# CLASS DESCRIPTIONS—MATHCAMP 2017

## Classes

**A Crash Course in Axiomatic Probability.** (Sam)

This class will be a brisk walk through some of the most fundamental topics in probability. We'll start from Kolmogorov's axioms and build our way through formal notions of independence, conditional probability, and random variables. By the end of the course, we'll have sufficient tools to prove some high-level theory and asymptotic results! We will mostly focus on discrete probability—so if you haven't seen much calculus that's fine—but we will touch briefly on continuous random variables. Throughout the class, we'll look at a few fun "applications" of probability. Typically, these will be applications to other areas of mathematics (like Graph Theory!).

*Prerequisites:* You should know the following words and corresponding symbols: (finite and countable) union, intersection, complement, and partition. Calculus will help for 20 minutes of this course. We'll also do one or two examples from graph theory, where knowing very basic terminology will be helpful (edge, vertex, clique, independent set, complete graph), but these are just fun asides and can be safely ignored.

**Advanced Complex Analysis.** (Yuval)

This class is mostly a continuation of Mark's Functions of a Complex Variables class, in which we'll be talking about some of my favorite theorems in Complex Analysis. The main goal is to understand various geometric properties of analytic functions: in particular, it will turn out that analytic functions are "conformal", meaning that they preserve all local geometric structure. This means that we expect analytic functions to be extremely rare and special, and we will prove that indeed they are: in fact, we will write down a list of all analytic functions from the unit disk to itself. Finally, we will see the Riemann Mapping Theorem, which tells you that everything I said above is wrong; analytic functions are actually ubiquitous, and we can find them *everywhere*.

Along the way, we'll see many other useful and beautiful theorems, which will allow us to count zeroes, compute integrals, and find maxima.

*Prerequisites:* Functions of a Complex Variable (both weeks).

**Algorithms to Generate Randomish Numbers.** (Sam)

Suppose you wanted to generate a truly random sequence of numbers. You sit down to start flipping a coin for forever, but shortly realize that this is boring. Instead, suppose you wanted to generate a sequence of numbers that looked random (so that you could fool your friend Tim!, say, into thinking that you were playing rock-paper-scissors randomly). How might you do this?

In this class, we'll introduce some of the methods for generating pseudorandom numbers. We'll also look at why some of the, were terrible!

*Prerequisites:* Basics of modular arithmetic .

**All Things Manifoldy.** (Apurva)

Who said that mathematicians are not real doctors, we perform surgeries all the time. In this class we'll take baby steps towards understanding manifolds. We'll learn some of the uber awesome techniques invented by topologists to study manifolds. We will perform surgeries on manifolds and do origami using simplices, and by the end of the class you'll be able to visualize (some) manifolds in higher dimensions.

Incidentally when Einstein tried to combine special relativity with Newton's gravity nothing seemed to work. It took him a decade to finally realize a beautiful solution to the conundrum: our universe is a 4 dimensional manifold and gravity is a measure of how the manifold curves. But what is a manifold?

*Prerequisites:* None.

**Analytic Number Theory.** (*Djordje Milicevic*)

Did you know that a large majority of the numbers with a quadrillion digits have at least 30 but no more than 40 prime factors? (Really.) If you have a large prime, how far is the next one anyway? Is it more likely to end in digit 1 or 7? And how hard — or how important — can it be to locate all zeros of a single function (the Riemann zeta-function) that you can earn a million dollars for doing so, and what does this have to do with throwing a fair coin?

Hardly any collection of questions appears more disparate than these, but actually they all have two thing in common: 1) they combine the beautiful and intricate multiplicative structure of the integers with the concepts and tools of calculus, the study of continuous change, and 2) we will talk about all of them this week! We will learn about arithmetic functions and their average orders, techniques of analytic number theory, characters, Riemann zeta function, and prime number theorem, and we will survey some landmark and contemporary developments in analytic number theory.

*Prerequisites:* A little familiarity with number theory (divisibility, primes) and with single-variable calculus. You should know what derivatives and integrals are, but you do not have to have a lot of experience working with them.

**Ancient Greek Mathematics.** (Yuval)

You probably know that Pythagoras proved the Pythagorean Theorem, and you may have heard that Euclid proved the infinitude of the primes. But did you know that Eudoxus basically invented the Dedekind Cut definition of the real numbers 2500 years before Dedekind, or that Archimedes was basically doing calculus 1500 years before Newton and Leibniz invented it? You may have heard that Archimedes yelled "Eureka" in the bathtub when he discovered the law of buoyancy that controls when things float and when they sink, but did you know that he subsequently used this idea to prove arguably the most intricate theorem before the 17th century? And you may have heard about the controversy around Euclid's famous "Parallel Postulate," but what does this postulate actually say?

In this class, we'll be learning about Ancient Greek math, from two separate perspectives: what they knew, and how they thought about it. In both cases, we'll see some surprises: they knew much more than you probably think they do, and they thought about math in a fundamentally different way from how we think about it today.

**Note:** This class has required homework, since I will be assigning you a (fairly small) amount of reading each day, which we will discuss the following day. In all cases, we will be reading bona fide Ancient Greek math; if you don't put in the time to carefully read and think about these, you will get very little out of the class the following day.

*Prerequisites:* None.

**And That Is Why Birds Shouldn't Drink Alcohol.** (Beatriz)

In the words of Kakutani, "A drunk man will eventually find his way home, but a drunk bird may get lost forever." In this class we will study random walks on a $d$-dimensional lattice $\mathbb{Z}^d$. We will prove that for $d \leq 2$, the random walk on $\mathbb{Z}^d$ is recurrent (meaning the walker returns to its starting point with probability one), yet for $d \geq 3$, it is transient (meaning there is a positive probability that the walker may not return to its starting point).

*Prerequisites:* Series.

**Banach-Tarski.** (*Andrew Marks*)

The Banach-Tarski paradox is a famous paradox about infinity. It states that a three dimensional ball can be cut into finitely many pieces which can be reassembled by rotations and translations into two balls each of which has the same size as the original ball! A consequence of the paradox is that not every set in three dimensions can be reasonably assigned a "volume".

We'll start the course by reviewing cardinality, which is the original way Cantor devised for measuring the size of infinite sets. We'll then prove the Banach-Tarski paradox using ideas from linear algebra and set theory. We'll end with some discussion of how the Banach Tarski paradox is related to Lebesgue measure – the modern way mathematicians have devised for measuring volume.

*Prerequisites:* Familiarity with matrices.

**Bernoulli Numbers.** (Lara)

The Bernoulli numbers $\{B_m\}$ are a mysterious sequence of rational numbers that arise naturally in many places. For example, in the Taylor series for trig functions and when studying Pascal's triangle. Once we know what Bernoulli numbers are, we'll use their generating function to better study them and to gain detailed information about series and insight into understanding the Riemann zeta function. We'll prove the von Staudt-Clausen theorem which will tell us for each prime, how $pB_m$ behaves $(\bmod p)$ and see the beautiful consequences this theorem has.

*Prerequisites:* Integration and series. To understand everything in the class, you'll need to know a bit of complex analysis, but you should follow most of the class without this.

**Calculus without Calculus.** (Tim!)

If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Angela is 5 cubits tall and Misha is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Angela's head to the top of Misha's head that touches the ground in the middle. What is the shortest length of string you can use?
- Apurva rides a bike around an elliptical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 20 meters way along the shoreline throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- Where in a movie theater should you sit so that the screen takes up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

*Prerequisites:* We won't use calculus (that's the point), but it would be good if you've seen it for context.

**Cap-Set Problem: Recent Progress.** (Paweł B., *camper teaching project*)

Have you ever played the card game SET? If not, run to the game lounge and play it. The picture on every card has four different features (number, shape, color, filling), each of which can take three values. We call three cards a *SET* when they are all different or all the same in each feature. The goal of the game is to find a SET among 12 cards on the table. As it turns out, it may not always be possible. One may wonder: "How many cards without a SET can you have on the table?"

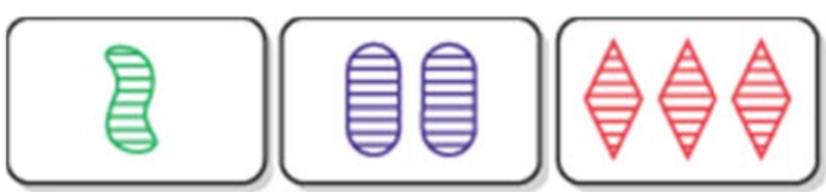


Figure 1. All 3 cards have different shapes, colors, and numbers and the same shading. Hence they form a SET.

A mathematician can imagine a SET card as a point in $\mathbb{F}_3^n$, $n$-dimensional space over $\{0, 1, 2\}$, for $n = 4$. Three points form a SET if they are collinear. The Cap-Set Problem asks, "What is the maximum number of points in $\mathbb{F}_3^n$ without a collinear triple?". For more than 30 years in the history of analytic combinatorics, nobody could even tell if the answer is of much smaller magnitude than $3^n$. However, in 2016, Ellenberg and Gijswijt in their groundbreaking paper showed that indeed it is. What is more, in contrast to the previous Fourier Analysis approaches, they use only elementary methods.

In this class, I will show you the whole argument that every subset of $\mathbb{F}_3^n$ without a collinear triple is exponentially smaller than $3^n$. The basic idea of this approach is a smart use of polynomials and ranks of matrices.

*Prerequisites:* Linear algebra (familiarity with vector space and its dimension and matrix and its rank). Understanding the meaning of $P(x)$, where $P$ is a polynomial in $n$ variables and $x$ is a point in $F_q^n$.

**Choosing Random Numbers.** (Misha)

If you want to pick a random number, but you don't care how, you might as well just roll a die. If you *do* care how, you had better do some math first. We'll do a few random things in this class, but the main goal is to see a very cute algorithm for doing what seems like magic.

You may know that factoring a 1000-digit number would take a computer thousands of years. But in this class, I will teach you how to factor a randomly chosen 1000-digit number in just a few minutes on a computer.

*Prerequisites:* Willingness to take on faith a few of my arbitrary claims about the values of infinite sums and products.

**Classification of Subgroups of $\mathrm{GL}_2(\mathbb{F}_\ell)$.** (Aaron)

What do the platonic solids have to do with 2 by 2 matrices over finite fields? It turns out that they precisely determine the "exceptional" subgroups of $\mathrm{GL}_2(\mathbb{F}_\ell)$. Here, $\mathrm{GL}_2(\mathbb{F}_\ell)$ denotes the group of invertible two by two matrices over the finite field $\mathbb{F}_\ell$. In this inquiry-based learning class, you will discover for yourself the classification of all subgroups of $\mathrm{GL}_2(\mathbb{F}_\ell)$. This classification will slickly interweave elegant ideas from linear algebra, group theory, geometry, and combinatorics, to determine all the subgroups of $\mathrm{GL}_2(\mathbb{F}_\ell)$.

*Prerequisites:* Linear algebra, group theory, finite fields.

**Cloudy with a Chance of the Continuum Hypothesis.** (Angela)

In the forecast next Wednesday and Thursday: clouds!

Sometimes a cloud is defined to be a visible mass of condensed water vapor floating in the atmosphere, typically high above the ground. Not today. Today, a cloud $C$ about a point $p$ is a subset of the plane such that for every line through $p$, $C$ intersects that line at only finitely many points. You might be wondering why you would care about sets like this. Well, it turns out that clouds are pretty weird.

Blurby seguey. The cardinality of the natural numbers is typically denoted $\aleph_0$, and the next biggest size of infinity (the smallest cardinality greater than $\aleph_0$) is denoted $\aleph_1$. The Continuum Hypothesis is a very famous statement that says that the cardinality of the real numbers is equal to $\aleph_1$; famous partially because it was the first statement that was proven to be independent of the ZFC axioms of set theory. Super famous.

Here's a simple statement about clouds: three clouds cover the plane. SURPRISE! It turns out this statement is actually equivalent to the Continuum Hypothesis. In this class we are going to prove the equivalence of these two statements. Get hyped. #gethypedforclouds

*Prerequisites:* None.

**Coloring Graphs on Surfaces.** (Marisa)

The Four Color Theorem tells us that if we want to properly color a graph drawn on the plane, we need at most four colors to do it. There is a much cooler theorem which says that if you want to draw a graph on *literally any other surface* and then properly color the vertices, you will need at most this many colors:

$$chr(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24\chi(S)}}{2} \right\rfloor$$

...which looks mildly unattractive, but which you could absolutely prove this week. And in this class, you will indeed prove this, and check out lots of other interesting properties of graphs on surfaces along the way. Works for both orientable and non-orientable, to boot.

*Prerequisites:* Intro Graph Theory or equivalent. We'll be using concepts from topology (e.g. All Things Manifoldy), but I'll cover all the key points in class.

**Coloring the Hyperbolic Plane.** (Ina)

Here's a problem: color a plane such that no two points a unit distance apart are the same color. How many colors do you need? Turns out, nobody knows! Since when this problem was posed in the 1950s, we've only known bounds. A newer line of research examines the problem in the hyperbolic plane, where even finding bounds becomes more complicated-but also, potentially, more achievable. In this class we'll begin with our current knowledge about this problem on the flat Euclidean plane. Then we'll move into hyperbolic space, exploring current research-either for fun or to give you tools to dig into this research yourself!

*Prerequisites:* None.

**Combinatorial Gems.** (*Alfonso & Kevin*)

Come join us in a discovery journey or some of combinatorics most precious gems. Each day we will propose a different problem and we will guide your exploration of it. We promise that every problem has an unexpected, mind-blowing, beautiful pattern at the end. If you are already familiar with some of the topics, ask Kevin or Alfonso if they are right for you. Do not look these problems up before class, or your risk spoiling the pleasure of discovering the answers yourself.

- *Tuesday:* **Wythoff's Game.** We have a plate with blueberries and strawberries in front of us. We take turns eating them. In your turn, you may eat as many berries as you want as long as they are all of the same kind (at least one) or exactly the same number of berries of both kinds (at least one). Then it is my turn. The player who eats the last berry wins. Will you beat me?
- *Wednesday:* **Treacherous Chords.** Draw a circle. Draw $N$ points on it. Join every pair of points with a line segment. In how many regions is the circle divided? If you compute this for $N$ from 1 to 5 and then make a guess for $N = 6$, your guess will almost certainly wrong.
- *Thursday:* **Pascal Parity.** Which numbers in the Pascal triangle are odd?
- *Friday:* **Nim.** Let's play a game. We have various plates with chocolates. We take turns eating them. In your turn, you may eat as many chocolates as you want as long as they are all on the same plate. Then it is my turn. The player who eats the last chocolate wins. Will you beat me?
- *Saturday:* **Error-correcting codes.** You place a coin on each square of a chessboard, some face up and some face down, any way you like. Then you tell Alfonso what your favourite square is. Alfonso will then flip one single coin of his choice. Then Kevin enters the room, and looking only a the chessboard, he can guess what your favourite square was. What was their trick? It is the same strategy that allows them to play 20 questions with a liar.

*Prerequisites:* None.

**Computer Aided Design.** (Elizabeth)

Computers are awesome! They can do so many cool things! In particular, if you can imagine some shape or machine, you can make a computer draw it in 3D. Once the computer knows what it is, then it can show you what it would like from any angle, and you can tweak it without having to redraw the whole thing. You can also turn it colors and zoom in on small details. Basically, anything you can do in your head, you can show to other people, with the computer. In addition, once you have told the computer about it, the computer can print out pictures or files that let machinists or machines make the part in real life. Computer aided design is useful for all kinds of things, from making robots to roller coasters to mathematical shapes.

*Prerequisites:* None.

**Constructive Logic.** (*Anti Shulman*)

In the early 20th century, a group of mathematicians staged a revolt against the prevailing orthodoxy of mathematical practice, and in particular against the unrestricted use of the law of excluded middle ("everything is either true or false"). Known as "constructivists", they insisted that any *proof* in mathematics should be a *construction*, and that non-constructive "existence proofs" should not be considered proofs at all. At the time, the revolt failed; but the late 20th and early 21st centuries have experienced a revival of constructive mathematics, based no longer on dogmatic arguments but on pluralist and pragmatic grounds, such as the increasing importance of computability and the need for flexibility to describe many different kinds of mathematics.

In this class we'll learn the basics of constructive logic, and what you can and can't do while avoiding the law of excluded middle. We'll see that some parts of mathematics (like elementary number theory) look just about the same constructively, some require small modifications here and there (like calculus), and others look completely different (like set theory). At the end we'll explore a few of the magical things that constructive logic makes possible, such as all functions being continuous, all existence being computable, and the use of true nilpotent infinitesimals instead of epsilon-delta limits.

*Prerequisites:* None.

**Crash Course on Representation Theory.** (Apurva)

One way to study groups is by making them do things to vector spaces and see what happens. Like every other useful thing in algebra this is inappropriately named as Representation Theory.

For finite groups, study it we shall.

*Prerequisites:* Group theory and linear algebra.

**Cryptography, and How to Attack It.** (Linus)

In a *normal* cryptography course, you'd learn how codes like RSA work.

In *this* course, you'll learn how to use math to hack the people who took the other course. You won't just learn RSA, Diffie-Hellman, and (maybe) more: you'll learn how to break the Vigenere cipher; crack subtly incorrect implementations of RSA; and (theoretically) break the entire Cryptocat iPhone app circa 2013.

In Week 2, we'll cover a random selection of more advanced topics in cryptography. The focus shifts somewhat from attacking bad crypto to "things Linus thinks are cool." Topics that might show up:

- Why the NSA has been encouraging everyone to use one specific prime $p$ for Diffie-Hellman
- Alice and Bob might want to date each other. Or maybe not. Alice is shy: if she loves Bob, then she *cannot* let Bob find out unless Bob loves her too. Bob is similarly shy. How can they find out whether the love is mutual?
- What does it mean for a sequence to "look random"?

*Prerequisites:* Have a good understanding of modular arithmetic. You should have internalized Euler's Theorem and the Chinese Remainder Theorem. For Week 2, introductory group theory – enough to say "$(\mathbb{Z}/5\mathbb{Z}) \cdot (\mathbb{Z}/12\mathbb{Z})$ is an abelian group isomorphic to $\mathbb{Z}/60\mathbb{Z}$.".

**Cubic Curves.** (Mark)

A curve in the $x, y$-plane is called a cubic curve if it is given by a polynomial equation $f(x, y) = 0$ of degree 3. Compared to conic sections (which have degree 2), at first sight cubic curves are unpleasantly diverse and complicated; Newton distinguished more than 70 different types of them, and later Plücker made a more refined classification into over 200 types. However, as we'll see, by using complex numbers and points at infinity we can bring a fair amount of order into the chaos, and cubic curves have many elegant and excellent properties. One of those properties in particular, which is about intersections, will allow us to prove a beautiful theorem of Pascal about hexagons and conic sections, and it will also let us define a group structure on any cubic curve - well, almost. We may have to leave out a singular ("bad") point first, but a cubic curve has at most one such point (which may be well hidden; for example, $y = x^3$ has one!), and most of them don't have any. Cubic curves without singular points are known as elliptic curves, and they are important in number theory, for example in the proof of Wiles' Theorem (a.k.a. "Fermat's Last Theorem"). However, in this week's class we probably won't look at that aspect at all, and no knowledge of number theory (or even groups) is required. With any luck, along the way you'll pick up some ideas that extend beyond cubic curves, such as how to deal with points at infinity (using "homogeneous coordinates"), what to expect from intersections, and where to look for singular points and for inflection points.

*Prerequisites:* Mild use of differential calculus, probably including partial derivatives; complex numbers; some use of determinants. Group theory *not* required.

**Decomposing and Factoring.** (Marisa)

Some math friends of mine once called up a professional football league (the American kind of football) and offered to design them a better schedule, and the league took them up on it.[1] This is how I imagine that conversation going.

> Mathematicians: Your schedule is a mess. We'd like to make it better.
> League: ??
> Mathematicians: Trust us. We have design theory.
> League: Fine. We have eight teams, and we want four games per week, for seven weeks, and no team can play two games on the same day, and every team will play every other team.
> Mathematicians: Here is a 1-factorization of $K_8$. Boom.
> League: Neat. Here is $1,000.

In this class, we'll be talking about design theory: from Kirkman's discovery in 1847 of what would (over Kirkman's strong objection) later be called Steiner triple systems, through perfect matchings in 1891 (when Petersen published his paper "The Theory of Regular Graphs"), to systems of distinct representatives and Hall's Theorem in 1935, and all the way up to 2012 when Alpach's Conjecture about cycle decompositions of $K_n$ was finally proved.

*Prerequisites:* Intro Graph Theory or equivalent; really, all you need to know is what $K_n$ and $C_n$ mean.

[1] http://www.nytimes.com/2001/02/03/arts/what-good-is-math-an-answer-for-jocks.html

**Discrete Derivatives.** (Tim!)

Usually, we define the derivative of $f$ to be the limit of $\frac{f(x+h)-f(x)}{h}$ as $h$ goes to 0. But suppose we're feeling lazy, and instead of taking a limit we just plug in $h = 1$ and call it a day. The thing we get is kind of a janky derivative: it's definitely not a derivative, but it acts sort of like one. It has its own version of the power rule, the product rule, and integration by parts, and it even prefers a different value of $e$. We'll take an expedition into this bizarre parallel universe. Then we'll apply what we find to problems in our own universe. We'll talk about Stirling numbers and solve difference equations and other problems involving sequences.

*Prerequisites:* Calculus (derivatives).

**Division Rings.** (Susan)

We know that rings have addition, subtraction, and multiplication. But where does division fit into this picture? Dividing, in the context of a ring, is essentially multiplying by a multiplicative inverse. But these inverses don't always exist. In the commutative setting, the field of fractions construction allows us to add inverses to a ring that did not previously have them.

In the noncommutative setting things are… not so nice. In this class, we'll see an example of a noncommutative domain that cannot be embedded into any division ring. We'll look at Ore domains, the closest noncommutative analogue of an integral domain, and see how we can expand it into an Ore Division Ring of Fractions. We'll see twisted Hilbert polynomials, which we can transform into Laurent division rings. We'll also explore the idea of inversion height, and encounter the mystery of what happens to the number three!

*Prerequisites:* Ring Theory.

**Elliptic Functions.** (Mark)

Complex analysis, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as cos and sin, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k),$$

where $\sigma_i(k)$ is the sum of the $i$-th powers of the divisors of $k$. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1^3 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1],$$

which you are welcome to check if you run out of things to do.)

*Prerequisites:* Functions of a Complex Variable (in particular, Liouville's theorem and the residue theorem).

**Euler Characteristic.** (Apurva)

Euler characteristic is a benign number that is computed by counting vertices and edges of graphs. This single number explains why a cyclone should have an eye, why you can eat a pizza without spilling all the toppings and why my hair looks messy no matter how much gel I apply. The entire branch of algebraic topology was invented to rationalize the existence of so powerful an invariant.

In this class we'll understand what it means for the Euler characteristic to be an invariant of surfaces and explore several geometric manifestations of it. We'll see proofs of the Sperner's lemma, Brouwer's fixed point theorem, hairy ball theorem and Gauss-Bonnet theorem using Euler characteristic.

*Prerequisites:* None.

**Evasiveness.** (Tim!)

We'll explore a conjecture in computer science that has been open for over 40 years, concerning the complexity of graph properties. One way to measure the complexity of a problem (like "Does this graph have a Hamiltonian cycle?") is by its time complexity — roughly, how long it takes a computer to solve it. Another important way to measure complexity is query complexity — roughly, how many questions you need to ask about the graph to answer the problem. The graph properties with maximum query complexity are called evasive, and the conjecture is that a huge class of graph properties — specifically, all those that are nontrivial and monotone — are evasive.

We'll trace the story of this conjecture through time from its conception in 1973, through a surprising appearance of topology in 1984, to the present day, including research from the past few years. Along the way, we'll see scorpion graphs, clever counting, collapsible simplicial complexes, transitive permutation groups, and hypergraph properties.

This class is directly related to my research, and in class we'll see a recent result of mine, along with its proof.

*Prerequisites:* Group theory (normal subgroups, quotient groups). If you haven't seen graph theory, talk to me first.

**Every natural is "Fibonacci": Games, Miles and Kilometres.** (Beatriz)

In this class we will prove Zeckendorf's theorem which states that every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This theorem has many very interesting application. In this class we will see how Fibonacci numbers can help us win the following game:

There is one pile of n stones. The first player may remove as many as they like on their first turn as long as they remove at least one and leave at least one. From then on, the next player may remove no more than their opponent did on their previous turn. The player who takes the last coin wins.

Also, because Mathcamp is so international, and I want to help you understand each other, we'll learn how to use Fibonacci numbers to easily convert from miles to kilometres and vice versa.

*Prerequisites:* None.

**Exclusion-Inclusion.** (*Po-Shen Loh*)

We all know that $|A \cup B| = |A| + |B| - |A \cap B|$, and there's some generalization of this to $n$ sets. So we will not discuss that at all, and talk about other things, like spheres and the exclusive-or operator. We will discover that lots of ideas are connected.

*Prerequisites:* $|a \cup b| = |a| + |b| - |a \cap b|$.

**Fast Matrix Multiplication.** (Yuval)

Some of my favorite algorithmic questions are related to a task that might seem pretty uninteresting, namely that of multiplying together two big matrices. In this class we will see how you can multiply together matrices way faster than you might have expected. Additionally, we'll talk about a big open problem in the field, which basically says that multiplying together two matrices is no harder than just looking at them. Finally, we'll talk a bit about how people are trying to prove this conjecture, and about some recent theorems that say all such attempts simply can't work.

*Prerequisites:* Know how matrix multiplication works! Also, some small background with algorithms will be helpful, but not necessary.

**Finite Fields.** (Aaron)

What do the rational numbers, complex numbers, and real numbers have in common, but not share with the integers? They are all fields; we can add, subtract, multiply, and divide elements in them. But which finite sets also have these properties? What possible sizes can such a finite set have? What are the possible subfields? These questions all have simple, beautiful answers which we will present in this course. Finite fields are crucially used throughout number theory, algebraic geometry, cryptography, and coding theory.

*Prerequisites:* Linear algebra, group theory.

**Finite Geometries.** (*J-Lo*)

There are infinitely many points in the Euclidean plane. Just think, all those points that no human being will ever use - what a waste! Suppose instead that we limit our geometrical landscape to having finitely many points - how much geometry could we reproduce? Can we meaningfully talk about distance? Lines? Circles? Angles? Trigonometry? Come to play around in these surprisingly small worlds and rewire your intuition for what various geometric concepts "look like" - and along the way, discover Elliptic Curve Cryptography, the state of the art in NSA-certified internet security protocols.

*Prerequisites:* None.

**Finite "Sets".** (Don)

What makes something finite? Is it that it has finitely many elements? What if it doesn't, techncially speaking, have any elements?

Come learn how Category Theory can tell us what properties really make an object finite - and then see some objects that, despite having infinitely many elements, have put in the hard work to truly earn the title of Finite.

*Prerequisites:* None.

**Fractals and Dimension.** (Steve)

The usual three dimensions are fun and all, but they get kind of boring after a while. One way to liven things up is to add more dimensions; billion-dimensional shapes are probably super cool! But you know what I like even more than big numbers? *Wrong numbers.* I want a two-and-a-half-dimensional shape. Or a $\pi$-dimensional shape. Or a shape with a decent number of dimensions, but for terrible reasons.

It turns out we can make this happen! The answer is *fractals*, a particularly weird and beautiful kind of shape. Fractals crop up throughout mathematics in all sorts of weird ways, and have lots of fascinating properties *besides* just being dimensionally weird. This class will be about what dimensions are, why fractals have silly numbers of them, and how awesome fractals are.

*Prerequisites:* None.

**From the Intermediate Value Theorem to Chaos.** (Beatriz)

The Intermediate Value Theorem is a theorem that – despite looking very simple or even obvious – has amazing consequences. In particular, it allows us to prove one of my favourite theorems, called Sharkovsky's Theorem. Sharkovsky's Theorem gives us very valuable information about the periodic points of continuous functions on the real line. In particular, we will show that period 3 implies chaos!

*Prerequisites:* None.

**Functions of a Complex Variable.** (Mark)

Spectacular (and unexpected) things happen in calculus when you allow the variable (now to be called $z = x + iy$ instead of $x$) to take on complex values. For example, functions that are "differentiable" in a region of the complex plane now automatically have power series expansions. If you know what the values of such a function are everywhere along a closed curve, then you can deduce its value anywhere inside the curve! Not only is this quite beautiful math, it also has important applications, both inside and outside math. For example, functions of a complex variable were used by Dirichlet to prove his famous theorem about primes in arithmetic progressions, which states that if $a$ and $b$ are positive integers with $\gcd(a, b) = 1$, then the sequence $a, a + b, a + 2b, a + 3b, \ldots$ contains infinitely many primes. This was probably the first major result in analytic number theory, the branch of number theory that uses complex analysis as a fundamental tool and that includes such key questions as the Riemann Hypothesis. Meanwhile, in an entirely different direction, complex variables can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow. Dirichlet's theorem is certainly beyond the scope of this class and heat conduction probably is too, but we should see a proof of the so-called "Fundamental Theorem of Algebra", which states that any nonconstant polynomial (with real or even complex coefficients) has a root in the complex numbers. We should also see how to compute some impossible-looking improper integrals by leaving the real axis that we're supposed to integrate over and venturing boldly forth into the complex plane! This class runs for two weeks, but it should be worth it. (If you can take only the first week, you'll still get to see a good bit of interesting material, including one or two of the things mentioned above.)

*Prerequisites:* Multivariable calculus (including Green's Theorem; if the Multivariable Crash Course doesn't get to Green's Theorem, it will be covered near the beginning of this class).

**Galois Theory Crash Course.** (Mark)

In 1832, the twenty-year-old mathematician and radical (in the political sense) Galois died tragically, as the result of a wound he sustained in a duel. The night before Galois was shot, he hurriedly scribbled a letter to a friend, sketching out mathematical ideas that he correctly suspected he might not live to work out more carefully.

Among Galois' ideas (accounts differ as to just which of them were actually in that famous letter) are those that led to what is now called Galois theory, a deep connection between field extensions on the one hand and groups of automorphisms on the other (even though what we now consider the general definitions of "group" and "field" were not given until fifty years or so later). If this class happens, I expect to be rather hurriedly (but not tragically) scribbling as we try to cover as much of this material as reasonably possible. If all goes well, we just *might* be able to get through an outline of the proof that it is impossible to solve general polynomial equations by radicals once the degree of the polynomial is greater than 4. (This depends on the simplicity of the alternating group, which we won't have time to show in this class but which may be shown in a separate week 5 class.) Even if we don't get this far, the so-called Galois correspondence (which we should be able to get to, and prove) is well worth seeing!

*Prerequisites:* Group theory; linear algebra; some familiarity with fields and with polynomial rings.

**Generating Functions and Regular Expressions.** (Linus)

To cheat at Mathcamp's famed week 4 puzzle hunt, I use regular expressions. For example, if I know a puzzle answer uses the letters d, u, c, and k in that order, I can use the regular expression d.*u.*c.*k to get a list of all English words it could be.

To count anything, e.g. the number of domino tilings of a $4 \times n$ rectangle, I use generating functions, a magic tool of combinatorics.

Learn how regular expressions and generating functions are the same thing, and use them together to instantly solve a bunch of problems like: - "What's the most chicken nuggets I can't order if they come in 5-piece and 8-piece boxes?" - "Why do rational numbers have repeating decimals?"

*Prerequisites:* None! There's no need to have even heard of a regular expression, generating function, or dduckkkkk before.

**Geometry in Motion.** (*Zach Abel*)

Come see how geometry folds and flexes like paper, de- and re-configures itself like transformers, and swings and hinges like a Strandbeest[2]. We'll look at some of my favorite geometric topics that involve some form of motion, and we'll frequently detour to look at recently solved or unsolved research questions in these areas (including some from my own research). Topics will include, but will not be limited to:

- **Flexible polyhedra** and **polyhedral flattening**: Imagine a polyhedron made with rigid metal faces that is only allowed to fold along the edges. Can such a shape be flexible? (*Yes!*) Can one of these be designed that can be folded flat, say for easy transport or storage? (*No!*) What if we allow folds anywhere, not just at the edges?
- **Polygon dissection**: If I give you an equilateral triangle and some scissors, can you cut it into a few polygonal puzzle pieces that can be rearranged into a square? (*Yes!*) (*Try it! Try using just 4 pieces!*) Is the same true in 3D for, say, a tetrahedron and a cube? (*No!*) What if the pieces are required to be hinged to each other as they reconfigure?
- **Mechanical linkages**: Picture a movable mechanical device made with rigid metal rods connected at rotatable hinges. If we affix a pen to one of the bars, what shapes can such a mechanism trace? A circle is easy with just one segment acting as a compass, but is a straight line segment possible? (*Yes!*) If we make the mechanism complicated enough, can we make a sketch of your face? (*Yes!*) What if edges are forbidden from crossing each other during the motion?

*Prerequisites:* None.

**Godel's Incompleteness Theorem.** (Steve)

In 1931, Kurt Godel proved that there are true sentences of arithmetic which cannot be proven from the standard axioms for arithmetic; moreover, he proved that *no reasonable system* of axioms would be free from this problem! This is totally wild: what exactly does it mean, and how on earth would one go about demonstrating it?

[2] If you don't know what this is, do yourself a favor: `https://www.youtube.com/watch?v=LewVEF2B_pM`

In this class we'll prove Godel's Incompleteness Theorem, and talk about what it does - and does not - mean. Historically, this was almost the beginning of modern logic, so there's also some great stories from the time during which it was proved and we might talk about those too.

*Prerequisites:* Comfort with formal proof, especially induction, and a little group, ring, or field theory.

**Group Theory.** (Mark)

How can you describe the symmetries of geometric figures, or the workings of a Rubik's cube? How do physicists predict the existence of certain elementary particles before setting up expensive experiments to test those predictions? Why can't fifth-degree polynomial equations, such as $x^5 - 3x + 2017 = 0$, be solved using anything like the quadratic formula, although fourth-degree equations can? The answers to these questions are mostly beyond the actual scope of this class, but they all depend on group theory. Knowing some group theory is at least helpful, and often crucial, in other parts of mathematics. So come get your feet wet (we'll consider taking off socks and shoes, but you shouldn't take any of that too literally.) We'll move fairly quickly and with luck, after doing fundamental concepts (and examples), we'll get to permutation groups, Lagrange's theorem, quotient groups, and maybe the First Isomorphism Theorem.

*Prerequisites:* None beyond the Mathcamp Crash Course.

**How Not To Prove That a Group Isn't Sofic.** (Viv)

Cayley's Theorem tells us that finite groups are all subgroups of finite permutation groups. A sofic group is a possibly-infinite group that we sort of maybe want to have the same property roughly speaking. The group can't always be a subgroup of a finite permutation group, so instead we just require that all finite subsets of our sofic group act kind of like finite subsets of permutation groups. This ends up being a super-useful definition, but we're left with a burning question: do non-sofic groups exist? We don't actually know the answer to this one. We'll spend the class talking about a hopeful candidate for a non-sofic group and one great way not to prove that it isn't sofic.

*Prerequisites:* Group Theory.

**How Not to Solve Tricky Integrals.** (Steve)

The function $e^{-x^2}$ doesn't have an elementary antiderivative, but the definite ("Gaussian") integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

can be solved in a variety of ways. One quite nice approach (due to Poisson) is via change of coordinates. It's a very easy trick, and it's natural to ask what other problems we could solve this way.

It turns out the answer is — *absolutely none*! In a precise sense, Poisson's trick only works once. Useful knowledge is overrated! Come see how to solve the Gaussian integral, and how not to solve tricky integrals in general!

If we have time, we'll see a broad generalization of Poisson's trick. This still doesn't work anywhere else, but it's really fancy.

*Prerequisites:* Multivariable calculus (specifically, change of variable in multiple integrals).

**How to Build a Calculator.** (Lotta)

I have a little box with buttons on it that's probably made of magic. When I push buttons like '2,' '3,' '4,' '+,' '3,' '4,' it returns '268.' How does it know? Probably there is a little person inside doing arithmetic by hand.

In this class, we will talk about circuits and gates and how you can put gates together to build an adder or anything you want.

*Prerequisites:* None.

**How to Count Using Topology.** (*Anand Deopurkar*)

Take an analog clock, stop it, and switch the minute hand and the hour hand. Most of the time, you will end up with an absurd configuration of hands that never arises naturally. But are there valid clock positions that remain valid after making such a switch? If there are, how many are there? We will answer this combinatorial question by converting it first into geometry and then into algebra, using an amazing gadget called the (co)homology ring. If time permits, we will count other things using this technique.

*Prerequisites:* None.

**How to Define the Square Root.** (Apurva)

If a function on real numbers $f : \mathbb{R} \to \mathbb{R}$ is not injective it is usually not possible to define an inverse, no matter how well behaved the function is. On the other hand, if a *complex differentiable* function on complex numbers $f : \mathbb{C} \to \mathbb{C}$ is not injective in many cases it is possible to define an inverse function by creating a new Riemann surface by gluing pieces of the complex plane together.

In this course we'll learn how to do this and in the process see how Riemann surfaces naturally come up and learn about their ramified coverings.

*Prerequisites: complex variables* you should be comfortable with what a complex differentiable function is.

**How to "Divide".** (Don)

Adding two sets together is pretty easy - we just take their union, with an extra copy of anything they share.

If $A + B$ is the same size as $A + C$, and $A$ is finite, it's possible, but not obvious, to show that $B$ is the same size as $C$ — that is, to subtract!

Multiplying two sets is also pretty easy - we take their cartesian product.

But, how do we divide? It turns out, entirely without a need for the axiom of choice, as long as we're clever enough. Instead, we get by with one secret ingredient - love [3].

*Prerequisites:* None.

[3] Where by love, I may or may not mean topological cobordism theory

**How To Get Rich Quick By Playing The Lottery.** (Aaron)

In 2004, Massachusetts wanted to make more money from the state lottery, so they devised a new game called cash winfall. In this game, instead of having only one large prize, there were many smaller prizes. When more people bought lottery tickets, the big prize "rolled down" into the smaller prizes, making these smaller prizes more valuable. In fact, so much money rolled down that on certain days, it was possible to buy lottery tickets for $2 with an expected return of 5$. We'll tell the story of how three groups of people exploited this lottery, and explain the math behind it, including a surprising appearance of the Fano plane.

*Prerequisites:* None.

**How to Pronounce "Lucas".** (Misha)

The French mathematician Édouard Lucas is kind of awesome. In between studying Fibonacci numbers, finding patterns in Pascal's triangle, and inventing an algorithm to look for Mersenne primes and perfect numbers, he invented the paper-and-pencil game Dots and Boxes, as well as the Tower of Hanoi puzzle (which he marketed under an anagram name, like Voltaire or Voldemort: N. Claus de Siam).

He's also probably tied with Euler for being the mathematician with the most mispronounced name. To pronounce his last name, say "Lu" to rhyme with "flew" and "cas" as though you're saying "car" in an exaggerated British accent.

This class will be an overview of many of the cool things Lucas did. Each day will be a separate, mostly independent topic in combinatorics, number theory, or the gray area in between.

*Prerequisites:* None.

**Hyperplane Arrangements.** (Kevin)

Suppose I want to cut a watermelon into pieces. With 0 cuts, I can make 1 piece: the entire watermelon. With 1 straight cut, I can make 2 pieces. With 2 cuts, I can make 4, and with 3 cuts, I can make 8.

I'm sure you see the pattern by now: with $k$ cuts, we can make $\frac{1}{6}(k^3 + 5k + 6)$ pieces. These cuts are an example of a *hyperplane arrangement*, which is simply a collection of $k$ hyperplanes in $n$-dimensional space. We'll learn some powerful techniques for counting the number of regions that a hyperplane arrangement determines, and we'll maybe even see some connections to graph theory. Throughout, partially ordered sets will abound!

*Prerequisites:* None.

**Hyperreal Numbers.** (Don & Tim)

In the beginning, there were integers, $\mathbb{Z}$, and they were good. We could add and subtract and multiply them, and for a time, that was enough.

Then, came the rational numbers, $\mathbb{Q}$, and the big spaces between the integers were filled in, and it was good. Now we could also divide by numbers other than zero, and between any two numbers, there were infinitely more. And for a time, that was enough.

Then, came the real numbers, $\mathbb{R}$, and even the tiniest spaces between the rational numbers were filled in, and it was good. Now, finally, our number line was complete, and for a time, that was enough.

And then, came the hyperreal numbers, $^*\mathbb{R}$, AND THEY WERE TOTALLY BANANAPANTS. Numbers got shoved in between real number where they had no business going, numbers started getting unboundedly infinite, and the number line basically exploded. And finally, we said, “enough is enough.”

In this class, we’ll learn about the crazy explosion of numbers known as “The Hyperreals.” We’ll learn how to build them, what their properties are, what they look like, and even what they can tell us about math that isn’t totally bananpants.

*Prerequisites:* None.

**I click “Random Page” in OpenProblemGarden.org until we get one we understand: then we spend 50 minutes trying to solve it.** (Linus)

See title.

*Prerequisites:* Probably combinatorics.

**Impossible Cohomology.** (Don)

Cohomology is a powerful tool, used to describe underlying properties of mathematical objects from topological spaces to groups and rings. It ultimately measures how a big thing is more complicated than the sum of its parts.

You may recognize this figure:


Each of the parts seems like a normal geometric object, but the way they fit together is wrong. With the power of cohomology, we’ll figure out exactly how wrong!

*Prerequisites:* None.

**Intersecting polynomials.** (Tim!)

You might think that everything there is to know about one-variable real polynomials has been known for hundreds of years. Except, in 2009, while bored at a faculty meeting, Kontsevich scribbled down a brand new fact about polynomials. You’ll discover it.

*Prerequisites:* None.

**Intro to Graph Theory.** (Marisa)

There is a theorem that says that for any map of, say, countries on your favorite continent, you can color the countries so that any two countries that share a border (not just meet at a point, but actually share some boundary) get different colors, and that the number of colors you will need is no more than 4. (Try inventing a complicated political landscape and coloring: no matter how crazy the scene, you'll always be able to color the map with four colors.)

Mathematicians have been pretty convinced about the truth of this Four Color Theorem since the late 1800s, but despite many false starts, no one gave a proof until 1976, when two mathematicians wrote a very good computer program to check 1,936 cases. (To this day, we have no human-checkable proof.)

In this class, we will definitely not prove the Four Color Theorem. You will, however, prove the *Five* Color Theorem, which is a whole lot shorter (and which was successfully proven by hand in 1890). Along the way, you'll meet many other cool concepts in Graph Theory.

Notice how I said "*you* will prove"? That's because the course will be inquiry-based: I won't be lecturing at all. You'll be working in small groups to discover and prove all of the results yourself!

*Prerequisites:* None!

**Laws of large numbers.** (Lara)

In this class, for $X_1, \dots, X_n$ independent and identically distributed random variables, we'll prove the convergence of expressions like $\frac{X_1+\cdots+X_n}{n}$ in different senses. We'll use these results to answer a few real-world problems such as: 1. How long do we expect it to take us to collect an entire set of stamps if we acquire one stamp at random per time interval? 2. How much should we pay to play a game where we win $2^j$ dollars if we get $j-1$ consecutive tails and it takes $2^j$ coin tosses to get heads for the first time?

*Prerequisites:* You should know what a random variable is, what it means to integrate a random variable and what it means for random variables to be independent.

**Linear Algebra.** (Don)

Linear algebra is the area of math that deals with vectors and matrices. It is one of the most useful methods in mathematics, both within pure math and in its applications to the real world. One could argue that most of what mathematicians (and physicists, and engineers, and economists) do with their time is try to reduce hopelessly complicated non-linear problems to linear ones that can actually be solved. Thus for many applied fields, the most important math to know is not calculus, but linear algebra. Obviously we can't cover all of linear algebra in one week, but this class will give you a basic background, as well as a preview of some of the most important results.

We're going to start out on the plane, where linear algebra springs out of geometry. We'll define linear maps and give an intuitive preview of the central themes of linear algebra. Then we'll leave our two-dimensional pictures behind and introduce the more general concepts of vector spaces, linear independence, dimension, kernels, images, eigenvectors, eigenvalues, and diagonalization. (If you don't know what any of these words mean, that's great: come to the class! If you know all of them, then you probably don't need this class.)

*Prerequisites:* None.

**Logic Puzzles.** (Don)

In a world where all people are either liars or truthtellers, suppose Armond says, "Burt is a liar," and Burt says, "Armond and Charlie are the same type of person." What can you tell me about Charlie?

A group of 4 campers and 4 staff need to cross a river, via a boat that can take 1 or 2 people. If the campers ever outnumber staff at a location, those staff will get covered in magic marker. Can they all make it across the river without anyone getting drawn on?

The ""Schedule" Making "Committee"" decides to meet at a particular day during the first four weeks of camp; it's Wednesday, Thursday, or Sunday of Week 1, Friday or Saturday of Week 2, Tuesday or Thursday of Week 3, or Tuesday, Wednesday, or Friday of Week 4. Two "members" of the "committee," identified as M and N, remember this list of dates. Further, M knows the week for the meeting, and N knows the day of the week for the meeting. They then make the following statements:

- M: "I know that N doesn't know the date."
- N: "I didn't know it before, but now I do."
- M: "Now I too know the date."

When is the "meeting"?

In this class, we'll look at puzzles like these, figure out how to solve them, and then go even further, studying the underlying logic behind the puzzles, and ultimately figuring out how to write them.

*Prerequisites:* None.

**Machine Learning (NO neural networks).** (Linus)

Emphasis on theory over practice. Possible topics:

- Learning a linear classifier (with noise) - Models of learning: PAC, Mistake Bound, Bayesian - VC dimension (a combinatorial property: "simple", "easy to learn" concepts tend to have low VC dimension)

*Prerequisites:* Thinking in n-dimensional space.

**Many Clubs Share People.** (Aaron)

I'll explain and discuss the classic problems of oddtown and even town. In oddtown, there are n people. These people form clubs where each club has an odd number of members, and for any pair of clubs, the number of people they have in common is n. How many clubs can there be? In eventown, there are n people. These people form distinct clubs where each club has an even number of members, and any pair of clubs share an even number of members. How many clubs can there be? The answers to these two questions are surprisingly different, but both involve linear algebra over the finite field with two elements.

*Prerequisites:* Linear Algebra.

**Math and Brains.** (*Nora Youngs*)

The brain is an incredibly complex organ, and even with impressive advances in neuroscience there are still many mysteries remaining around how it functions. In this class, we'll take a look at some of the mathematical models which have been used to understand how neurons work, and how they interact in both small and large networks. We'll also try to see how these models answer questions like: What is a memory? and: How does learning happen? Phrased another way, in this class we will think about what might be the best mathematical way to think about thinking.

*Prerequisites:* The willingness to suspend reality in order to construct a feasible mode.

**Mathcamp Crash Course.** (Tim!)

This course covers fundamental mathematical concepts and tools that all other Mathcamp courses assume you already know: basic logic, basic set theory, notation, some proof techniques, how to define and write carefully and rigorously, and a few other tidbits. If you are new to advanced mathematics or just want to make sure that you have a firm foundation for the rest of your Mathcamp courses, then this course is *highly* recommended.

Here are some problems to test your knowledge:

(1) Negate the following sentence without using any negative words ("no", "not", etc.): *"If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."*
(2) Given two sets of real numbers $A$ and $B$, we say that $A$ *dominates* $B$ when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two, disjoint, non-empty sets $A$ and $B$ such that $A$ dominates $B$ and $B$ dominates $A$.
(3) Prove that there are infinitely many prime numbers.
(4) Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then $f$ is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)
(5) Define rigorously what it means for a function to be increasing.
(6) Prove that addition modulo 2017 is well-defined.
(7) What is wrong with the following argument (aside from the fact that the claim is false)?

    **Claim:** On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

    **Proof:** We proceed by induction on $n$. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k + 1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all $n$. QED.
(8) Explain what it means to say that the real numbers are uncountable. Then prove it.

If you are not 100% comfortable with most of these questions, then you can probably benefit from this crash course. If you found this list of questions intimidating, then you should *definitely* take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

*Prerequisites:* None.

**Mathematica Workshop.** (Beatriz)

Mathematica software is a very useful tool that allows us to efficiently solve symbolic and algebraic problems. We'll learn how to perform several different computations, plot graphs, and create dynamic visualization using Mathematica.

*Prerequisites:* None.

**Mathematics in Crisis: The St. Petersburg Paradox.** (Sam)

By the early 18th century, probability was starting to develop as a field of its own right. Core to this development was the concept of $E$xpectation (Expected Value). The hope was that expectation would become the tool for making rational decisions: the decisions a mathematician would make. In 1713, however, Nicolas Bernoulli came along and proposed what is now known as the St. Petersburg paradox: a game that no mathematician would pay even $100 to play, but for which mathematics dictated that anyone should jump at the chance to pay any amount of money to play. Many of the initial resolutions to this paradox (posed by serious mathematicians) might seem crazy today. For example, an event having different **mathematical** and **moral** probabilities.

In this class, we'll briefly trace out the early history of probability, leading up to the crisis that was the St. Petersburg Paradox. We'll then spend the rest of the class talking about the historical resolutions proposed.

*Prerequisites:* You should know what expected value is.

**Math Writing Workshop.** (Beatriz)

This class is suitable for all levels. What you learn from this class with be tailored to your needs. Whether you need help with the general basics of mathematical writing or writing your Mathcamp project, you will benefit from this class.

Writing a solution to a problem set is very different from writing a report or an article to be published in a journal or a blog. Good writing is an essential part of mathematics that is very often overlooked and very difficult to master. We will talk about what constitutes good mathematical writing and do some hands-on activities. The goal of this class is to help improve your ability to write rigorous and elegant proofs. The emphasis will be not on how to figure out the answer but on how to turn your solution into a rigorous argument that is clear, concise, and effective.

*Prerequisites:* None.

**Metric Space of Metric Spaces.** (Steve)

A *metric space* is just that - a set of *points* together with a notion of *distance*. We demand that this distance notion not be too silly (e.g. the distance from $a$ to $b$ should be the same as the distance from $b$ to $a$), but otherwise anything goes.

This means we can have very weird metric spaces. Some of these are just oddly shaped; others have "points" that, strictly speaking, aren't points. For instance, we can have metric spaces whose points are *lines* in other, more familiar spaces. And, because metric spaces are fun, we can put metric spaces in metric spaces - there is a nice metric space whose "points" are themselves metric spaces!

This class will begin by studying metric spaces in general, and some properties and constructions of metric spaces - compactness, completions, isometries and homeomorphisms, etc. From there we will move on to the metric space of metric spaces, and have fun playing around with such a strange object. Finally, we'll talk a bit about how this space is used in mathematics - in particular, how it gets used in *algebra* (because when you have a geometry of geometries, obviously that's all about algebra).

Time permitting, we'll look at some other fun metric spaces - for instance, the *pseudoarc*, a "line" that can't be cut into two shorter "lines," or the Urysohn space, a metric space which is "as big as possible" in a precise sense.

*Prerequisites:* Comfort with proofs.

**Multivariable Calculus Crash Course.** (Mark)

In real life, interesting quantities usually depend on several variables (such as the coordinates of a point, the time, the temperature, the number of campers in the room, the real and imaginary parts of a complex number, . . . ). Because of this, "ordinary" (single-variable) calculus often isn't enough to solve practical problems. In this class, we'll quickly go through the basics of calculus for functions of several variables. As time permits, we'll look at some cool applications, such as: If you're in the desert and you want to cool off as quickly as possible, how do you decide what direction to go in? What is the total area under a bell curve? What force fields are consistent with conservation of energy?

*Prerequisites:* Knowledge of single-variable calculus (differentiation and integration).

**My Favorite Magic Trick.** (Don)

I've always liked magic, especially mathematical magic — a good mathematical magic trick is essentially a live-action puzzle, with particularly good flavortext. There are a number of tricks that I hold dear to my heart, but this class is about my favorite one: the one where you do all the work. Come perform this trick, and then figure out how it works!

*Prerequisites:* None.

**Non-Classical Logic.** (Joshua, *camper teaching project*)

Mathematics has this long-standing tradition of creating new systems. When the reals weren't enough, we invented the complex numbers. When geometry was too strict, we invented topology. But what happens if we want to change the very laws of logic? In this class, we'll do just that.

For the most part, we'll stick to a type of logics called modal logics. These logics can help us make sense of statements like "Misha must be the crowman," among other statements. We might also touch on other interesting types of logics near the end of class.

*Prerequisites:* None.

**Not Your Grandparents' Algorithms Class.** (Sam)

Have you ever found yourself in a panic, wondering what to do if you ended up as a contestant on the not-totally-made-up hit game show *Wait, wait. Who'd I just marry?* This class will study algorithms for making progressively more and more complicated decisions; by the end of the week, we'll have just the right set of ideas to placate any fears about that game show!

It turns out that, in *Wait, wait. Who'd I just marry?*, the decisions you make are complicated. You get new information in various rounds, and the "matchmakers" employed by the show provide less than truthful insight (to add drama and boost ratings, naturally). These decisions are tedious, so we'll start by figuring out how to solve optimization problems that are about as mathematically nice as possible: linear programs. To solve linear programs we'll take an atypical path and study the ellipsoid algorithm: an algorithm with beautiful geometric intuition that's important theoretically, and kind-of sort-of useful practically! Unlike a traditional algorithms class, we'll spend approximately zero time focusing on running time beyond vague notions about whether or not an algorithm is efficient in a formal sense. Instead, we'll emphasize the ideas behind a whole bunch of cool algorithms!

*Prerequisites:* Basic comfort with working in $\mathbb{R}^n$, including set notation. We may briefly use modular arithmetic on day 5.

**Penrose Tilings.** (Steve)

It's easy to tile the plane with squares; it's also pretty boring. More complicated tiles lead to more interesting tilings — tiling with hexagons is pretty neat — but of particular interest are those tilings where *every part looks unique*: although they tile the plane with few shapes, the resulting tiling has few or no symmetries at all. It is not obvious that (interesting) tilings of this type exist, but they do. One particularly famous one is the *Penrose tiling*, which is a tiling (or rather, class of tilings) with very little actual symmetry but lots of tantalizing structure; if you haven't seen this before, it's worth googling (it's really quite pretty). We'll look at a couple simple examples of "symmetry-less" tilings, and then dive into a mathematical explanation of the Penrose tilings and their various interpretations. Come for the tilings, stay for the projections of five-dimensional cubes!

*Prerequisites:* None.

**Permutation Statistics.** (Kevin)

Here's a classic conundrum. The first passenger to board a full flight can't remember what seat to sit in and picks a random seat. Every passenger thereafter sits in the correct seat if available, or a random seat otherwise. You're the last passenger to board. What's the probability you end up in the correct seat?

Here's a more difficult one. Suppose you and 99 of your best "friends" are incarcerated by Don for being too janky, and Don offers to release you if you meet the following challenge. Don secretly places each of your names in 100 different boxes then lays the boxes out in a row in his office. One by one, you and your 99 friends can enter Don's office and peek inside 50 boxes, but you can't tell your friends what you see. If you and all 99 of your friends each manage to find your own name, you'll be released!

Everyone individually has a 1/2 chance of succeeding; if you pick randomly and independently, then there's a vanishingly small chance that you all win the game. But you can decide a strategy ahead of time so that you actually have a decent shot of being released!

In this class, we'll use generating functions to study permutation statistics, things like average number of cycles in a permutation, chances of two numbers being in the same cycle, and maybe some more exotic properties of permutations as well, and we'll see how they answer these questions.

*Prerequisites:* Generating functions are useful to have seen, but I can help catch you up if necessary.

**Planar Algebras.** (*Noah Snyder*)

We write mathematics, like we write words, on a line. This means you can multiply on the left or on the right, but not on top or bottom. In planar algebra, you can use the whole page to write your expressions multiplying in any direction you like. Planar algebras play important roles in operator algebras, knot theory, and quantum groups. One nice thing about planar algebra is it's a place where very some recent research is accessible without a lot of background, in particular most of the material in this class will be taken from papers published in the past 35 years, some as recent as last year!

*Prerequisites:* Linear Algebra (vector spaces, bases, and dimension).

**Prime Numbers.** (Lara)

How far out do we have to go to ensure we've found the nth prime? What can we say about the gaps between consecutive primes? What happens if we sum the reciprocals of all the primes?

In this class we'll try to gain insight into these questions and into the mysterious prime counting function $\pi(x)$. Starting right at the beginning with Euclid's proof that there are infinitely many primes, we'll use the technique in this proof to find an upper bound for the nth prime that will shed some initial light on the behaviour of $\pi(x)$. We'll then develop more machinery that will give us sharper bounds on $\pi(x)$ and see the connections it has with sums like $\sum_{p\le n} \log p$. Finally we'll come full circle and give another proof there are infinitely many primes, this time with a better understanding of what 'infinitely many' means here.

*Prerequisites:* Some understanding about how to play with integrals and series.

**Problem Solving: Convexity.** (Misha)

You see lots and lots of olympiad problems about inequalities. These are often solved by applying obscure theorems due to Scottish mathematicians.

In the real world, mathematicians study inequalities too, but their approach is different. Inequalities that arise from convex functions are much more important.

So in this class, we will take the best of both worlds and use convexity to solve olympiad math problems.

*Prerequisites:* None.

**Problem Solving: Diophantine Equations.** (Misha)

This is a class about solving equations which end words such as "where $x$, $y$, and $z$ are integers". Sometimes this involves reasonable techniques like "consider the prime factors of both sides". Sometimes this involves bizarre techniques like "take the equation modulo 19".

We'll solve problems in class together, and then I'll leave you more problems to solve for homework, and hopefully you will end up walking away with an answer to the question "How do you know to use 19?"

*Prerequisites:* You should be comfortable with modular arithmetic.

**Problem Solving Discussion.** (Misha)

So how do you actually solve olympiad problems?

This is day 2 of a class in which we'll pick apart a competition problem, discuss different solutions to it, and try to answer one question: how would you come up with those solutions?

You don't need to have been to day 1 of this class (which happened in Week 1), or to remember what happened if you were there. You *do* need to think about the following math problem, which comes from the 2012 IMO:

Let $k$ and $n$ be fixed positive integers. In the liar's guessing game, Amy chooses nonnegative integers $x$ and $N$ with $0 \leq x \leq N$. She tells Ben what $N$ is, but not what $x$ is. Ben may then repeatedly ask Amy whether $x \in S$for arbitrary sets $S$ of integers. Amy will always answer with yes or no, but she might lie. The only restriction is that she can lie at most $k$ times in a row. After he has asked as many questions as he wants, Ben must specify a set of at most $n$ positive integers. If $x$ is in this set he wins; otherwise, he loses.

Prove that:

(a) If $n \geq 2k$ then Ben can always win.
(b) For sufficiently large $k$ there exist $n \geq 1.99k$ such that Ben cannot guarantee a win.

*Prerequisites:* Think about the problem in this blurb. (This is also what I mean by the required homework: this is a one-day class, so the required homework must be done by the time you get to class.).

**Problem Solving Discussion.** (Misha)

So how do you actually solve olympiad problems?

This is the first day of a discussion class that will be held on several days of camp. We'll pick apart a competition problem, discuss different solutions to it, and try to answer one question: how would you come up with those solutions?

This Saturday, we'll meet to discuss a problem many of you have seen already: problem 6 from this year's Qualifying Quiz.

(See `http://www.mathcamp.org/quiz` for the text of the problem if you haven't seen it.)

*Prerequisites:* Think a little bit about the problem we're going to discuss.

**Problem Solving: Geometric Transformations.** (Misha)

In this class, we will learn how to use geometric transformations to solve math competition problems. The following topics will be covered:

(1) Translation and central symmetry (Tuesday)
(2) Rotation and reflection (Wednesday)
(3) Similarity and spiral similarity (Thursday)
(4) Inversion (Friday)

In class, we will learn about how to use these transformations, and how to spot when they can be used, by solving problems together. There will be problems left to solve on your own. You won't need to solve these to keep up with the class, but you should, because solving problems on your own is critical to learning problem-solving.

*Prerequisites:* The equivalent of a high school geometry class.

**Problem Solving: Linear Algebra.** (Misha)

Most high school math contests (the IMO included) do not use any topic considered to be too advanced for high school, such as linear algebra. This is a shame, because there have been many beautiful problems about linear algebra in undergraduate contests such as the Putnam Math Competition.

In this class, we will look at linear algebra from a new perspective and use it to solve olympiad problems.

*Prerequisites:* Linear algebra. In particular, you should know what eigenvalues are.

**Problem Solving: Probabilistic Method.** (Tim!)

"When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth." — Sherlock Holmes. This is a good way to way to solve crimes, and a good way to solve math problems. If you need to prove that some Mathcamp staff is a spy, calculate the probability that a randomly-chosen staff is a spy. If the probability is greater than 0, then you can safely conclude that a traitor walks among us (even though you might not know who it is).

Perhaps the most surprising thing about this method is that it is actually useful! In fact, the principle above is all you need to solve this problem:

- Prove that there exists a graph with 1,000,000 vertices such that every set of 40 vertices has a pair of adjacent vertices and a pair of nonadjacent vertices.

One might be worried that a probability-based proof to this problem might not be air-tight because it leaves things to chance, but fear not — even though the proof uses probability, the final result is true with absolute certainty.

In addition to this strategy, we'll see more probability-based approaches to solving problems (even problems whose statements often don't reference probability at all!). Part of the class time will consist of campers working on problems in groups and presenting solutions.

*Prerequisites:* None.

**Problem Solving: the "Just Do It" method.** (Linus)

Example Problem: Is there a sequence of integers $a_1, a_2, a_3, \ldots$ which contains every integer exactly once, where the sequence of differences $a_i - a_(i+1)$ also contains every integer exactly once?

The "Just Do It" method turns some frightening-looking combinatorics problems, like this one, into jokes.

[If this is a two-day class, the second day will be 3 chilis and focus on harder variants, such as transfinite Just Do It.]

*Prerequisites:* None.

**Pseudo-Telepathy via Representation Theory of Finite Groups.** (Jalex)

Your friends Alice and Bob are experimental quantum physicists. They claim that they have established high-fidelty quantum entanglement between their labs. This is a valuable resource: it can be used for things like teleporting quantum states and secure distribution of cryptographic keys. You'd like to verify or falsify their claim. If you were an experimentalist, this would be straightforward: you could just take the particles into your lab and do some experiments—these would be only as complicated as the ones Alice and Bob have already done.

However, you're a mathematician; you'd prefer not to get your hands dirty. Instead, you'll call them up on the telephone to ask them some questions about systems of linear equations over finite fields. If they answer all of your questions correctly, you can be pretty sure that their claim is correct. How is this possible? Answering this question will lead us to talk about the foundations of quantum mechanics, a planar algebra for equations in finitely generated groups, and the representation theory of the extraspecial groups of order $p^5$.

(Based on joint work with Andrea Coladangelo. If we have time at the end of the class, we'll discuss some low-hanging open questions!)

*Prerequisites:* Linear Algebra (be able to prove that a vector space is classified up to isomorphism by its dimension) Group Theory (be able to define specific groups using generators and relations).

**Quadratic Field Extensions.** (Lara)

In this class we'll figure out why fields of the form $\mathbb{Q}\sqrt{d}$ are important and what we can say about them. We'll see what it means to be an integer in such a field and work with rings of such integers. We'll explore primes and unique factorisation, but instead of ring elements, we'll see that we should- and will-play with ideals instead. One question we'll answer is: which integers can be written as the sum of two squares?

*Prerequisites:* Ring theory. The material from Susan's week 2 class is definitely sufficient. Talk to me if you're unsure.

**Quadratic Reciprocity.** (Mark)

Let $p$ and $q$ be distinct primes. What, if anything, is the relation between the answers to the following two questions?

(1) "Is $q$ a square modulo $p$?"

(2) "Is $p$ a square modulo $q$?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

*Prerequisites:* Some basic number theory (if you know Fermat's Little Theorem, you should be OK).

**Quandles.** (Susan)

So... you know associativity? That thing that makes the operations in groups and rings at least reasonably nice? Well I say, phooey to associativity! Who needs it?

In this class we'll learn about a weird-looking nonassociative algebraic structure called a Quandle. We'll talk about how quandles behave, how they misbehave, and how they're related to knot theory. If you want just a little bit more brain-bending algebra before the end of camp, then this is the class for you.

*Prerequisites:* None.

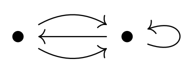
**Queueing Theory.** (Misha)

We will look at continuous versions of Markov chains and their applications to queueing theory: the study of how long you will wait in line.

*Prerequisites:* None—in particular, my class on Markov chains is *not* a prerequisite.

**Quivers.** (*Asilata Bapat*)

In of algebra and representation theory, directed graphs like to go by the fancy name of *quivers*. Here is an example.



Given a quiver, we play the following game: we put a vector space at each vertex, and a corresponding linear map at each arrow, and then try to classify how many fundamentally different such configurations we can construct.

This simple idea leads us surprisingly quickly to some classic (solved and unsolved!) problems in algebra, as well as some topics of current research. In this class we'll see how to do algebra on quivers, and explore lots of concrete examples. Along the way we'll invoke some tools from ring theory, and get a sneak preview into the world of homological algebra.

*Prerequisites:* Linear algebra (you should know about kernels and cokernels of linear maps, and direct sums of vector spaces). Familiarity with rings would be useful but not necessary.

**Ramsey Theory.** (*Lynn Scow*)

If you color each integer between 1 and 100 either red or blue, it may not be surprising that you are able to find a sequence of the form

$$a, \quad a+d, \quad a+d+d$$

where all three of these numbers are red, or all three are blue. Our next question is: did we need all one hundred numbers for this result? Results like these fall within the realm of Ramsey theory. In this course we will become familiar with different Ramsey-type problems and the tools with which to build solutions. We will leave plenty of room to experiment with small examples, and paper and colored pencils will be provided! Ramsey theory can be thought of as the theory of existence of patterns/order/regularity in large and complex structures. The cool thing about the theory is that it doesn't just say that you that these patterns probably exist, it guarantees that you could find them! And you are guaranteed to have fun doing so! Actually, that is not guaranteed by the theory, but it is probably true. Bring your love of counting!

*Prerequisites:* Addition/multiplication mod $n$, definitions of vertices and edges from graph theory, some ideas from counting like binomial .

**Real Analysis.** (*Nic Ford*)

If you've taken a calculus class that was anything like mine, you probably learned about limits and continuity in a way that might have seemed a bit unsatisfying. Something like "when $x$ gets really close to 3, $f(x)$ gets really close to 6" or "$f$ is continuous because when you draw the graph you never have to lift your pencil off the paper". Descriptions like this can be a nice way to understand the general concept that words like "limit" are trying to express, but they're pretty useless for actually proving anything. How close is "really close"? When, exactly, does $f(x)$ have to be close to 6? How would anyone even begin to write a proof that some function is continuous?

In this class, we'll talk about how to make concepts like these precise, starting with exactly what we mean when we talk about a real number in the first place. We'll start going back through the stuff you learned in calculus class, giving meaning to definitions and proofs to theorems, and when we're done you'll have seen what it really means for a function to be continuous, for a sequence or a series to converge, or for a limit to exist.

*Prerequisites:* You should be comfortable with proofs and with the basic language of set theory. In particular, you should know what it means to prove something by contradiction, to prove a statement containing the phrase "if and only if", and what it means to write something like $C = \{x : x \in A \text{ and } x \notin B\}$. It will also help if you know what it means for a set to be "countable". Calculus is not strictly required, but it will be easier to follow if you've been exposed to the ideas of limits and continuity in some form already.

**Riemann and Series.** (Lara)

The Riemann Hypothesis is one of the biggest open problems there are in mathematics. We'll begin this course by understanding what this hypothesis says. However, we'll have a far less lofty goal than proving it.

Instead the goal of this class is to find the values taken by the Riemann Zeta function (the function on which the hypothesis is based) at positive even integers. We'll develop the theory of Taylor series, see some of its applications and the problems it can be used to solve, finishing this section with understanding Euler's intuition about what $\zeta(2)$ should be. We'll then get acquainted with Fourier series and use them to prove that Euler was right and to come up with a recursive formula for $\zeta(2n)$.

*Prerequisites:* A bit of familiarity playing with integrals.

**Riemann Surfaces.** (Aaron)

Riemann surfaces form a beautiful breeding ground for ideas from many fields of math such as algebraic geometry, number theory, symplectic geometry, dynamics, and complex analysis. A Riemann surface is a surface which looks like the complex numbers if you zoom in around any point. For example, the sphere and the torus are Riemann surfaces. In the first couple days, we will use Riemann surfaces to give slick proofs of theorems from complex analysis, like Liouville's theorem and the open mapping theorem. By the end of the course, we will obtain a bound on the number of maps from any compact Riemann surface (other than the sphere or a torus) to itself.

*Prerequisites:* Complex analysis, some familiarity with point-set topology may be helpful but is not required.

**Ring Theory.** (Susan)

Ring theory is a beautiful field of mathematics. We cut ourselves loose from our usual number systems—the complexes, the reals, the rationals, the integers, and just work with . . . stuff. Stuff that you can add. And multiply. Rings are structures in which addition and multiplication exist and act as they "should." Polynomials, power series, matrices, real-valued functions on a set—wherever you have some way of defining an addition and a multiplication, you've got a ring.

Somehow, in throwing away the numbers that gave us our initial intuition about how addition and multiplication should work, we are left with a tool that is immensely powerful. Ring theory is the backbone of fields such as algebraic geometry, representation theory, homological algebra, and Galois theory.

This class will be a quick introduction to some of the basics of ring theory. We will cover the ring axioms, homomorphisms of rings, integral domains, and basic commutative localization theory.

*Prerequisites:* None.

**Set Theory.** (Steve)

Sets appear everywhere in mathematics — it's very difficult to do math without sets. What about studying sets without math?

It turns out that sets are all we need to do math! We start with the emptyset, and build progressively more complicated sets with some basic operations (taking powersets, taking unions, ...) and a couple more complicated operations. It turns out that from this modest beginning, we can build all of mathematics!

Set theory is the study of the "universe One particularly interesting question, which was asked in various ways during the early 20th century, is: what sets do we *actually need* in mathematics? One way to phrase this is to ask about *parts* of the whole universe $V$, which aren't the whole thing but still have "enough sets" that they satisfy the ZFC axioms (and maybe more!), and at the same time are easier to understand than $V$. This is called *inner model theory*, and is one of the main research areas in modern set theory. The goal of this class is ultimately to do some inner model theory, and see why it's super cool.

Still undecided? Inner model theory studies *iterable weasels.* Seriously. Google it. Weasels. Tell your friends!

*Prerequisites:* Comfort with proofs.

**Shannon Capacity of Graphs.** (Yuval)

What do umbrellas have to do with text messages? As it turns out, quite a bit! In this class, we will use graphs to understand communication, and then use communication to understand graphs. In the end, we will use umbrellas to understand both communication and graphs.

*Prerequisites:* Basic graph theory (Marisa's Week 1 class certainly suffices, but come talk to me if you haven't taken it; you might be prepared anyway.).

**Simplicity itself: $A_n$ and the "other" $A_n$.** (Mark)

The monster group (of order roughly $8 \cdot 10^{53}$) gets a lot of "press", but it's not the largest finite simple group; it's the largest *exceptional* finite simple group. (Reminder: A simple group is one which has no normal subgroups other than the two "trivial" ones; by using homomorphisms, all finite groups can be "built up" from finite simple groups. The complete classification of finite simple groups was a monumental effort that was completed successfully not far into our new millennium.)

What about the unexceptional finite simple groups? They come in infinite families, and in this class we'll look in some detail at two of those families: the alternating groups $A_n$ and one class of groups of "Lie type", related to matrices over finite fields. (If you haven't seen finite fields, think "integers mod $p$" for a prime $p$.) By the way, the simplicity of the alternating groups plays a crucial role in the proof that in general, polynomial equations of degree 5 and up cannot be solved by radicals (there is no "quintic formula").

We'll prove that $A_n$ is indeed simple for $n \geq 5$, and we should be able to prove simplicity for the other class of groups also, at least for $2 \times 2$ matrices.

*Prerequisites:* Basic group theory and linear algebra; familiarity with finite fields would be helpful, but not really necessary.

**Smol Results on the Mobius Function.** (Karen, *camper teaching project*)

Our goal is to give an overview of classical analytic number theory techniques. In particular, we'll discuss the Mobius function in detail. The Mobius function appears in many unexpected places, such as in connection to the Riemann zeta function and the roots of unity. There are various open problems about its "randomness." It is also a useful tool for extracting values from certain types of summations and for analyzing other arithmetic functions. We will define many fundamental concepts about multiplicative functions and see how to use them to prove cute (smol) facts such as "the Mobius function can be expressed as the sum of the primitive roots of unity."

*Prerequisites:* None.

**Solving cubic and quartic equations without the mess.** (Aaron)

Probably you are well acquainted with the quadratic formula. It is similarly possible to solve cubic and quartic equations in terms of radicals, but unfortunately, these formulas are incredibly messy.

How might you come up with these formulas yourself? In this class, we'll explain how to solve cubic and quartic equations using radicals via pure thought, without any messy equations.

The method I'll describe is heavily motivated by Galois theory, though you don't need to know any Galois theory to appreciate it.

*Prerequisites:* None.

**Special Relativity.** (Lotta)

According to xkcd[4], Special Relativity is very philosophically exciting and doesn't require that much mathematical background.

Special relativity is about things that go really really fast. Almost as fast as the speed of light! When things go fast, classical physics breaks down and unintuitive things start happening. Things appear to become shorter and they experience time differently. Even stranger, two people moving at different speeds relative to each other may disagree on the order that events happened. In this class, we will work from the two postulates of special relativity and derive all these strange effects.

*Prerequisites:* None, but being familiar with classical mechanics will help you appreciate the awesomeness.

**Surreal Numbers.** (Jalex)

Typical constructions of the reals go like this:

- Start with 0.
- Put in 1.
- Use addition to get $\mathbb{N}$
- Use subtraction to get $\mathbb{Z}$.
- Put in fractions to get $\mathbb{Q}$.
- Complete the Cauchy sequences to get $\mathbb{R}$.

That's a big number of steps! In this class, we'll start from the following definition:

- A Number is an ordered pair (L — R) where L and R are sets of numbers, and no number in L is bigger than any number in R,

and get not only the reals, but a much richer extension that has a proper subfield of every cardinality.

*Prerequisites:* None.

**Symmetries of Spaces.** (Apurva)

This course is about Lie (pronounced Lee) groups in low dimensions. Lie groups (or matrix groups) are groups which arise as symmetries of Euclidean spaces with extra structures.

In this course we'll study matrix groups $O(2)$ = isometries of $\mathbb{R}^2$, $O(3)$ = isometries of $\mathbb{R}^3$, $SL_2(\mathbb{R})$, $SL_2(\mathbb{C})$ = symmetries of of $C^2$, $O(1,3)$ = symmetries of the space-time $\mathbb{R}^{1,3}$, $Spin(3)$ = symmetries that give rise to electron spin = unit quaternions, etc.

We'll try to see how these correspond to observables and transformations in physics like spin, time dilation and space contraction.

[4] https://xkcd.com/1861/

*Prerequisites: groups:* you should know what a group is and what homomorphisms are.
*linear algebra:* you should know the definitions/statements of the following terms: linear transformations, eigenvalues, eigenvectors and the cayley-hamilton theorem.

**Systems of Differential Equations.** (Mark)

Many models have been devised to try to capture the essential features of phenomena in economics, ecology, and other fields using systems of differential equations. One classic example is given by the Volterra-Lotka equations from the 1920s:

$$\frac{dx}{dt} = -k_1\,x + k_2\,xy\ ;\ \frac{dy}{dt} = k_3\,y - k_4\,xy\,,$$

in which $x, y$ are the sizes of a predator and a prey population, respectively, at time $t$, and $k_1$ through $k_4$ are constants. There are two obvious problems with such models. Often the equations are too hard to solve (except, perhaps, numerically); more importantly, they are not actually correct (they can only hope to approximate what really goes on). On the other hand, if we're approximating anyway and we have a system $\frac{dx}{dt} = f(x,y)\ ;\ \frac{dy}{dt} = g(x,y)$, why not approximate it by a linear system such as $\frac{dx}{dt} = px + qy\ ;\ \frac{dy}{dt} = rx + sy$ ? Systems of that form can be solved using eigenvalues and eigenvectors, and usually (but not always) the general behavior of the solutions is a good indication of what actually happens for the original (nonlinear) system if you look near the right point(s). If this sounds interesting, come find out about concepts like trajectories, stationary points, nodes, saddle points, spiral points, and maybe Lyapunov functions. Expect plenty of pictures, and probably an opportunity for some computer exploration using *Mathematica* or equivalent. (If you don't want to get involved with computers, that's OK too; most homework will be doable by hand.)

*Prerequisites:* Linear algebra (eigenvectors and eigenvalues), calculus, a little bit of multivariable calculus (equation of tangent plane).

**The Baire Category Theorem.** (Lara)

In this class we'll prove the Baire category theorem, which tells us that a countable intersection of dense open sets is dense and explore some of its amazing consequences. Examples of these are: 1. Not only are the irrationals uncountable, but they can't even be written as the countable union of closed sets. 2. There exists functions continuous at all the rationals but discontinuous at all the irrationals, but not vice versa. 3. Any infinitely differentiable function on $[0, 1]$ that has some derivative vanish at each point of the interval must be a polynomial.

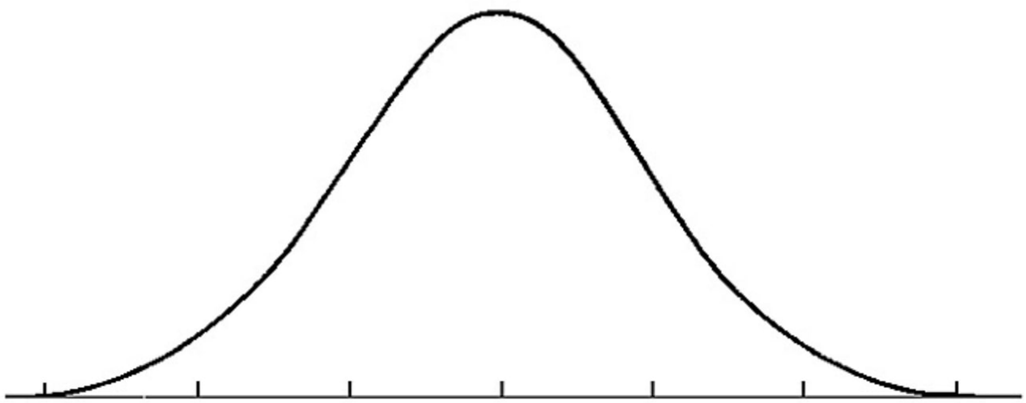
*Prerequisites:* You should know what a metric space is and be comfortable with the idea of open sets and closed sets and what the interior and closure of sets are.

**The Bell Curve.** (Mira)

> I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error." The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason.
>
> Sir Francis Galton, 1889

Human heights; SAT scores; errors in scientific measurements; the number of heads you get when you toss a million coins; the number of people per year who forget to write the address on a letter they mail.... what do all of these (and numerous other phenomena) have in common?

Empirically, all of these phenomena turn out to be distributed according to "the bell curve":



The bell curve, known in the 19th century as the "Law of Error", is now usually called the *normal* or *Gaussian* distribution. It is the graph of the function $e^{x^2/2}/\sqrt{2\pi}$ (scaled and translated appropriately). We will see how Gauss derived this function from a completely backward argument – a brilliant leap of intuition, but pretty sketchy math. We'll see how the great probabilist Laplace explained its ubiquity through the Central Limit Theorem. (Maybe you've learned about CLT in your statistics class ... but do you know the proof?) We'll talk about how the normal distribution challenged the nineteenth century concept of free will. Finally, we'll look at some other mathematical contexts in which the normal distribution arises – it really is everywhere!

*Prerequisites:* Integral calculus. (There will be a lot of integrals!).

**The Combinatorial Nullstellensatz.** (Yuval)

You might well ask, "What is the combinatorial Nullstellensatz?" That would certainly be a question. It may well even be a question that we will address in this class. When I say "we", I refer to you, the students who will attend this class, and Yuval, who will teach it. However, "we" does not include me, Ari, the person writing this blurb at Yuval's request, because I do not know what the combinatorial Nullstellensatz is. I suspect that it similar to the plain old Nullstellensatz, but somewhat more

[5] Here is a worse, but potentially more informative, blurb: The Combinatorial Nullstellensatz is one of the most versatile tools I know, and it can be used to solve a huge range of problems and prove a huge range of theorems. This is particularly surprising because it's nothing more than a very simple statement about how the roots of polynomials can be arranged. In this class, we will be focusing on the applications of this theorem, primarily ones from algebra, graph theory, and number theory.

combinatorial. Note that the word "Nullstellensatz" comes from the Tagalog word "Null", meaning "zero", and "stellensatz", meaning "stellensatz". Anyway, you should come to this class, unless you're not interested in it, but honestly I think I'll be okay either way[5].

*Prerequisites:* None.

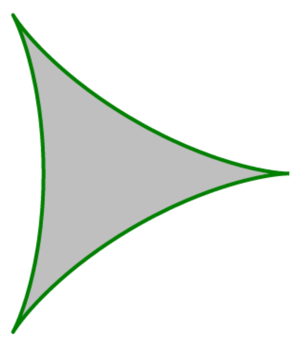
**The Fundamental Group.** (Aaron)

The fundamental group is an algebraic invariant we can attach to a geometric space which tells us about the holes in that space. For example, the real line has trivial fundamental group because it has no holes, while the circle has nontrivial fundamental group. As an application, we will prove the Borsuk-Ulam theorem, which implies that at any time, there are always some two points on exact opposite sides of the earth, with the same temperature and barometric pressure. We will use this to show you can always slice any ham sandwich, (however lopsided) with a single cut, so that there is the same amount of both pieces of bread and ham on each side of the slice.

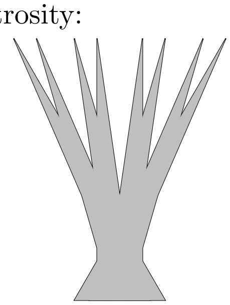
*Prerequisites:* An understanding of open and closed subsets of $\mathbb{R}^n$. Group theory is not a prerequisite though it may be helpful to take simultaneously if you have never seen it before.

**The Kakeya Conjecture.** (Yuval)

If you've ever carried a ladder, you know that turning a corner can be very tricky. But in the early 20th century, the Japanese mathematician Sōichi Kakeya tried to understand just *how* tricky this is. In this class, we'll try to answer this question. Along the way, we'll encounter one of the most important open problems in analysis, we'll see how polynomials can magically make our problems disappear, and we'll find out why this nice, simple shape



is much *worse* than this horrifying monstrosity:



*Prerequisites:* None.

**The Kakeya Maximal Conjecture.** (Yuval)

Do you like the Kakeya Conjecture, but feel that it's too plain? If so, you might be interested in taking it... to the *max*.

In this class, we'll learn about the Kakeya Maximal Conjecture, which is an important strengthening of the Kakeya Conjecture. Much of the recent progress on the Kakeya Conjecture (including the proof of the conjecture for the case $n = 2$ that we saw on the last day of class in Week 1) has actually been progress on the Kakeya Maximal Conjecture.

In this class, we'll learn what the Maximal Conjecture is, and why it's such a useful framework for proving Kakeya-type results.

*Prerequisites:* The Kakeya Conjecture.

**The Logistics of Zombies: Cobwebs and Chaos.** (Beatriz)

A zombie outbreak has been detected at Math Camp. Fearing for their lives, Math campers are trying to predict how the zombie population will evolve in time. After doing some research, they chose the logistic map as their model. The logistic difference equation is given by

$$x_{n+1} = rx_n(1 - x_n) \text{ where } 0 < r \leq 4,$$

where $x_n$ represents the fraction of zombies in our population after $n$ days, and $r$ is the growth rate of the zombie population at Math Camp.

After a brief introduction to discrete dynamical systems, we will explore the logistic map and see that it exhibits many different types of behaviour depending on the value of the parameter $r$. It turns out that this rather innocent looking equation is an excellent example of a very simple non-linear dynamical equation that can exhibit chaotic behaviour.

Since we are interested in long term behaviour, doing computations by hand becomes very tedious and time consuming. So in order to study the logistic map, we will use Mathematica to write an interactive program that will allow us to graphically explore the various properties of this map.

We will write the code to generate both cobweb diagrams and a time series plot of the iterates of the logistic map, like the ones below, for different parameters $r$ and different stating points $x_0$. This will allow us to see how changing the parameter $r$ gives rise to several different behaviours.

We will also construct a bifurcation diagram which is an excellent way to summarize the range of different long-term behaviours that arise as $r$ increases. Using a bifurcation diagram we will see that for $r > 3.57$, chaotic behaviour arises, but there are also what we call "windows of stability", in which for some values of $r > 3.57$ some stable orbits occur, giving rise to oscillating populations.

*Prerequisites:* Mathematica Workshop.

**The Mathematics of Voting.** (*Mira & Ari*)

**Note: this class is a Superclass! It meets for two periods a day, plus up to one hour of TAU (though we may not always need that full hour). On the other hand, this class does not assign homework: you will spend a large part of your time in class solving (fun) problems, but you won't need to do any work outside of class.**

Everyone knows that elections involve choices. But it turns out that the most important choice is one that most voters don't even think about: it is the choice of *voting system*, including what information gets collected from the voters and how this information is used to determine the winner (or set of winners). For instance, do the voters get to list only their first choice of candidate or do they get to rank all the candidates? Do we split the country into geographic districts each of which elects a single representative, or should everyone in the country have a say in the composition of the legislature as a whole? And of course, once the votes are in, what algorithm do we use to select the winner(s)?

*Voting theory* is the study of voting systems and their properties, which are often completely unintuitive and pathological. The choice of voting system can have a huge effect on the outcome of an election, so this topic is obviously important from a political point of view. But it is also really fun and cool math!

On Tuesday and Wednesday in class, we will introduce the basics of voting theory, explore different single-winner systems, and prove some depressing theorems showing that no voting system can have all the nice properties you want it to have. If you've seen some voting theory before, much of this will be review.

After that, we will move on to less standard topics: gerrymandering (Thursday), apportionment (Friday), and partisan symmetry (Saturday). During some of the classes, you will be doing computer simulations and/or working with real US data using a "geographic information system" (i.e. software for manipulating and analyzing maps). Even if you've taken voting theory at Mathcamp (or elsewhere) in the past, these topics are likely to be new to you.

Wednesday depends on Tuesday, but otherwise the topics are more or less independent, so you can pick and choose which days of the class you want to attend. However, the schedule given above is subject to change, so it you really want to see a particular topic, talk to Mira or Ari.

*Prerequisites:* None.

**The Pseudoarc.** (Steve)

Counterexamples are the best! We write down a totally reasonable statement — like "You can't cut a ball into pieces, rearrange them, and get two balls of the same size as the original" — and then break them with weird ideas. This class is about one particular counterexample: the *pseudoarc*.

Here's a basic fact about lines: I can cut a line into two smaller lines. A reasonable guess — written a bit informally — is that any "line-like" shape can be cut into two smaller "line-like" shapes. It turns out this is wildly false: the pseudoarc is a kind of lineish thing which can't be cut into two smaller pieces. The fact that such a weird beast exists at all is surprising; even more surprisingly, in a precise sense *most* shapes are like the pseudoarc!

In this class we'll define the pseudoarc and sketch its basic properties and the proofs of these properties. Do you like weird shapes of doom? Come to this class!

*Prerequisites:* Metric spaces.

**The Stable Marriage Problem.** (Beatriz)

Imagine we run a dating agency, and we must match $n$ men with $n$ women. We ask each man to rank the women in order of preference; similarly, each woman is asked to rank the men. Is there an algorithm to guarantee the best possible matches? In fact, to attract more clients, our agency offers one million dollars to those whose matched partner leaves them for another client; can we build an algorithm that ensures we will never have to pay up? We'll also learn why it's not a good idea for women to wait for a man to propose. Can our algorithm be adapted to solve other similar problems, such as the roommate problem?

*Prerequisites:* None.

**The Szemerédi Regularity Lemma.** (Yuval)

The Szemerédi Regularity Lemma is one of the deepest and most powerful tools in combinatorics. Roughly speaking, it tells us that all "big" graphs look "the same", and that we can actually more or less forget about where the edges of our graphs are. This may sound stupid, but its power cannot be overstated.

In this class, we'd learn the Regularity Lemma and see some of its applications in number theory, for which it was originally developed. By the end of the class, you'll have some ideas that go into the famous (and famously difficult) Green-Tao Theorem on arithmetic progressions in the primes.

*Prerequisites:* Intro to Graph Theory.

**The Traveling Salesman Problem: Recent Breakthroughs.** (Sam)

Suppose you wanted to visit all of the 559 Starbucks in Washington in as short a time as possible; this problem can be phrased as finding a Hamiltonian tour of minimum cost on a graph (i.e. finding the shortest giant cycle visiting each of the 559 Starbucks exactly once). This example is one of the most fundamental applications of the Traveling Salesman Problem[6] (TSP). Starbucks aside, the TSP is an important problems in mathematics and computer science that has taunted researchers for decades. It is hard (in a formal sense) to come up with an algorithm that exactly solves the TSP, and so instead one searches for approximation algorithms: clever ways of finding approximate solutions with theorems bounding those approximate solutions as "not too bad."

The state of the art approximation algorithm dates back to 1976. In the 40+ years since, considerable effort has gone into trying to improve on that approach. In this course, we'll discuss some of the newest and most important breakthroughs. Doing so will involve involve a hodgepodge of ideas from graph theory, combinatorics, and CS theory. We'll also see how exciting

$$1.4999999999999999999999999999999999999999999999999996$$

is!

Finally: the goal of this class is to read a recent, exciting paper about the TSP. In class we'll work through the motivation, background, and key ideas in the paper. By the end of the course, however, you should be set to read and internalize a pretty awesome research paper!

*Prerequisites:* You should feel fairly good about graph theory, including the phrases: complete graph, Eulerian graph, Hamiltonian graph, tree, and matching. You should also know, be willing to learn on the fly, or take on faith that a graph is Eulerian if and only if it is connected and every vertex has even degree.

**Topological groups.** (Aaron)

Like groups? Like topology? Then you'll love topological groups! An topological group is just a topological space which is also a group.

We'll prove some neat facts about topological groups, such as showing that open subgroups are closed, finite index closed subgroups are open.

*Prerequisites:* Group theory.

[6] OK, not really.

**Topological Tic Tac Toe.** (Beatriz)

The game of Tic-Tac-Toe is famously boring: it has a simple perfect strategy, and if two players play this strategy, the game is guaranteed to end in a draw every time. The problem is that the topology of the game board – a flat square – is too simple to allow for sufficiently many possible moves to make the game interesting. So let's allow the game board to have a more interesting topology. For example, what happens when we play Tic-Tac-Toe on a torus? A Klein bottle? A Möbius band? Or some other 2-dimensional surface? Is there still a perfect strategy? How many different first moves are there? Can two Tic-Tac-Toe games on different surfaces be equivalent?

*Prerequisites:* None.

**Trail Mix.** (Mark)

Is your mathematical hike getting to be a bit much? Would you like a break with a class that offers a different topic each day, so you can pick and choose which days to attend, and that does not carry any expectation of your doing homework? If so, why not come have some Trail Mix? Individual descriptions of the topics for the five days can be found below.

**Trail Mix Day 1: Exploring the Catalan Numbers.** What's the next number in the sequence $1, 2, 5, 14, \ldots$ ? If this were an "intelligence test" for middle or high schoolers, the answer might be 41; that's the number that continues the pattern in which every number is one less than three times the previous number. If the sequence gives the answer to some combinatorial question, though, the answer is more likely to be 42. We'll look at a few questions that do give rise to this sequence (with 42), and we'll see that the sequence is given by an elegant formula, for which we'll see a lovely combinatorial proof. If time permits, we may also look at an alternate proof using generating functions.

*Prerequisites:* None, but at the very end generating functions and some calculus may be used.

**Trail Mix Day 2: Integration by Parts and the Wallis Product.** Integration by parts is one of the only two truly general techniques for finding antiderivatives that are known (the other is integration by substitution). In this class you'll see (or review) this method, and encounter two of its applications: How to extend the factorial function, so that $(\frac{1}{2})!$ ends up making sense (although the standard terminology used for it is a bit different), and how to derive the famous product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

which was first stated by John Wallis in 1655.

*Prerequisites:* Basic single-variable calculus.

**Trail Mix Day 3: The Prüfer Correspondence.** Suppose you have $n$ points around a circle, with every pair of points connected by a line segment. (If you like, you have the complete graph $K_n$). Now you're going to erase some of those line segments so you end up with a tree, that is, so that you can still get from each point to each other point along the remaining line segments, but in only one way. (This tree will be a spanning tree for $K_n$). How many different trees can you end up with? The answer is a surprisingly simple expression in $n$, and we'll go through a combinatorial proof that is especially cool.

*Prerequisites:* None.

**Trail Mix Day 4: The Jacobian Determinant and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.** How do you change variables in a multiple integral? In the "crash course" in week 1 we saw that when you change to polar coordinates, a somewhat mysterious factor $r$ is needed. This is a special case of an important general fact involving a determinant of partial derivatives. We'll see how and roughly why this works; then we'll use it to evaluate the famous sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ . (You may well know the answer, but do you know a proof? If so, do you know a proof that doesn't require Fourier series or complex analysis?)

*Prerequisites:* Multivariable calculus (the crash course is plenty); some experience with determinants.

**Trail Mix Day 5: A Tour of Hensel's World.** In one of Euler's less celebrated papers, he started with the formula for the sum of a geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and substituted 2 for $x$ to arrive at the apparently nonsensical formula

$$1 + 2 + 4 + 8 + \cdots = -1\,.$$

More than a hundred years later, Hensel described a number system in which this formula is perfectly correct. That system and its relatives (for each of which 2 is replaced by a different prime number $p$), the $p$-adic numbers, are important in modern mathematics; we'll have a quick look around this strange "world".

*Prerequisites:* Some experience with the idea of convergent series.

**Trig Functions by Hand.** (Misha)

When you learn about trig functions, you typically memorize a few of their values (for 30° or 45°, say) and if you want to know any of the other values, you get pointed to a calculator.

Has that ever seemed unsatisfying to you? If so, take this class, in which we'll see that finding some of these values is as easy as solving polynomials, and approximating all of them is as easy as multiplication. If time allows, we'll learn how to compute inverse trig functions, and also how to quickly find lots of digits of $\pi$.

*Prerequisites:* Be familiar with the formula $e^{ix} = \cos x + i \sin x$.

**Turing and His Work.** (Sam)

It's hard to understate how remarkable of a person Alan Turing was. His contributions to mathematics are as broad as they are significant: he was instrumental in breaking encrypted Enigma messages, laid much of the groundwork for theoretical CS, wrote computer programs before anything even vaguely resembling today's computers existed, and towards the end of his life, started working in mathematical biology. He is described as "shy and diffident" and "fairly clumsy," but also as a "a warm, friendly human being" who "was obviously a genius, but [an] approachable and friendly genius [who was] always willing to take time and trouble to explain his ideas" and who was "funny and witty." He was an avid fan of chess, both writing perhaps the first computer program to play chess and inventing his own form of chess-sprinting, and ran marathons at a near-Olympian level.

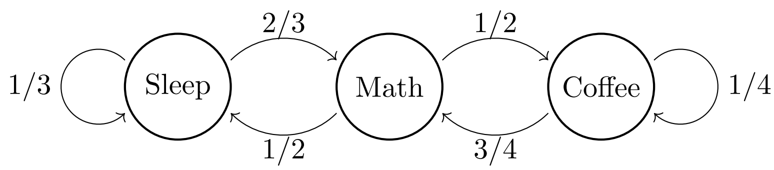
This class is a seminar in which we'll try to get to know who Turing was as a person, through a variety of lenses. Day 1 will be primarily social history; we'll look at some of the major events in his life and get a better read on his personality. Then we'll transition to seminar, where we'll read and discuss some of his papers. Finally, we'll switch gears again, and try to learn about Turing Machines from one of Turing's papers!

**Caveat Emptor:** This is a seminar style course, so "homework required" means that some reading (15 minutes to an hour) will be required. The reading is vital to the class. On the plus side, you get to read actual mathematics papers, read papers written by Turing, and learn directly from Turing through them!

*Prerequisites:* None!

**Underhanded Tricks with Markov Chains.** (Misha)

When I am not at Mathcamp, the activity I am doing is updated from hour to hour by the following rules (numbers on arrows denote probability):




This is an example of a Markov chain, and questions we might ask about it include the following:

- What fraction of the time am I doing work?
- How long will it take me to get home?
- What's the probability I'll go for a whole day without coffee?

All of these questions can be answered in a boring way: by solving systems of linear equations. In this class, we'll learn to solve them in more exciting ways: by defining a betting game about what I'm doing, granting me the power of time travel, or transforming the Markov chain into an electric network.

*Prerequisites:* None; I'll make some offhanded references to linear algebra and graph theory, but you will not need either one to follow the class.

**Unique Factorization Domains.** (*Alfonso & Kevin*)

You know that every integer can be written as a product of primes in a unique way. But, are you sure this is true? It turns out that proving the uniqueness part is not easy at all, even though we all take this fact for granted since kindergarden!

In this class you learn how to prove this rigorously, and you will also study other "number systems" where the same result is true, and where it fails. Sometimes the uniqueness part fails. Sometimes some numbers cannot be written as product of primes at all! Pathological examples are delicious.

This is an IBL class, where you will be doing most of the work yourselves, while we help you. There will be a daily handout, and we will expect that you finish some problems during TAU if you could not do them during class (since otherwise you will get lost on the next day).

*Prerequisites:* The class class will be easier if you know what a ring is and what an ideal is. You can still take the class if you do not know this, as long as you are willing to work hard (perhaps with our help during TAU).

**Urban Planning.** (*Luke Joyner*)

Cities, both real and imaginary, are as complex as anything humans have ever created. On the one hand, they are consummate examples of structure and planning, as profound and intricate as a good fugue. On the other hand, they are the messy artifacts of time and people doing unpredictable things, artifacts of politics and culture and power and resistance and greed... they can become characters in our lives, friends even. How can we look at cities to understand them, appreciate them... imagine them? This class will be a project-based introduction to cities and urban design, including a dive into the geometry and topology of city grids and networks. It will include some interesting math (mostly in problems to work on, rather than lecture), but will not be exclusively math, because while math bubbles up everywhere when you think about cities and places, it's necessary to look at them in other ways too. So we will walk all those places, sometimes randomly, sometimes with intent, and use what we've learned, and our own life experience, to try and redesign Tacoma's streets by the end of the week. Cause we're here, and we're crazy like that.

*Prerequisites:* Basic understanding of Graph Theory (Note: drawing skills are not required, but drawing by hand or on the computer will be an important part of our final project; I will lead an optional seminar on drawing techniques for anyone interested on Wednesday or Thursday.) .

**Using quaternions to describe symmetries of Platonic solids.** (*David Morrison*)

TBA

*Prerequisites:* TBA.

**Vertex-Transitive Polytopes.** (Viv)

We're going to spend this class studying highly symmetric shapes, for differing definitions of "highly symmetric" and "shapes." For starters, we'll define regularity and prove that the number of regular polytopes in $n$ dimensions is given by one of my all-time favorite sequences:

$$1, \infty, 5, 6, 3, 3, 3, 3, 3, 3, 3, \ldots$$

Then we'll talk about relaxing regularity to vertex-transitivity. This gives us a lot more leeway to think about shapes that we'd really like to be able to call highly symmetric; for example, a soccer ball is vertex-transitive but not regular. We'll talk about what happens when we try to construct vertex-transitive shapes that have many vertices, and what happens with shapes in many dimensions. Along the way, we'll discuss discrete Gauss-Bonnet and Euler characteristic.

*Prerequisites:* None.

**Wallpaper Patterns.** (Susan)

Your wallpaper is a fascinating mathematical object! Well, maybe not your wallpaper in particular—you may not even have wallpaper. However, any repeating pattern that we use to decorate a wall is an example of a mathematical object called a "wallpaper pattern."

In this class we will be discussing the classification of wallpaper patterns. We will explore a beautiful topological argument that shows that there are exactly seventeen distinct types of wallpaper pattern.

Expect lots of drawing, cutting, pasting, folding and smershing—this is a hands-on class!

*Prerequisites:* None.

**Weak Separation.** (Kevin)

Here's a fairly unremarkable-looking combinatorial definition. Label $n$ points around a circle in order from 1 to $n$, and let $S$ and $T$ be $k$ element subsets of the points. We say $S$ and $T$ are *weakly separated* if we cannot find a chord with endpoints in $S - T$ and a chord with endpoints in $T - S$ that cross. Symbolically, if $S$ and $T$ are $k$ element subsets of $1, 2, \ldots, n$, then they are weakly separated if there do not exist $a, c \in S - T$ and $b, d \in T - S$ so that $a < b < c < d$ or $b < c < d < a$.

It turns out this harmless definition hides many secrets. In 1998, it was conjectured that if you try to collect as many $k$ element subsets of $1, 2, \ldots, n$ as you can so that any two are weakly separated from each other, you'll always end up with exactly $k(n - k) + 1$ of them. This so-called "purity conjecture" took over a decade to prove, and along the way this innocent idea of weak separation made a crucial appearance in the nascent study of cluster algebras, among several other modern topics.

In this class, we'll get a glimpse of what cluster algebras are and how weak separation gets involved in one particularly beautiful cluster algebra. We'll also develop the combinatorics of wonderful objects called plabic graphs to start shedding some light on the purity conjecture.

*Prerequisites:* None.

**Welzl's theorem on graph homomorphisms.** (Jalex)

A *graph homomorphism* $f : G \to K$ is a function from the vertex set of $G$ to the vertex set of $K$ such that if $(x, y)$ is an edge of $G$, then $(f(x), f(y))$ is an edge of $K$. Say that $G < K$ if there exists a homomorphism from $G$ to $K$ but there does not exist a homomorphism from $K$ to $G$. In this class, we'll prove a theorem of Welzl: The partially order set of finite graphs with $<$ is dense. In other words, if $G < K$, then we can always find $H$ such that $G < H < K$.

*Prerequisites:* Know what a graph is. Know what a function is.

**What is Homology?** (Apurva)

Enough said.

*Prerequisites:* Linear algebra,
when I inadvertently utter the phrase 'topological space' in class you should be happy and not sad.

**What's It Like to Live in a Hyperbolic World?** (Linus)

The video game HyperRogue takes place in a hyperbolic plane. I learned what the hyperbolic plane is like from playing this game. Come to the computer lab and play this game for one hour.

*Prerequisites:* None.

**What's The Deal With $e$?** (Susan)

The continued fraction expansion of $e$ is

$$1+\cfrac{1}{0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\ddots}}}}}}}}}}}}}}$$

Okay, but seriously, though, why?!?! Turns out we can find a simple, beautiful answer if we're willing to do a little integration. Or maybe a bit more than a little? Come ready to get your hands dirty—it's gonna be a good time!

*Prerequisites:* Familiarity with integration by parts, and basic partial fractions. Campers who are already familiar with continued fractions can skip the first day.

**Young Tableaux.** (Kevin)

A *standard Young tableau* (SYT) is a way to fill a portion of a grid of boxes with the first $n$ positive integers so that rows and columns are increasing. For example, here are all the SYT with three boxes:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

Notice that the shape in the middle corresponds to two different SYT. A natural question to ask, then, is how many different SYT there are of a given shape.

In this class, we'll study the wonderful world of combinatorics associated with counting the number of SYT. We'll see a beautiful bijection between pairs of SYT of the same shape and permutations. We'll also meet the celebrated hook length formula of Frame, Robinson, and Thrall, which was the subject of recent controversy for trivializing a USAMO problem from 2016.

*Prerequisites:* None.

## Colloquia

**Algebra and the Internal GPS.** (*Nora Youngs*)

When you think of neuroscience experiments, one thing which may come to mind is a scientist in a lab coat running rats through a maze. But how exactly are those rats learning to navigate that maze? Part of the answer lies with a set of neurons called place cells, which are so named because they are specifically active for certain locations. We'll consider an algebraic way to extract useful geometric information from the neural data of place cells, and explore how that information can tell us something about the animal's environment.

**Covering Spaces and Square Dancing.** (*Alfonso Gracia-Saz*)

A covering space of a topological space (for example a surface or a curve) is what you get when you "unfold" it. For instance, you can unfold a circle entirely and get a line, or unfold it partially and get ... another circle. You could also unfold a torus, and get another torus, a cylinder, or a plane. You can unfold almost anything, like a Klein bottle or $GL(n)$, but you cannot unfold a Hawaiian ring. Interestingly, when we unfold a topological space, paths that started and ended at the same point end up wandering in space, creating something called monodromy.

Covering spaces have many applications in daily life, such as Lie groups, quantum field theory, or square dancing. What does square dancing have to do with covering spaces? Usual square dances have 8 dancers, but there is a 12-dancer variant called "hexagon dancing". SD callers often go through a lot of trouble to explain the rules for hexagon squares, and are usually at a loss to figure out when a choreography that resolves in regular squares will resolve in hexagon squares. Their lives would be so much simpler if they simply said "hexagon squares are a triple cover of the quotient of regular square dancing by a $\mathbb{Z}/2\mathbb{Z}$ symmetry and a choreography that resolves in regular squares also resolves in hexagons if and only the path of every boy composed with the inverse of the path of his girl has winding number around the center congruent to 0 mod 3." In other words, this is a real-life application: a question posed by dancers that algebraists managed to solve.

This talk will be illustrated with shiny animations, courtesy of Ryan Hendrickson. No topology or dancing knowledge will be assumed.

**Down the Rabbit Hole.** (*Anti Shulman*)

Come enter a world where everything you think you know about mathematics is in doubt: where not every subset of a finite set has to be finite; where a real number need not be either positive, negative, or zero; and where the Intermediate Value Theorem and Extreme Value Theorem can fail to hold. This is the land of constructive mathematics, where we deny the law of excluded middle ("everything is either true or false") and forbid proof by contradiction ("if something isn't false, it must be true").

Why would we do such a thing? (Other than to annoy your calculus teacher, I mean.) It turns out that like Alice's rabbit hole, ours is inhabited not only by weird creatures, but also by magic. In constructive mathematics, every function is continuous, everything that exists can be found by a computer, and we can do calculus with true infinitesimals rather than epsilon-delta limits. Arguably, constructive mathematics reflects the "real world" even better than classical mathematics does!

(This colloquium will be just a taste of constructive mathematics; to learn more about it, come to my class this week.)

**Farey Fractions and Plane Geometry.** (*Noah Snyder*)

How do you find the best way to approximate an irrational number using rationals? How good are these approximations? These questions can be answered using certain properties of the Farey sequence. These patterns in the Farey sequence can in turn be proved using plane geometry. No background will be assumed (in particular, this talk should be understandable to people who haven't seen continued fractions while still being interesting to people who have.)

**Games People (Don't) Play.** (Steve)

Let's play a game! *Hackenbush* is the best game for people who don't like art—we start with a pretty drawing, and then get rid of it, and the first person who can't make less art loses.

Hackenbush is a really mathematically interesting game. One easy thing we can say about a game is whether a given player has a winning strategy. But there's more we can do: it turns out we can assign *numbers* to games, measuring how *much* a given player wins the game, and do arithmetic with these numbers by combining the games in certain ways. Hackenbush is a particularly good example of this. So Hackenbush is *also* the best game for people who like adding numbers.

But we can only get *some* numbers this way. I want more numbers! It turns out the right thing to do here is consider *infinitely long* Hackenbush games. Actually playing an infinite game isn't really something we can do, but they turn out to be very mathematically interesting and useful, and Hackenbush is one of the easiest impossible games to play. So infinite Hackenbush is the best game for people who don't exist.

In this colloquium I'm going to pretend not to exist—come pretend not to exist with me!

**Hydras.** (Susan)

The Lernean Hydra was a legendary monster with many heads, poisonous breath, and an all-around bad attitude. The hero Heracles was sent to kill the beast, but found that whenever he cut off one of its heads, two would grow back in its place. What's a hero to do? We will attempt to slay a different kind of Hydra. In the Hydra game, we start with a rooted tree (our Hydra), and in each turn, we remove a "head". On the $n^{\text{th}}$ turn, $n$ new Hydra heads will grow back in its place. Heracles's story has a happy ending—he was able to kill the Lernean Hydra with an extremely clever plan of attack. What sort of cleverness do we need to kill our Hydra? Come and find out!

**Many Campers Split Pizza.** (*Asilata Bapat*)

How can we split a circular pizza among $n$ campers and make sure everyone gets an equal share? The usual way is to slice it by diameters at equal angles, so that the number of pieces is a multiple of $n$.

But this is not the only easy solution! In this talk we will discover some other surprising ways to solve this problem, with the help of some Euclidean geometry, some calculus, and some pictures.

**Opinionated Primes.** (*Zach Abel*)

Many mathematicians can tell you about their favorite prime number, but you've never met a prime number like this one[7] that will tell you, at length[8], about its favorite mathematician! *Wait, that's a massive number. How do you know it's actually prime?* My computer said so. *But how does the computer know?* It asked the Fibonacci numbers. *And how did you* find *a prime so large, anyway?* By setting Pascal's triangle on fire. *Who is this prime's favorite mathematician? What does that even mean, anyway?* You'll have to ask it!

These questions (and answers!) will be discussed in detail during the talk, as well as all the prime-number mischief they unlock.

---

[7] 4555555565555555555555656555666666656665656566666666666665555465656555554455555555566655656665566666666656666666666454554655555655
556555555566655666556656566666565666656663444455666656666566556565666566656565655566666665655656444565556566556566666556666566
5666565656665655556566655655455555656666666666656666566556566665566565555655555656564445545555666666665666666566656566656566656555565
5565656654555455565666566666565555555676666655565656556565656566645445555566566664334545666666654545555665655556565556545455
5565656564222212455555553111235656566665555656665645555555556566643334222123344332222222555666565665666665454555655655554232222
2222321224322212556565656666656564545555656565441211111111232212111211245656666566666565455565655534312121111112121111111122
112245556665656565655555565654321212122211121222121233311211255556666666555565565653312121222122121211212112343111212345555655
564555555553211112122121222112121121213332111212555555556544555665321121222222122322122111122331211212214545555455455555664222
133222222223332112111121232222112112245455444555565211122323212333422221221112332232322221244444445556531121212232221223231
121121122232222222111134453444455521112122332222112233212211211222212121121344544455545411222212223221212222222212112122222
2112112444444444622113321212222321124331121221122112122122114454444455211223312121232112122222211232122122222211234444445
4522122332121212221212112212234432212121112212444445454112233222111112111212221212456432222112112123444444522113342222212
11112111212212555544321211222134534454431222333222211211212222222234555553432222112244544445321122333333212232321212223455
65454333121221244544444432112333332122224432323233245555555433322223245544444221222432222221356666534445555454555534332213
4544344421211223322222224665655555555555445443443321234455434444211222323222223566665655555445555544432222335555344432212122
32321225566665333333334445432212221345555444421212122321222467765321122222223445422222212355554334422212122222213566742233
3222322345432322222244554543431111112122222257743545332223334553322332232545555333312111213322122565555322222122356632222
22222245554533321111212563212367665422121122356663223221222455454344121121123352224777543225322223566642353222325555443331
21121215442225767553335521224567642223222335555543332121111245233367777765444323356677422222223334545544331111211233345667
877776544344566667522244343245445433312111112343566777787876555666767675324444433554544333121211113444566777777777667667
66754244544334544554341111211124543577778778877776766668652344444444445443331111112112565667777787777767666666542234444344
44543432112111112365566777776776767666454442223434423444443431111211212146457777777767767663222222234344244444443321121212
12134336767777777776664442222334443333444434321211211222211366777777677767666762223444353333444434321121121211212366777677
6777766665442344344332233443431111121122222225677777776766566443222234444321233443311221221222222156676777777663433322222
24443222224433221121122122221246677777776654344442212344432222233433212112212212212365677776776665553222223553323222233333
1112111212222123566677777766565532233444432222211343211112111221212125666677776766655434334452322333212432112111212112211
14666566776666665665544442222322212342111121212212121136765556666667676665444332332322222432121112121212112115766655666677
77655434343333332222322212112112221121214776655555656654544442444423222213312111121121212121213676665445544444343222554332
3231233111212121222212211167766655555554433222366633212222332112212212222212112467756655444432112125656533222123221121111
1121221212125666554433221121124665665332222332112122121212121212433443333222221113256665665433223321212112121111211243433
332212111121125767666734444223211111211112221115644332222212222212586765666534545434121121122122212138765332322222222222486
76666633454553311121221211121137876566566454444334866666666523334453321111112122121268876777675544454358776666662232343432
211111111121126889767766666555436887777766412223343222111111212124788986777666555446987888877622221333222211111211112457
9998676766555557987888888731122223223221111111213258999866766665558988888888885112111321132211111112223168989976776666689
8898888886311212212212122111122313117999988567676698988889888841111111211123121232242122179989986567668988888888886212121
1121212222242333121129887665454555567777889897411121112121222223213311111385435446665634443446888861212121121122233222121
1111148357534666553355334589882222111221221223121112111111665655446654334653455889422122121222122312111211111115665555254
42333534555887222111112222222311111211111134645433312213543354457932212111211221211111111111111344544434111113445554458721
21122122211221111121111123445344422112113523544227322212122112221122111111111122354434221111112433642312122121222221222221
1111112112235533311212111145566343122121211212212121111111111111254431111111111256655442122221212221222212111111211112243
1111111111114665544412221212121211111111212111121112121211111111135656644311211112212121211111111211111121111112121121156
576543212112111121212111211111111211112111111111111256655344211211121112111111111111211112111111211111111114676542112112111
121111111111111111211111211111211221112454311211121211221211111111211211111111111111121111211221112121212212111121111111111
111111121211112121121222121212121121111112111111111111111211111111111211112111211212112122112111211211111111111111111121111111
211112112111212211111121212111111111111111111111112111111121212122121121121212111111111111111111111111112111212121111211212222
1212111111111111111111111121111112111111212112212121212112211111121111111111111111111111111212121122121212121112111211221111
11111111111111111111111121111121211121212221111111111111111111111111111111111111111211211111121111111

[8] A 5,671-digit length, to be exact…

**Rational Tangles.** (Tim!)

Imagine four people standing in a circle holding the ends of two ropes. A caller gives them instructions to move around, twisting the ropes into a tangle. After some time, a magician enters the room. The caller tells the magician a single rational number: with just that information, and without looking at the ropes, the magician gives instructions for how to undo the knot.

This magic trick is just one of the things we'll learn about rational tangles, which are a systematic approach, proposed by Conway, to describing knots. In this colloquium, we'll see how to build complicated knots from simple pieces, and how to use group theory to untangle the secrets of their hidden structure.

**Squaring the Circle.** (*Andrew Marks*)

The idea of dissecting a set and rearranging its pieces to form another set dates back to the ancient Greeks. One application of this idea is finding formulas for areas of polygons. For example, we can dissect a parallelogram into a triangle and trapezoid and then rearrange them to form a rectangle. This idea can be used to show that the area of a parallelogram is its base multiplied by its height. Building on these ideas, mathematicians have been investigating the general problem of when we can show that two shapes have the same volume by cutting them into congruent pieces.

In two dimensions, it turns out that this always works for polygons—a famous theorem of Wallace-Bolyai-Gerwien states that any two polygons of the same area can be chopped into smaller congruent polygons. This theorem realizes the dream of the ancient Greeks, and has a beautiful proof using pictures.

The analogous problem in three dimensions was one of Hilbert's famous problems. Breakthrough work of Dehn from 1901 showed that there are two polytopes of the same area which are not dissection congruent. Dehn proved this by introducing an important geometrical invariant called the Dehn invariant.

More recently, mathematicians have been thinking about similar problems for shapes that are not polygons. In 1925 Tarski asked if a disk can be partitioned into finitely many sets which can be reassembled to form a square of the same area. This question became known as Tarski's circle squaring problem. It remained open until Laczkovich gave a positive solution in 1990. At the end of the talk, we'll say a little about Laczkovich's solution, and our recent result joint with Spencer Unger that gives an explicit way to square the circle.

**The Icosian Game.** (Misha)

Can a knight visit all 64 squares of a chessboard in 63 jumps, then come back to the start? What if we ask the same question for a $4 \times 4$ board? What if we're instead walking around the vertices of a dodecahedron?

In this colloquium, we will figure out when the answer to such a question is definitely "yes", and when it is definitely "no". In between, there will be a disturbingly large range of cases where we can only say "I don't know". But that's okay, because I'll also explain why, if you could always solve this problem easily, then you'd be able to win a million dollars, steal billions of dollars, and break all of mathematics as we know it.

**The Party Problem and Ramsey.** (*Lynn Scow*)

The Party problem states that at any party with at least 6 people, there will be 3 among these 6 who all know one another, or 3 among these 6 who are mutually strangers. Actually, it's not a problem! It's fun to meet new people, and it's fun to catch up with old friends! What is the minimal number of people ($n$) that need to be at the party to find 4 who all know one another, or 4 who are mutually strangers? In this talk I'll given an introduction to what are called Ramsey numbers (the numbers $n$ above) and mention some extensions of this idea to other situations.

**Twenty-Seven Lines.** (*David Morrison*)

Let $f(x, y, z)$ be a polynomial of degree 3 in 3 variables, and suppose that at every point of the corresponding cubic surface $S = \{f(x, y, z) = 0\}$ the tangent plane to the surface $S$ is well-defined. Suppose we seek straight lines in space which lie completely within $S$. Under suitable conditions, there are exactly 27 such lines! We will see why this is true, and investigate the intricate combinatorial structure which the lines posses. (For example, each line meets exactly 10 other lines.)

The "suitable conditions" include: (1) thinking carefully about the behavior of $S$ "at infinity" and the possibility that some of the lines may be located "at infinity," and (2) solving equations over the complex numbers rather than over the real numbers. (Over the real numbers, the conclusion is "at most 27 lines.")

In spite of these caveats, there exist polynomials with real coefficients having exactly 27 real lines, and we will see an example.

**Using Math to Protect Democracy.** (Mira)

Every 10 years, the US has a census to determine the number of representatives each state should get in Congress. Then the legislature in each state comes up with a *districting plan*: a way of splitting the state into the correct number of regions (districts) of equal population, each of which will elect one representative to Congress.

The problem is, the specific choice of districting plan can have a huge effect on the results. For instance, suppose a state of 400,000 people is allotted four representatives. There are two political parties, A and B, and Party B happens to be in power at the time of the census. It does some polling and adopts the following districting plan:

| District | A supporters | B supporters |
|---|---|---|
| 1 | 85,000 | 15,000 |
| 2 | 45,000 | 55,000 |
| 3 | 45,000 | 55,000 |
| 4 | 45,000 | 55,000 |

Now B can count on winning 3 out of 4 districts in the next election, even though 55% of the the voters in the state support A!

This kind of thing happens all the time; both Republicans and Democrats do it. It seems so obviously wrong that you wonder how it can be legal. But it turns out that determining whether a districting plan is "fair" or "neutral" is much more complicated than you might think. Right now, many people, including lawyers, political scientists, computer scientists, and mathematicians are working hard to figure out a solution to the gerrymandering problem before the next census in 2020. You can help! Come to this talk to learn how.

**Why are drums shaped by number theory sometimes louder than others?** (*Djordje Milicevic*)

Simple harmonics, such as monochromatic light waves or heart rhythms or standing patterns of a vibrating string, are basic building blocks of analysis: a compound signal like sunlight or the sound of your favorite instrument is composed of (many) single-color bands or single-pitch tones.

On more general spaces, flat ones or those with some curvature, the role of simple harmonics is played by eigenfunctions, objects central in contexts ranging from spectral geometry, a field whose spirit was captured by Mark Kac's famous question "Can you hear the shape of a drum?", to quantum mechanics, where they represent "pure quantum states" and where their concentration of mass is closely related to geometry and dynamics.

After describing some of these fundamental modes and what they can tell us about the underlying spaces, we will discuss what eigenfunctions have to do with number theory (things like primes, or divisors, or Fermat's Last Theorem) and how additional symmetries of arithmetic or geometric nature can drive their exceptional behavior not generically observed or predicted by physical models.