CLASS DESCRIPTIONS — WEEK 3, MATHCAMP 2018

Contents

9:05 Classes

History of Math. (Sam, Tuesday–Saturday)
Do you remember a time when variables didn't exist? When we had to communicate equations with sentences like:

> Take absolute number on the side opposite to that on which the square and simple unknown are. To the absolute number multiplied by four times the [coefficient] of the square, add the square of the [coefficient of the] unknown; the square root of the same, less the [coefficient of the] unknown, being divided by twice the [coefficient of the] square is the [value of the] unknown [Quoted from Katz, from around 700CE in India]

If so, you're probably about 400 years old. Think about that: for the overwhelming majority of mathematical history, symbolic notation simply did not exist. This class is about moments like that transition: moments when the way we do mathematics fundamentally changed. Topics may include:

- The emergence of mathematics
- Greek mathematics and proof
- The development of symbolic notation
- Analytic geometry, and the recognition that curves and equations captured **the same** thing
- Cauchy and a rigorous foothold for calculus
- The emergence of probability in the 17th century
- The St. Petersburg paradox, and a fundamental challenge to mathematical tools

Chilis: 🌶🌶
Homework: Optional
Prerequisites: None

Linear Programs & Convex Optimization. (Linus, Tuesday–Saturday)
What do these problems have in common?

- You play rock-paper-scissors, but winning with rock is worth 2 points instead of 1. What is the Nash equilibrium?
- Is there an English anagram between only vegetable words and only fruit words? "Parsnip lima" → "Apple raisin" almost works, but it changes an 'm' to an 'a' and "lima" isn't really a vegetable.

- Can you place overlapping L-shaped triominoes into a 5-by-7 square grid so that every square is covered the same number of times?

The answer: they can each be expressed as a system of linear inequalities. These systems are a rare intersection between (a) super useful and (b) super beautiful theory. Probable topics:

– Every system of linear inequalities has a "dual" system ("LP duality")
– How a computer can solve systems of linear inequalities quickly
– Applications of LP duality in combinatorics (such as Hall's Marriage Lemma)

Chilis: 🌶🌶🌶

Homework: Recommended

Prerequisites: Ability to program *not* needed; this class is not about writing computer code. Be comfortable working with vectors in n-dimensional space. Basic linear algebra recommended but not required.

Related to (but not required for): Guess Who? (W2); Max-Flow Min-Cut (W3)

Partially Ordered Sets. (*Mike Orrison*, Tuesday–Saturday)
Relations on and between sets are ubiquitous in mathematics, and one of the most useful kinds of relations is that of a partial order. Familiar examples include the less-than-or-equal-to relation on the set of real numbers, the divides relation on the set of positive integers, and the is-a-subset-of relation on the set of all subsets of a set.

This course will introduce some foundational ideas associated with partially ordered sets. We will then use this material to begin working with one of my favorite ideas in mathematics, Möbius inversion, which is an eye-opening generalization of inclusion-exclusion and a discrete analog of the Fundamental Theorem of Calculus.

Chilis: 🌶🌶

Homework: None

Prerequisites: Matrix algebra (adding and multiplying matrices) and the ability to create and work with unions, intersections, and direct products of sets.

Systems and Signals Analysis. (Jeff, Tuesday–Saturday)
One of the simplest ways to model a real world phenomenon is to model it with a function of a single variable. For instance, if I want to understand a piece of music, I could use the function

$$\rho(t) := \text{the air-pressure density of the sound wave that passes through my ear at time } t.$$

Or if I wanted to give you an understanding of a lighthouse in the distance, we could study the electromagnetic field strength as a function of time. The list goes on. We'll call all of these types of functions *signals.*

While this provides a simple description of things we see and hear in the real world, this does not give a good description of how humans observe the world. With most phenomenon, people perceive the frequency content of the signal instead of the time content: when we listen to music, we perceive pitches instead of air density. When we watch a movie, we perceive colors at different intensities instead of the electromagnetic wave.

One of the mathematical tools we'll develop in this class is the *Fourier transform*, which takes a signal and produces the frequency data of that signal. We'll then talk about *systems,* which describe models which modify signals — think like how a mute on a trumpet changes the tones the trumpet produces. Finally, we'll classify a wide variety of systems from simple measurements.

Chilis: 🌶🌶

Homework: Optional

Prerequisites: You should know how to integrate and differentiate trig functions, as well as understand and give a solution to the differential equation $\frac{df}{dt} = f$. Complex numbers and Euler's formula are useful, but not strictly necessary.

Trees! (Shiyue, Tuesday–Saturday)
What do a tree and a language with only two letters have in common? How can you have a bigger group as a subgroup of a smaller group? Can you decompose a ball into a finite number of sets and reassemble them into two balls identical to the original? If you think that the group axioms are boring, how else could you define a group? Is a subgroup of a free group free? What?? Free? What am I talking about anyways?

Here is the magic word that will chain together all these questions: trees! All of this seeming nonsense falls into a serious math subject called "geometric group theory", where the interplay between geometric spaces and groups reveals big secrets about the world that we live in and the groups that you happen to ponder upon. We will start with trees, graphs, free groups of finite rank, group action on trees, Cayley Theorems, and tons of examples such as Coxeter groups. And in the process, we will answer all of the above questions!

Chilis:

Homework: Recommended

Prerequisites: Definition of a group; some additional group theory knowledge is useful but not necessary.

Related to (but not required for): Group Theory (W1); Intro Graph Theory (W1); Visualizing Groups (W4)

10:10 Classes

Axiomatic Music Theory. (J-Lo, Tuesday–Saturday)
What makes some combinations of notes more pleasant to listen to than others? Why does the chromatic scale have 12 notes? And what's up with those same four chords being used in every pop song?

At first glance, music might seem like a collection of completely arbitrary facts that just coincidentally combine in consonant ways. But on the contrary, many of these complicated musical constructions can be derived as corollaries from just a couple of basic conditions that we think music ought to satisfy; you will reach these conclusions for yourselves by working through a set of problems in groups. The rich, complex intricacy? That's actually number theory under cover.

Chilis:

Homework: Recommended

Prerequisites: None (knowing some group theory may help, but this can also be picked up as you go along)

Commutative Algebra and Algebraic Geometry(1/2). (Mark, Tuesday–Saturday)
In its classical form, algebraic geometry is the study of sets in n-dimensional space that can be described by polynomial equations (in n variables). This is both a very old and a quite active branch of mathematics, and for over a century now it has relied heavily on commutative algebra — that is, on the properties of commutative rings and related objects. We'll start by looking at some of those, including prime and maximal ideals and a review of quotient rings, and we'll see how the algebra can be used to give us information about the geometric sets. For instance, we'll show that if a set can be given by polynomial equations, then a finite number of such equations will do. We may also see how to translate the idea of dimension into the language of algebra. There may well be cameo appearances

by the axiom of choice (in the guise of Zorn's lemma) and a bit of point-set topology (on a space whose points are ideals!), but you don't need to know any of those things going in. In the second week, I hope to prove Hilbert's famous Nullstellensatz (Theorem of the Zeros), arguably the starting point for modern algebraic geometry, at least for the case of two variables.

Chilis:

Homework: Recommended

Prerequisites: Familiarity with polynomial rings, ideals, and quotient rings.

Related to (but not required for): Algebraic Number Theory (W1); Intro Ring Theory (W2); Intersecting Curves (W2); The Outer Automorphism of S_6 (W2); Representation Theory (1/2) (W3); Rational Points on Elliptic Curves (W4)

Convergence Issues; or: Monsters in Real Analysis. (Ben, Tuesday–Saturday)
During the 1800s, infinite series became increasingly important in analysis, especially series of functions, such as Fourier series. These were useful for solving differential equations, but some mathematicians working with them began to notice problems. Abel discovered a Fourier series which was discontinuous at some points, despite being a sum of continuous trigonometric functions. Dirichlet found that he could change the value of one series by rearranging its terms, a result that Riemann vastly generalized. Finally, in 1872 Weierstrass presented a function that was continuous everywhere, but differentiable *nowhere*. These struck one prominent mathematician, Poincaré, as being about as far from "honest functions" as you could possibly get: he called them "monsters."

Well, this course is about the monsters. We'll start with Riemann's theorem on rearranging series (the one referenced above) and then move on to working with sequences and series of functions. In this work, we'll see a few common features of our monsters, and learn about the helpful concept of uniform convergence. Then, once we learn a few of the standard results about uniform convergence, we'll be ready to talk about the biggest monster I mentioned above: Weierstrass's Nowhere Differentiable Function.

Chilis:

Homework: Recommended

Prerequisites: This course focuses on ideas from calculus (limits, series, derivatives, integrals), so you should know what all of these are.

Related to (but not required for): What *should* integration be? (W1); Metric Space Topology (W2)

Low-Dimensional Zoology. (Larsen, Tuesday–Saturday)
A *manifold* is a shape that looks like n-dimensional Euclidean space, if you only look at small parts of the shape. For example, a sphere is a 2-dimensional manifold — just ask the many people throughout history who thought the Earth was flat. A torus is another good example, but a pair of intersecting planes is not, since the intersection doesn't look anything like a single plane. In other words, manifolds are the most natural kind of shape from a topological point of view, since they only differ from each other on the "global" scale.

But what manifolds are out there? How can we characterize them and tell them apart? Depending on how you count, there is only one 1-dimensional manifold: a circle. In higher dimensions, there are progressively more distinct manifolds, so to keep things simple we'll focus on "low" dimensions, in this case meaning dimensions up to 3. We will define various constructions and distinguishing properties that will help us completely classify 2-dimensional manifolds, and we'll do (some of) the same for 3-dimensional manifolds.

Chilis:

Homework: Recommended

Prerequisites: Knowledge of point-set topology could help, but isn't necessary.

Related to (but not required for): DIY Hyperbolic Geometry (W1); How Curved is a Potato? (W2)

The Continuum Hypothesis (1/2). (Susan, Tuesday–Saturday)
In 1874, Georg Cantor proved that there are more real numbers than natural numbers, essentially proving for the first time that there are different sizes of infinity. This result suggests an obvious follow-up question: is there an infinity between the size of the naturals and the size of the reals, or are these infinities in some sense right next to each other? The statement that there are no intermediate infinities became known as the continuum hypothesis.

For almost a hundred years, mathematicians struggled to either prove or disprove the continuum hypothesis. Finally, in 1963 Paul Cohen proved that the continuum hypothesis was independent of Zermelo-Fraenkel set theory. It could neither be proved nor disproved from our standard set theory axioms.

But how do you prove that you can't prove something? This proof involves a bizarre technique known as forcing, in which you take a tiny toy universe, add some objects, and take the closure under being-a-set-theoretic-universe. In this class we'll build the machinery required for forcing, and use it to prove the independence of the continuum hypothesis.

Chilis:

Homework: Required

Prerequisites: Some knowledge of the ordinal numbers, particularly ω_1. Stupid Games on Uncountable Sets could function as a prereq.

Related to (but not required for): Stupid Games on Uncountable Sets (W1)

11:15 Classes

Axiomatic Geometry. (Misha, Wednesday–Saturday)
In 1899, Hilbert came up with a list of axioms for Euclidean geometry.

We'll talk about why we need them, and about the horrible, no-good, actually-kind-of-fun things that happen when we leave out some of them.

Chilis:

Homework: Recommended

Prerequisites: None

Related to (but not required for): Machine Geometry; or, Area and Coarea (W1)

Conflict-Free Graph Coloring. (Pesto, Wednesday–Saturday)
You put some cell towers on a graph, each broadcasting at some frequency. Every vertex in the graph is a cell phone that needs to be able to listen at at least one frequency, but can't listen at a frequency if two towers adjacent to it are both trying to broadcast at that frequency. How many cell towers and how many distinct frequencies do you need?

This problem defines a version of graph coloring called "conflict-free" graph coloring. For this new version of graph coloring, we'll prove an analogue of the most important unsolved problem in graph theory, generalize the four-color map theorem, and prove that we (probably) can't solve it efficiently in general.

Chilis:

Homework: Recommended

Prerequisites: None.

Related to (but not required for): Intro Graph Theory (W1)

Oversize Airline Luggage. (Tim!, Wednesday)
Most airlines have the same funny rule for how large your luggage can be: The total length + width + height can be at most a specified value. You show up to the airport with a bag that's too large. But then you have an idea: You decide to put your bag inside another bag, but at an odd angle so that the outer bag has a smaller length + width + height than the inner bag. Can you do it?

Chilis:

Homework: None

Prerequisites: Some calculus

Representation Theory (1/2). (Aaron, Wednesday–Saturday)
Representation theory is a field of math aiming to describe the symmetries of your favorite shapes. It can be pithily summarized as group theory meets linear algebra.

In the first two days, we'll introduce the basic notions from representation theory and describe the amazing coincidences satisfied by the representations of a group. These will be proved in week 2.

In days 3 and 4, you'll work on problems in class, applying the aforementioned coincidences to compute all the representations of symmetry groups platonic solids, and many other fun problems.

Chilis:

Homework: Required

Prerequisites: Linear algebra and group theory. From group theory, familiarity with group actions is essential. One should also be familiar with symmetric groups, conjugacy classes, commutators, and quotient groups, among other things. From linear algebra, one should be familiar with linear transformations, eigenvalues, trace, and direct sum, among other things.

Related to (but not required for): Symmetries and Polynomials (W1); Group Theory (W1); Intro Ring Theory (W2); Linear Algebra (2/2) (W2); The Outer Automorphism of S_6 (W2); Commutative Algebra and Algebraic Geometry(1/2) (W3); Galois Theory (W3); Commutative Algebra and Algebraic Geometry (2/2) (W4)

Required for: Representation Theory (2/2) (W4)

Riemann's Explicit Formula. (Kevin, Wednesday–Saturday)
Riemann's explicit formula is

$$\sum_{n=1}^{\infty} \frac{\pi_0(x^{1/n})}{n} = \int_0^x \frac{dt}{\log t} + \sum_\rho \int_0^{x^\rho} \frac{dt}{\log t} - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t}$$

where π_0 is the normalized prime counting function and ρ ranges over all nontrivial zeros of the Riemann zeta function.

We'll explore how a crazy formula like this could possibly arise, and we'll see how the Riemann zeta function's zeros end up so intricately connected to the prime numbers.

Chilis:

Homework: Recommended

Prerequisites: Calculus

Shuffling and Card Tricks. (*David Roe*, Thursday–Saturday)
Have you ever wondered how many times you need to shuffle a deck of cards until it is completely randomized? What does that even mean? How many card tricks do you know that rely on math rather than sleight of hand? Come to this class if you want to become mathematical card shark!

Chilis:

Homework: None

Prerequisites: None.

1:10 Classes

An Introduction to q-Analogues. (*Maria Gillespie*, Thursday–Saturday)
A q-analog of an expression P is a polynomial or rational expression in the variable q for which setting $q = 1$ gives P. The theory of q-analogs comes up in combinatorics, algebra, geometry over finite fields (including the popular game SET), and quantum mechanics, and we'll discuss all of these aspects but focus mostly on the applications to combinatorics.

Classes will be lecture-based but with class participation; I may pause occasionally and let the students try an example before resuming lecture. It will be a one-hour class for each of three days.

Chilis:

Homework: None

Prerequisites: Comfort with multiplying polynomials together is essential. Students should also be familiar with basic combinatorial formulas involving binomial coefficients, factorials, and Pascal's triangle. A bit of familiarity with generating functions may be helpful but is not required.

Divisor Theory of Graphs. (*David Perkinson*, Tuesday–Saturday)
This is a course on the divisor theory of graphs and the abelian sandpile model. Topics would include: chip-firing and the "dollar game" on a graph, the discrete Laplacian, the Jacobian and Picard groups of a graph (the cokernel of the Laplacian — computing their structure gives an excuse for appreciating the Smith normal form of an integer matrix), tree-bijections, the matrix-tree theorem, orientations of graphs, and the Riemann-Roch theorem for graphs due to Baker and Norine. If there is time, I would also talk about the "dual" version of this theory — the abelian sandpile model.

The material is described in detail in my text with Scott Corry (to appear this year), which can be found at

`http://people.reed.edu/~davidp/divisors_and_sandpiles/`

An outline of a course I taught on this subject at Reed can be found at

`http://people.reed.edu/~davidp/374/`

Classes will be 1.5 hours long.

Chilis:

Homework: Recommended

Prerequisites: some linear algebra, including determinants.

Galois Theory. (Viv, Tuesday–Saturday)
You may know a formula for solving quadratic equations, but what about polynomials of higher degree? For many years, the question of solving quintic equations plagued the mathematical community. Ultimately, it was answered by a hotblooded young punk named Evariste Galois, who described a startlingly beautiful connection between field theory and group theory shortly before being killed in a duel. We'll learn about this connection and how it applies to quintics.

Chilis:

Homework: Recommended

Prerequisites: Group theory

Related to (but not required for): Algebraic Number Theory (W1); Symmetries and Polynomials (W1); Intro Ring Theory (W2); Representation Theory (1/2) (W3); Representation Theory (2/2) (W4)

Generating Functions, Catalan Numbers, and Partitions. (Mark, Tuesday–Saturday)
Generating functions provide a powerful technique, used by Euler and many later mathematicians, to analyze sequences of numbers; often, they also provide the pleasure of working with infinite series without having to worry about convergence.

The sequence of Catalan numbers, which starts off $1, 2, 5, 14, 42, \ldots$, comes up in the solution of many counting problems, involving, among other things, voting, lattice paths, and polygon dissection. We'll use a generating function to come up with an explicit formula for the Catalan numbers.

A *partition* of a positive integer n is a way to write n as a sum of one or more positive integers, say in nonincreasing order; for example, the seven partitions of 5 are

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, \quad \text{and} \quad 1+1+1+1+1.$$

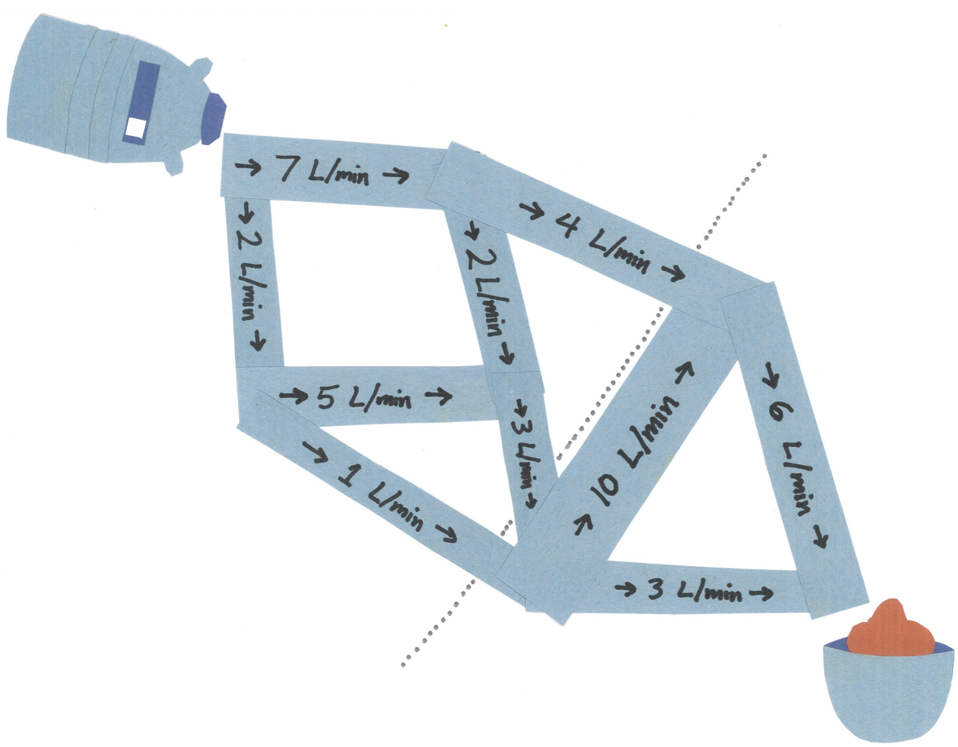
The number of such partitions is given by the partition function $p(n)$; for example, $p(5) = 7$. Although an "explicit" formula for $p(n)$ is known and we may even look at it, it's quite complicated. In our class, we'll combine generating functions and a famous combinatorial argument due to Franklin to find a beautiful recurrence relation for the (rapidly growing) partition function. This formula was used by MacMahon to make a table of values for $p(n)$ through $p(200) = 3972999029388$, well before the advent of computers!

Chilis: 🌶🌶 → 🌶🌶🌶

Homework: Recommended

Prerequisites: Summation notation; geometric series. Some experience with more general power series may help, but is not really needed. A bit of calculus may come in handy, but you should be able to get by without.

Max-Flow Min-Cut. (Tim!, Tuesday–Saturday)
The JCs have built a network of pipes to carry liquid nitrogen, so that they can make more ice cream! Each pipe in the diagram below is labeled with its capacity, the number of liters of nitrogen it can transport per minute.



How fast can this system transport nitrogen from the dewar to the bowl? It can't be more than 8 liters per minute, because there's a bottle-neck: at most 8 liters per minute can flow across the dotted line. In fact, there's a way to route nitrogen through the pipes that achieves *exactly* that rate (such a routing is called a *flow*). Can you find that maximum flow?

More generally, is there an efficient algorithm to find the maximum flow for any network? And, is there always a "bottle-neck" (usually called a *cut*) that matches the maximum flow? We'll find the answer to these questions (the answer is yes to both!). Then, we'll use the algorithm to help solve other seemingly unrelated problems!

Chilis: 🌶🌶

Homework: Recommended

Prerequisites: Graph theory

Related to (but not required for): Intro Graph Theory (W1); Linear Programs & Convex Optimization (W3)

Spectral Graph Theory. (Laura, Tuesday–Wednesday)
Spectral graph theory is a way of turning problems about graphs into linear algebra by associating a matrix to a graph (called the adjacency matrix) and studying its eigenvalues. We'll look at a classic application of this technique to Moore graphs of girth 5, which can be defined as regular graphs (all vertices have the same number of neighbors) where any two non-adjacent vertices have a unique common neighbor. Surprisingly, the only possible degrees such a graph can have are 2, 3, 7, or 57 (and it's not even known whether one of degree 57 exists)! The proof is an elegant application of linear algebra to studying graphs!

Chilis: 🌶🌶

Homework: Recommended

Prerequisites: Basic linear algebra. Familiarity with bases, eigenvalues, eigenvectors, and trace.

Related to (but not required for): Intro Graph Theory (W1); Triangulations and Flip-Graphs (W1)

Colloquium

The Threshold Density Theorem. (*David Perkinson*, Tuesday)
Imagine a system in which particles of energy are randomly added to sites in a finite network. When the number of particles at a site reaches a certain threshold, the site becomes unstable and fires, sending one particle to each of its immediate neighbors in the network. These neighbors, in turn, might then exceed their thresholds, so adding a single particle can set off a cascade of firings. We wait until all firing activity has subsided before randomly adding the next particle. Since the network is finite, we will eventually reach a point at which a particle is added and the system is no longer stabilizable. What total amount of energy do you expect to be in the system then? This talk will present Lionel Levine's threshold density theorem, which provides an answer to this question.

Change of Perspective in Mathematics. (*Mike Orrison*, Wednesday)
Change of perspective is ubiquitous in mathematics. Consider, for example, change of coordinates, Bayes' Theorem, combinatorial proofs, or even the simple fact that $2 + 3 = 3 + 2$. In this talk, I'll offer some personal reflections on the unifying role that change of perspective plays when we learn, share, create, and discover mathematics. I'll also describe how conversations over the years with teachers, undergraduates, and my own children have shaped my understanding of the power of change of perspective.

How to Win the Lottery. (*David Roe*, Thursday)
From 2005 through 2012, three groups in Massachusetts earned millions of dollars playing a lottery game called Cash Winfall. I'll describe how the game worked, explain why it was exploitable, give connections with projective geometry and share stories from my participation in one of the pools. I'll also reaffirm the common wisdom that almost every other lottery is not worth playing.

What is a q-analog? (*Maria Gillespie*, Friday)
A q-analog of a mathematical quantity P is an expression involving q such that setting $q = 1$ yields the original quantity P. While the definition is a simple one, the theory of q-analogs is a new and growing field of study that connects many different areas of mathematics. We'll discuss some common q-analogs and their many appearances in mathematics: in combinatorics, number theory, algebra, quantum physics, and the popular game of SET.

Visitor Bios

Michael Orrison. Michael is a professor of mathematics at Harvey Mudd College, where he has been since 2001. His teaching interests include linear algebra, abstract algebra, discrete mathematics, and representation theory. His research interests include voting theory and harmonic analysis on finite groups. In particular, he enjoys finding, exploring, and describing novel applications of the representation theory of finite groups. He also enjoys cooking, watching movies, and coaching and refereeing soccer.

David Perkinson. David Perkinson is a professor of mathematics at Reed College in Portland, OR. His mathematical interests include the Abelian sandpile model, combinatorics, and algebraic geometry. He loves doing research with undergraduates, having advised over 40 undergraduate theses and routinely working with students on summer research projects. His book "Divisors and Sandpiles: An Introduction to Chip-Firing", coauthored with Scott Corry, will appear this summer. He has taught mathematics at the African Institute of Mathematical Sciences in South Africa, Ghana, and Cameroon. His hobbies include board games and playing jazz guitar and the mbira, a musical instrument from Zimbabwe.

Maria Gillespie. Hi, I'm Maria Gillespie, and I'm a combinatorialist! I love working with discrete and algebraic mathematical objects like permutations and words, Young tableaux, polynomials, and generating functions. I also like geometry and sometimes combine it with the combinatorics that I do. Looking forward to meeting all the Mathcampers during my visit!

David Roe. David Roe's first mathcamp was more than half a lifetime ago. He studies computational and algebraic number theory, p-adic representation theory, while hiking, climbing, playing board games, rafting, baking, berry-picking, reading science fiction, roleplaying games, and traveling to awesome places in his free time.