SOME PROBLEMS IN COMBINATORIAL GAME THEORY

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This is a follow-up to my colloquium on Combinatorial Game Theory. Here are a few other problems in combinatorial game theory, ordered from easiest to open problem. Your task is to find the winning strategy.

1.- Another berry game. We have a row of blueberries and raspberries. In his turn, a player has to eat one or more berries, but they all have to be of the same kind, and they all have to be consecutive berries from the right end of the row. For instance, if the row is *RBBRRRRBBBB*, then the player may eat one, two, three, or the four right-most blueberries. If the row is *BRBBRRRB* then the player has to eat the right-most blueberry. Remember that your goal is to eat the last berry.

2.- The M&M game. We start with two bags of M&Ms, one with 19 and one with 20. In her turn, a player has to eat all the M&Ms in one of the bags, and then split the M&M from the other bag between the two, leaving at least one on each bag, and not necessarily evenly. The she gives the two bags to the other player. The player who receives two bags with one M&M each loses as he can no longer move.

3.- The Euclid game. Two players start with a pair of distinct positive integers, a and b. The first player subtracts a positive multiple of the smaller integer from the larger one so as to leave a positive remainder. He thus creates a new pair of positive integers. The second player repeats this process with the new pair of integers. The players continue taking turns in this fashion, and the player who creates a pair of equal integers wins. For instance, if the original pair is $(24, 7)$, then the legal moves are $(17, 7)$, $(10, 7)$, and $(3, 7)$. For which pairs (a, b) does the first player have a winning strategy?

4.- Wythoff Game. We have a plate of blueberries and a plate of raspberries. In her turn, a player may do one of the followings:

- eat as many blueberries as she wants, at least one,
- eat as many raspberries as she wants, at least one, or
- eat as many blueberries and raspberries as she wants, the same amount of both, and at least one

The player who eats the last berry wins.

For which original positions does the first player have a winning strategy and what is it? This problem is hard, but I assure you that the pattern is beautifully mind-blowing.

5.- The princess and the roses. Princess Alice has two suitors: Laura and Renata. They alternate days in trying to gain her heart by bringing her roses. They pick up the roses from the same garden. Each day, the corresponding suitor will bring her one or two roses, but never two roses of the same colour. The suitor to pick the last rose from the garden will win Alice's heart. Assuming that initially there are 3 blue roses, 4 red roses, 5 yellow roses, and 6 white roses, who will win?

Can you solve the same problem if there are roses of k colours, for arbitrary values of the initial number of roses of each colour? I recommend you start with one colour, then two colours, and so on.

Warning: The problem is not too hard for up to five colours. However, for six or more colours, this is an open problem! This means that nobody knows the answer or even whether there is a simple way to describe the winning strategy in general.

References

If you find these problems interesting, a good reference to learn more is:

- E. Berlekamp, J. Conway, and R. Guy: *Winning Ways for your Mathematical Plays*, 2001, A K Peters, Ltd.

If you have any questions or want hints, you may reach me at `alfonso@mathcamp.org`.