.

The Outer Automorphism of S_6. (Aaron)

What do mystic pentagons, platonic solids, pentads, nobbly wobblies (yes, the dog toys), and linear fractional transformations have to do with each other? They all describe the outer automorphism of S_6!

In this IBL class, you'll work through problems together to discover how these are all examples of the same phenomena, and even prove that S_6 is the only symmetric group with an outer automorphism.

Prerequisites: Group theory.

The Quantum Spring. (Apurva)

How do groups and algebras show up in physics? What does the set of 2×2 trace 0 matrices have anything to do with the quantum mechanics? How do Linear Operators create and annihilate particles? Why is Linear Algebra the answer to all your prayers?

Prerequisites: Linear Algebra, specifically, Eigenvalues and Eigenvectors.

The Stable Marriage Problem. (Marisa & Shiyue)

N single men and N single women want to pair up and get married. These are their names and preferences:

- Jeff: Susan > Shiyue > Jessica > Viv > Ania
- Ben: Ania > Susan > Viv > Shiyue > Jessica
- Agustin: Ania > Viv > Shiyue > Susan > Jessica
- Pesto: Jessica > Shiyue > Viv > Ania > Susan
- Tim! : Jessica > Ania > Susan > Viv > Shiyue

- Ania: Jeff > Pesto > Tim! > Agustin > Ben
- Jessica: Agustin > Ben > Jeff > Tim! > Pesto
- Susan: Agustin > Pesto > Tim! > Ben > Jeff
- Shiyue: Agustin > Tim! > Jeff > Ben > Pesto

- Viv: Agustin > Pesto > Ben > Tim! > Jeff

Is it possible to make everybody happy? Obviously not since almost everybody wants to marry Agustin. But is it possible to at least create a stable situation? For instance, it is a bad idea for Agustin to marry Shiyue and for Viv to marry Pesto, because then Agustin and Viv would prefer each other rather than staying with their partners, so they will run away together. How can we at least avoid having a run-away couple? Is there more than one way to do it? What is the best way to do it? And what if Shiyue and Viv decide that marrying each other is better than marrying Agustin?

Prerequisites: None.

Topological Tverberg's Theorem. (Viv)

The convex hull of a set X of points is the smallest set C containing X and all lines between points in C. Given four points in the plane, I can *always* partition them into two sets whose convex hulls intersect. And if I live in any Euclidean space and I'm given enough points, I can do the same thing. And if, before starting my partitioning process, I draw my convex hulls however I like, rather than through such silly procedures as "following the definition", I can *still* do the same thing. And if I'd like to split my set into arbitrarily many subsets instead of two, I can still do the same thing... provided I'm partitioning into a prime power number of subsets.

If this seems really weird, that's because it's really weird. We'll prove these statements and discuss the recent astonishing counterexample to the composite case.

Prerequisites: Linear Algebra.

Trail Mix. (Mark)

Is your mathematical hike getting a little too strenuous? Would you like to relax a bit with a class that offers an unrelated topic every day, so you can pick and choose which days to attend, and that does not expect you to do homework? If so, some Trail Mix may be just what you need to regain energy. Individual descriptions of the four topics follow.

Trail Mix Day 1: Perfect Numbers. Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called *perfect*, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, called Mersenne primes — a search that has largely been carried out, with considerable success, by a far-flung cooperative of individual "volunteer" computers.

Prerequisites: None.

Trail Mix Day 2: Intersection Madness. When you intersect two ellipses, you can get four points, right? So why can't you get four points when you intersect two circles? Well, actually you can, and what's more, two of the four points are always in the same places! If this seems paradoxical (and, I hope, interesting), wait until we start intersecting two cubic curves (given by polynomial equations of degree 3). There's a "paradox" there too, first pointed out by the Swiss mathematician Cramer in a letter to Euler, and the resolution of that paradox leads to a "magic" property of the nine intersection points. If time permits, we'll see how that property (known as the Cayley-Bacharach theorem) gives elegant proofs of Pascal's hexagon theorem and of the existence of a group law on a cubic curve.

Prerequisites: None, although a little bit of linear algebra might show up.

Trail Mix Day 3: Integration by Parts and the Wallis Product. Integration by parts is one of only two truly general techniques known for finding antiderivatives (the other is integration by substitution). In this class you'll see (or review) this method, and two of its applications: How to extend the factorial function, so that there is actually something like (1/2)! (although the commonly used notation and terminology is a bit different), and how to derive the famous product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \quad ,$$

which was first stated by John Wallis in 1655.

Prerequisites: Basic single-variable calculus.

Trail Mix Day 4: The Nine-Point Circle. There is some beautiful geometry hidden in and around every triangle. In particular, there are several points that can qualify as "centers" of the triangle, but that are different unless the triangle is equilateral. One of those points is the center of a circle that goes through nine related points, so it's not surprising that it's called the nine-point circle. If you haven't seen this (and the Euler line) but you like plane Euclidean geometry, you're in for a treat.

Prerequisites: None.

Trees! (Shiyue)

What do a tree and a language with only two letters have in common? How can you have a bigger group as a subgroup of a smaller group? Can you decompose a ball into a finite number of sets and reassemble them into two balls identical to the original? If you think that the group axioms are boring, how else could you define a group? Is a subgroup of a free group free? What?? Free? What am I talking about anyways?

Here is the magic word that will chain together all these questions: trees! All of this seeming nonsense falls into a serious math subject called "geometric group theory", where the interplay between geometric spaces and groups reveals big secrets about the world that we live in and the groups that you happen to ponder upon. We will start with trees, graphs, free groups of finite rank, group action on trees, Cayley Theorems, and tons of examples such as Coxeter groups. And in the process, we will answer all of the above questions!

Prerequisites: Definition of a group; some additional group theory knowledge is useful but not necessary.

Triangulations and Flip-Graphs. (Viv)

Let's say I have two triangulations of a polygon and I want to have some way of getting from one to the other. I can do what's called a diagonal flip, where I remove one diagonal, look at the quadrilateral I've just made, and fill in the other diagonal of that quadrilateral. Can I get from my first triangulation to my second one using only these flips? How many flips will I need? And, most importantly, what does any of this have to do with hyperbolic geometry? We will explore the answers to all of these questions!

Prerequisites: None!

Tropical Plane Curves. (Shiyue)

Happy Halloween! We saw some creature named tropical geometry going around TAU. But what is that? Tropical Geometry is an emerging subfield of algebraic geometry. Technically speaking, it provides a modern degeneration technique to replace algebraic varieties with combinatorial objects. Classical algebraic geometers study the interplay between polynomials and their zeros. But zeros could be hard to study. In 1990s, people discovered that transforming all our polynomials into tropical semiring, which is $\mathbb{R} \cup \{\infty\}$ with the usual addition and multiplication replaced with taking the min/max and addition, will turn polynomials into piecewise linear functions and their zeros into polyhedra. Now we can do combinatorics to tackle these algebraic geometry problems! This course will cover basic arithmetic in tropical semiring, tropical plane curves, Bezout's Theorem for tropical plane curves, etc.

Prerequisites: Some ring theory will be useful.

Uranus has at least 2 storms. (Jeff)

True Life Story of Mathematics: At UC Berkeley, we're required to take a qualifying oral exam in our second year. The exam is moderated by 3 mathematicians, and an outside member not from the mathematics department whose job is to make sure the 3 faculty members on your committee are treating you fairly. They're also suppose to lob you some easy questions, like "what made you study this field," or "what is your favorite theorem?"

One of my subjects was differential topology, which is what I mostly study now. My outside committee member asked me the following wonderful question:

> "Describe a result from your field in a way which anybody can understand."

I feel like differential geometry is full of these types of results, so I brought out my favorite result: that a generic smooth function on the sphere has 2 critical points. I tried to package this in a way which was simpler to state.

> "On any planet, at any given moment, there are at least 2 storms."

I thought this was a clever way of stating the theorem. By replacing the construction of gradient vector field with wind, and by classifying critical points of a vector field as a storm, I figured I had discovered some elegant way of getting at the crux of the field. To which, the professor (who was a tenured member of the astronomy department) looked at me and stated.

> "Uranus has no storm systems."

I ended up passing my qual, but not before the 3 mathematicians members of the committee spent 10 minutes trying to convince this astronomer that Uranus did indeed have 2 storms.

In this class, we'll prove that Uranus has at least 2 storms, and many other things about differential manifolds, by using the tools of *Morse Homology*, which provides incredibly strong inequalities for the number of critical points on a manifold.

Prerequisites: Linear algebra is a must. Comfort with multi-variable calculus, or faith in pictures will be helpful. This is the full-fat, homological version of the discrete Morse theory class we covered in Combinatorial topology, so attending that class will give a nice framing for this story, but is not strictly necessary.

Visualizing Groups. (Sara)

If you know the basics of group theory, but want to gain a better intuitive understanding of groups, this is the class for you. We will focus on *pretty pictures*, more commonly referred to as *Cayley graphs*. On Day 1, we will define Cayley graphs, learn how to visualize subgroups and cosets, and learn a simple way to recognize normal subgroups using a Cayley graph. On Day 2, we will look at direct and semidirect products. Semidirect products are very cool. They allow you do things such as construct nonabelian groups out of abelian groups. Without semidirect products, you might find yourself saying: "Bippidy bazinga! I have a rotation group of a square (order 4) and a reflection group of a square (order 2), and I want to put them together to make the symmetry group of a square D_8, but the direct product isn't powerful enough to product a group as janky as D_8. Wow, this really makes me sad." If you want to learn more about groups and not be sad, consider taking this class.

Prerequisites: Group theory and a little graph theory. No knowledge of Cayley graphs or semidirect products necessary.

Wedderburn's Theorem. (Mark)

Have you seen the quaternions? They form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$, with basis $, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j.$$

Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).

Prerequisites: Some group theory and some ring theory; familiarity with complex roots of unity would help.

What *should* integration be? (Steve)

We know how to find the area under the graph of a "reasonable" function: first consider approximating the area as a bunch of thin vertical rectangular strips, and now just compute the limit as those strips get thinner and thinner (= as our approximation to the area gets more and more precise). For reasonable functions, this limit exists and tells us the area under the graph. This is the **Riemann integral**.

But what about *unreasonable* functions? Consider, for example, the function D given by setting $D(x) = 0$ if x is irrational and $D(x) = 1$ if x is rational. If you try to graph D, you'll quickly see why it's not integrable via Riemann sums. However, there is a heuristic argument (which we'll see in class) that the area under D should in fact be well defined (and equal 0). So we ask: is there a method of integration more general than the Riemann integral, which will let us tackle weird functions like D?

Similarly, the Fundamental Theorem of Calculus tells us that a wide class of antiderivatives can be given by Riemann integrals; however, it turns out that there are functions which have antiderivatives but which are not Riemann integrable. So again, we can ask: is there a method of integration which will let us find antiderivatives of a broader class of functions than the Riemann integral?

It turns out that the answer is *yes*. In this class we'll dive into this strange world. We'll focus on presenting the fundamental ideas behind generalizing the Riemann integral, with a specific focus on two generalizations: the **Lebesgue integral** and the **gauge integral** (or *Henstock-Kurzweil integral*).

Prerequisites: Calculus - specifically, comfort with integrals, Riemann sums, and the epsilon-delta definition of limit.

What is a "Sylow" anyways? (Ben)

Recently, in 2004, the Classification of Finite Simple Groups was completed, finishing off decades of work spread out over hundreds of articles. We won't be able to prove this theorem in class (among other things, I don't know if anyone in the world knows all of the details), but we can look at one of the useful tools involved in the classification!

The Sylow Theorems are a few results arising from group actions, proven by the Norwegian mathematician Peter Sylow. These theorems provide a way to show that all groups of order 15 are cyclic, that there are no nonabelian simple groups of order less than 60, among other consequences.

What are these theorems? How can we use them? What are simple groups? What are a few other facts about groups that come in handy when looking for simple groups? What are some fun facts about Sylow, and how do you pronounce his name? These are some of the questions we'll investigate in this class.

Prerequisites: Group Theory.

Would I ever Lie Group to you? (Apurva)

Lie Groups are groups which are also manifolds. The easiest examples of Lie groups come from Linear Algebra as symmetries of vector spaces. We'll study these Matrix Groups and understand their connections with their geometry and physics.

Prerequisites: Linear algebra, Group Theory.

Wreath Products. (Viv)

You may have heard of a couple of ways to take products of groups: for example, directly, or semi-directly.

Wreath products are another way! They're also my favorite way. We'll define them and discuss examples, like the lamplighter group or the Rubik's Cube group.

Prerequisites: Group Theory.

Yes, you can square the circle. (Misha)

Since ancient times it has been an open problem to use compass and straightedge to construct a square and a circle with equal area.

That is, it has been an open problem until it was proved to be impossible in 1882.

But we'll do it anyway: in the hyperbolic plane.

Prerequisites: None.

Yes, you can trisect angles. (Ben)

Trisecting arbitrary angles with compass and straightedge alone has been known to be impossible since the development of Galois Theory.

On the other hand, since antiquity we have known how to trisect arbitrary angles, and divide them into 5 pieces (or six, seven, and so on) by use of a construction called a "quadratrix." This isn't a curve we can construct using compass and straightedge, but use of it in mathematical arguments dates back to at least 400 BCE.

In this class, we will learn about what this curve is and about its history, and how to use it to trisect angles. If we have time, we'll discuss other uses of the quadratrix and how we can approximate it with compass and straightedge.

Prerequisites: None.

Colloquia

Accidental Mathematics. (*Matt Stamps*)

Growing up, I always loved learning about world-changing scientific breakthroughs that were discovered by accident. Penicillin, artificial sweeteners, X-rays, and synthetic dyes (to name just a few) were all stumbled upon by scientists with other goals in mind. More recently, I have come to wonder why anecdotes about accidental discoveries in mathematics are not as commonplace. Is it because mathematicians like to inflate their egos by attributing all of their theorems to their superior intellects? Is it because society prefers to maintain the popular perception of mathematics as a series of dry logical deductions speckled with astonishing strokes of genius? Or is it is because of something else entirely? Regardless of the answer, I argue that mathematics happens accidentally all the time. In this talk, I will share some accidental discoveries from my own work involving Penrose tilings, circle packings, chordal graphs, lecture hall partitions, lattice polytopes, and polynomial rings.

A hex on your fixed points! (*Katie Mann*)

This talk is about two of my favorite things:

(1) board games, and
(2) topology.

The board games part is about Hex, a 2-player game that uses a hexagonal grid. It's easy to learn and to play, but hard (at least for a human!) to figure out a winning strategy. The topology part is a famous theorem of Brouwer that says every continuous map from a disk to itself has a fixed point. Or, as I like to put it: if you stir a bowl of soup, some molecule in the soup always ends up in exactly the same place it started. Brouwer's theorem has lots of deep and beautiful proofs and consequences. But we're going to prove it using Hex.

Building machines that learn and think like people. (*Josh Tenenbaum*)

Increasingly our lives are filled with "artificial intelligence technology": machines that do things we used to think only humans could do. But we don't yet have any real AI. We don't have machines with anything like the flexible, general-purpose commonsense intelligence that lets humans do everything they do to get around in the world. What's missing, and how could we build it?

Recent successes in artificial intelligence and machine learning have been largely driven by methods for sophisticated pattern recognition, including deep neural networks and other "deep learning" methods. But human intelligence is more than just pattern recognition. At the heart of human common sense is our ability to model the world: to explain and understand what we see, to imagine things we could see but haven't yet, to solve problems and plan actions to make these things real, and to build new models as we learn more about the world. I will talk about our recent work attempting to reverse-engineer these capacities, drawing on several kinds of mathematical and computational tools: probabilistic (Bayesian) inference, probabilistic programming, program synthesis, and video game engines. I will show examples of how these tools let us build mathematical models of human intelligence, and also make AI systems that are smarter in more human-like ways.

Change of Perspective in Mathematics. (*Mike Orrison*)

Change of perspective is ubiquitous in mathematics. Consider, for example, change of coordinates, Bayes' Theorem, combinatorial proofs, or even the simple fact that $2 + 3 = 3 + 2$. In this talk, I'll offer some personal reflections on the unifying role that change of perspective plays when we learn, share, create, and discover mathematics. I'll also describe how conversations over the years with teachers, undergraduates, and my own children have shaped my understanding of the power of change of perspective.

Dynamical Limits: Sometimes a Picture is Worth Zero Words. (*Scott Kaschner*)

A dynamical system is a mathematical system that evolves over time. In this talk, I will give the history and current status of a project involving a sequence of dynamical systems; it's literally a mathematical system that evolves over time evolving over time. There will be a lot of fractal geometry and, despite my efforts to avoid both, some trigonometry and C++ code. Dynamics researchers are often scientifically inspired by computer-based experimental research; this talk will illustrate how non-directed experimentation in mathematics can lead to interesting (and proved!) mathematical results.

***Not* the Continuum Hypothesis.** (Steve)

After Cantor showed that the reals are uncountable, a natural question — which he asked — becomes: "Is there anything 'between' the reals and the naturals?" That is, could there be a set of real numbers which is uncountable but still not as big as $\mathbb{R}$ itself? (In symbols: is $2^{\aleph_0} > \aleph_1$?)

The claim that there is no such "intermediate cardinality" became known as the **continuum hypothesis** (CH), and a lot of early effort went into proving or refuting it (by Cantor and others). The eventual solution, by Paul Cohen in 1963 (following work of Kurt Godel in 1951), was incredibly surprising, beautiful, and fundamentally changed set theory and indeed all of logic.

...But that sounds hard, so let's make Susan do it. Instead, I want to talk about something else. Long before the problem itself was solved, an important development occurred when people realized that (in a precise sense) any counterexample to CH would have to be *really ugly*. This marked the birth of **descriptive set theory** — the study of shapes that aren't quite as bad as they could be. Descriptive set theory provides techniques for showing that "reasonable definable" sets of real numbers have nice properties, such as *measurability* (a technical property which prevents Banach-Tarski-paradox-y stuff from happening) and *not being a counterexample to CH.* The key tool here is the notion of a *determinacy principle*: that "niceness properties" correspond to certain games having winning strategies for one player or the other (= being determined), and so we can show that a set of reals has nice properties by showing that various games related to it are determined.

I'll present the initial descriptive set theory of the continuum hypothesis, via the **perfect set game**.

How to Win the Lottery. (*David Roe*)

From 2005 through 2012, three groups in Massachusetts earned millions of dollars playing a lottery game called Cash Winfall. I'll describe how the game worked, explain why it was exploitable, give connections with projective geometry and share stories from my participation in one of the pools. I'll also reaffirm the common wisdom that almost every other lottery is not worth playing.

Is Mathematics Biology's Next Microscope? (*Angela Gallegos & Kamila Larripa*)

Mathematical biology is poised for explosive growth as biology becomes more quantitative. The need for mathematical and computational approaches to organize this information is acute. The best models not only shed light on how a process works, but might predict what may follow or propose new experiments to try. The mathematics can be relatively simple; it is the novel application that allows us to peek into a biological system and suggest directions for future experiments. We will discuss some of our favorite examples of mathematical modeling applications in our work including cancer treatment and population biology.

Quantum Hack. (J-Lo)

If someone were to develop a viable, decently-sized quantum computer, internet security as we know it today would cease to exist. Come to learn what mathematical computations a quantum computer can do that a classical computer can't — and why there may still be hope for our non-quantum internet to become quantum-secure.

Quantum Mathematicians. (*Scott Strong*)

Mathematical physics is a branch of applied mathematics dealing with physical problems. Mathematical physicist Robbert Dijkgraaf, who was interviewed by Numberphile in 2017 [1], discusses how the first quantum revolution has left mathematicians catching up with both its concepts and language. In fact, he asserts the need for "quantum mathematicians." Perhaps he says this because we stand at the precipice of a paradigm shift that will bring about great change in the way we work with information acquisition, communication, and simulation. [2]

While mechanics and electromagnetism tend to be compulsory courses due to their alignment with single and multivariate calculus, quantum mechanics is often seen as a specialized class. For physicists it's perceived as heavily historical, while it is considered highly technical by mathematicians. Since it is important that young mathematicians gain interest in the field, I will provide an experimental lecture that aims to start from knowledge of single-variable calculus and create models leading to a quantum mechanical prediction.

[1] `https://youtu.be/m6rfpQXzXu0`
[2] `https://youtu.be/kcTGzE_AtBc`

The Lost Mentor and the Sleepy JC. (Misha)

Suppose that a mentor is lost on a hike in the woods. If they call you on the phone, can you give them directions, even if you don't know where in the woods they are? (And even if you don't have a map?)

The Most Beautiful Equation in Math. (*Po-Shen Loh*)

What is e? What is π? What is i? What is 1? What is 0? What do they all have to do with each other, and why?

The Threshold Density Theorem. (*David Perkinson*)

Imagine a system in which particles of energy are randomly added to sites in a finite network. When the number of particles at a site reaches a certain threshold, the site becomes unstable and fires, sending one particle to each of its immediate neighbors in the network. These neighbors, in turn, might then exceed their thresholds, so adding a single particle can set off a cascade of firings. We wait until all firing activity has subsided before randomly adding the next particle. Since the network is finite, we will eventually reach a point at which a particle is added and the system is no longer stabilizable. What total amount of energy do you expect to be in the system then? This talk will present Lionel Levine's threshold density theorem, which provides an answer to this question.

Two equals one: Street-fighting mathematics and science for better teaching and thinking. (*Sanjoy Mahajan*)

With traditional science and mathematics teaching, students struggle with fundamental concepts. For example, they cannot reason with graphs and have no feel for physical magnitudes. Their instincts remain Aristotelian: In their gut, they believe that force is proportional to velocity. With such handicaps in intuition and reasoning, students can learn only by rote. I'll describe these difficulties using mathematical and physical examples, and discuss how street-fighting mathematics and science—the art of approximation—can improve our thinking and teaching, the better to handle the complexity of the world.

What is a q-analog? (*Maria Gillespie*)

A q-analog of a mathematical quantity P is an expression involving q such that setting $q = 1$ yields the original quantity P. While the definition is a simple one, the theory of q-analogs is a new and growing field of study that connects many different areas of mathematics. We'll discuss some common q-analogs and their many appearances in mathematics: in combinatorics, number theory, algebra, quantum physics, and the popular game of SET.