# CLASS DESCRIPTIONS — WEEK 2, MATHCAMP 2018

Contents



## 9:05 Classes

**Haskell.** (🌶🌶, Larsen, Wednesday–Saturday)
This is not a math class. (Not on the surface, anyway.) We will learn how to code with a programming language called Haskell, which was named after a mathematician. Haskell is "lazy" and "purely functional" — in other words, it is structurally very different from the programming languages you're probably familiar with. In this class, we'll type some code that looks like this, and learn why we do it that way:

```
data NaturalNumber = Zero | S NaturalNumber
instance Eq NaturalNumber where
  Zero == Zero = True
  Zero == S y  = False
  S x  == Zero = False
  S x  == S y  = x == y
```

*Homework:* Required

*Prerequisites:* None

**Hilbert's Paper, Scissors, and Rock.** (🌶🌶, Steve, Wednesday–Saturday)
Consider the property "has the same area as." For complicated shapes this probably has something to do with integrals; for polygons, however, your instinct is probably that we're just talking about *cutting and rearranging*  that two polygons have the same area if and only if we can slice one into a bunch of pieces and rearrange them to get the other. Now, some of you may have heard about the Banach-Tarski paradox and related results and be worried, but if we keep everything nice and simple it turns out we don't get into trouble: if $A$ and $B$ are polygons, then $A$ and $B$ have the same area if and only if we can cut $A$ into finitely many pieces via straight line cuts and reassemble those pieces to get a copy of $B$. That is, two polygons have the same area if and only if they are **scissors congruent**. Note that scissors congruence, unlike the integral approach, doesn't actually require us to ever measure an area, but rather defines "same-area-ness" entirely on its own.

In 1900, Hilbert published a list of 23 problems. The third problem  the first to be solved, within a year and by Hilbert's own student Dehn  asked whether scissors congruence can define *volume* correctly, in addition to area. That is, given polyhedra $A, B$ of equal volume, can we always cut $A$ into finitely many pieces via straight plane cuts and reassemble those pieces to get a copy of $B$?

In this class we'll see Dehn's answer to Hilbert's playful question, "Scissors beats paper; does it also beat rock?"
*Homework:* Optional
*Prerequisites:* Proof by induction

**How Curved is a Potato?** (🌶🌶🌶, Apurva, Wednesday–Saturday)
Ans: $4\pi$.

Every potato has total Gaussian curvature of $4\pi$ and so does the surface of the Earth. What is even more interesting is the fact that Gaussian curvature is entirely *intrinsic* to the surface and does not depend upon how it is embedded within an ambient space. Gauss called this the Theorema Egregium. To truly understand this phenomenon mathematicians had to invent manifolds and the field of Riemannian geometry.

In this class, we'll learn how linear algebra and calculus come together to make differential geometry, define principal, mean, and Gaussian curvatures and understand their geometric significance. By the end of this class, you'll have a much deeper appreciation of linear algebra.

*Homework:* Recommended
*Prerequisites:* The only things we'll need from calculus are partial derivatives and vector valued functions. You should be familiar with or be willing to learn on the fly - Taylor series expansion in 2 variables, eigenvalues and eigenvectors of 2x2 matrices.
*Related to (but not required for):* Machine Geometry; or, Area and Coarea (W1); DIY Hyperbolic Geometry (W1); Metric Space Topology (W2); Low-Dimensional Zoology (W3); Differential Topology (W4); The Fundamental Group (W4)

**Intro Ring Theory.** (🌶🌶🌶, Lara, Wednesday–Saturday)
As my awesome colleague and friend Susan so aptly put it: to study rings, 'we cut ourselves loose from our usual number systemsthe complexes, the reals, the rationals, the integers, and just work with…stuff. Stuff that you can add. And multiply.'

Rings naturally arise in many areas of math and serve as a powerful tool for problem-solving, allowing us to understand familiar structures from a fresh perspective.

We'll give lots of examples of rings to get a sense of how ubiquitous they are, then we'll play with rings to our hearts' content! We'll learn how to tell which rings are 'the same' and how to map between rings, as well as the many wonderful things special subsets called 'ideals' can help us do.
*Homework:* Recommended
*Prerequisites:* None.
*Related to (but not required for):* Algebraic Number Theory (W1); Symmetries and Polynomials (W1); Representation Theory (1/2) (W3); Representation Theory (2/2) (W4)
*Required for:* The Class Number (W2); Commutative Algebra and Algebraic Geometry(1/2) (W3); Galois Theory (W3); Commutative Algebra and Algebraic Geometry (2/2) (W4)

**Random Matchings in Cubes.** (🌶🌶, *Fedya Manin*, Friday–Saturday)
Congratulations! You've been commissioned to bake a cubical raisin pudding for Marisa and Alfonso's wedding. Unfortunately, after it's done, you find out that Alfonso is very particular about raisins, and wants them to be arranged in a perfectly rectilinear grid throughout the pudding. Fortunately, your pet mole Alfred is willing to burrow through the pudding and put every raisin in the right spot. Assuming the original placement of the raisins was totally random, how long will it usually take Alfred to do this? How does this time depend on the total number of raisins? The answer depends on the dimension of the pudding (and Alfred); dimension 2 is the hardest.

*Homework:* Optional
*Prerequisites:* some calculus

**de Bruijn Sequences.** (, Pesto, Wednesday–Thursday)
de Bruijn sequences are sequences of 0s and 1s containing all possible subsequences of a given length exactly once: for instance, 0001110100 contains every possible sequence of 3 0s and 1s.

For what lengths do such sequences exist, and how easily can we find them if they do?

Bonus homework: how were they used as an early error-correction code in Sanskrit poetry, 2500 years before error-correction codes even existed?

*Homework:* Optional

*Prerequisites:* None.

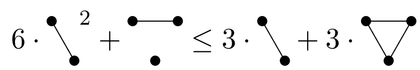
*Related to (but not required for):* Intro Graph Theory (W1)

## 10:10 Classes

**Flag Algebras.** (, Misha, Tuesday–Saturday)
We prove graph theory results from 1907 by methods developed in 2007. (Later, we'll solve some more recent problems, as well.)

Flag algebras are a tool in graph theory introduced by Razborov to talk about how densities of small subgraphs in large graphs relate to each other. We do some graph theory, then we do some algebra to automate our graph theory, then we do some more algebra to automate our algebra.

Selling point: we get to write inequalities where our variables are graphs. Here's an example:



*Homework:* Recommended

*Prerequisites:* You will be lost without graph theory. Some ideas from abstract algebra and linear algebra will show up.

*Related to (but not required for):* Intro Graph Theory (W1)

**Hardness.** (, Linus, Tuesday–Saturday)
What does it mean for a math problem to be hard? Is solving a Sudoku puzzle hard? Is Portal 2 hard?

I know the answers aren't "yes" and "no" respectively. I know this because I can make a Portal 2 level that makes the player enter a Sudoku solution and only lets them beat the level if it works.

So, Portal 2 is at-least-as-hard as Sudoku. This class is all about at-least-as-hard. It's complexity theory, the same kind of math as the P versus NP problem.

*Homework:* Recommended

*Prerequisites:* Be comfortable with me saying the word "algorithm" a lot

**Intersecting Curves.** (, J-Lo, Tuesday–Saturday)
The following statement is extremely concise and elegant: A line and a parabola intersect at exactly two points.

Unfortunately it also appears to be wrong. We could just declare it to be wrong and leave it at that, or we could start changing our assumptions until we can force it to be true.

This course takes the second approach. Working through a collection of examples in groups, you will determine the hypotheses you need in order to force the above statement (and many others like it) to be true, and explore how these elegant (and now true) statements have algebraic implications, such as to Pythagorean triples and elliptic curves.

*Homework:* Recommended

*Prerequisites:* None

*Related to (but not required for):* Commutative Algebra and Algebraic Geometry(1/2) (W3); Rational Points on Elliptic Curves (W4)

**Linear Algebra (2/2).** (, Mark, Tuesday–Saturday)
A continuation of the week 1 class. If you're thinking of joining, ask me to get an idea of what you might need to catch up on.

*Homework:* Recommended

*Prerequisites:* None.

*Related to (but not required for):* Representation Theory (1/2) (W3); Representation Theory (2/2) (W4)

**MCMC.** (, Mira, Tuesday–Saturday)
MCMC does not stand for "Mathcamp Mathcamp", but it stands for something almost as exciting: "Markov Chain Monte Carlo". It is a technique for sampling from very complicated probability distributions. That might not sound particularly glamorous, but in fact, MCMC lies at the heart of the Bayesian revolution in statistics that began in the 1990's and is still going strong. The revolution is changing the way scientists use mathematical models in their work. There is literally no branch of the natural or social sciences where researchers aren't currently using MCMC.

If the above sounds intriguing but you have no idea what any of the words mean (Markov chain? Monte Carlo? Probability distribution? Bayesian?), that's totally OK – we're going to start from scratch. We'll define exactly what MCMC does and why this is so important. We'll analyze it mathematically and see how and why it works. We'll talk about it in the broader context of the Bayesian revolution. And we'll give lots of examples from different branches of science, including real data sets for you to play around with. For those who want to go more in depth, I will also be offering a couple of MCMC-based projects later in camp.

*Homework:* Recommended

*Prerequisites:* No hard prerequisites, but if you have never done any statistics, the Week 1 class on Statistical Modeling will give you some much-needed context and help you hit the ground running.

*Related to (but not required for):* Statistical Modeling (W1); Street Fighting Mathematics (W1); Two equals one: Street-fighting mathematics and science for better teaching and thinking (W1); NLP (W4)

## 11:15 Classes

**Intro Number Theory (2/2).** (, Mark, Tuesday–Saturday)
A continuation of the week 1 class. If you're thinking of joining, ask me to get an idea of what you might need to catch up on.

*Homework:* Recommended

*Prerequisites:* None.

*Related to (but not required for):* Rational Points on Elliptic Curves (W4)

**Metric Space Topology.** (, Jeff, Tuesday–Saturday)
One of the oldest notions in mathematics is the idea of distances. I think before we even had a notion

of numbers, angles or lines, we had an idea of what it meant for two things to have a distance between them.

- We have real-world notions of distance: "San Francisco and Los Angeles are 559 km apart."
- We have relational ideas of distance: "The Penguin is a distant relative of the velociraptor", or " I think that my friends from elementary school have grown apart since then".
- We have correlative notions of distance: "The English King James Bible and American King Bible are closer than the New American Bible", or "the Dow Jones Industrial Average tracks closely with Standard and Poor's 500 index".

A metric space is a framework for looking at all of these ideas together, allowing us to develop a general theory which attacks many different kinds of problems. We'll also get an introduction to topology — the study of the shape of space when you're allowed to deform your geometry — through the lens of metrics.

*Homework:* Recommended

*Prerequisites:* None

*Related to (but not required for):* DIY Hyperbolic Geometry (W1); How Curved is a Potato? (W2)

*Required for:* Systems and Signals Analysis (W3); Convergence Issues; or: Monsters in Real Analysis (W3)

**PS: Inequalities.** (🌶🌶🌶🌶, Pesto, Tuesday–Saturday)
High-school olympiads usually try to choose problems relying on as little prior knowledge as possible. In inequalities problems, they usually fail completely; training is necessary to solve most and sufficient to solve many of them. We'll go over the common olympiad-style inequalities, and solve problems like the following:

(1) Prove that if $a$, $b$, and $c$ are positive and $ab+bc+cd+da=1$, then $\frac{a^3}{b+c+d}+\frac{b^3}{a+c+d}+\frac{c^3}{a+b+d}+\frac{d^3}{a+b+c}\geq\frac{1}{3}$.
(2) [USAMO 2004] Prove that if $a$, $b$, and $c$ are positive, then $(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3)\geq(a+b+c)^3$

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to olympiad-style problems you'll have solved as homework the previous day.

*Homework:* Required

*Prerequisites:* None.

**The Class Number.** (🌶🌶🌶, Kevin, Tuesday–Saturday)
A special case of Fermat's Last Theorem says that there are no nonzero integers such that $x^3+y^3=z^3$. If $\zeta_3$ is a third root of unity, we can factor this as $(x+y)(x+y\zeta_3)(x+y\zeta_3^2)=z^3$. If we know that factorization is unique once $\zeta_3$ gets involved, we can make a lot of progress towards proving this case!

If we play this game with $\zeta_{23}$, we don't have unique factorization. But by studying the *class number*, a finer measure of the structure of rings of integers, we can overcome this obstacle!

Can we, in fact, prove all of Fermat's Last Theorem this way? We're still doomed with $\zeta_{37}$, but at least we tried.

This class will pick up right where Shiyue's Algebraic Number Theory class stops. If you're not sure what that means, come talk to me after Week 1 ends.

*Homework:* Recommended

*Prerequisites:* Week 1 Algebraic Number Theory or equivalent.

*Related to (but not required for):* Algebraic Number Theory (W1); Intro Ring Theory (W2)

**Topological Tverberg's Theorem.** (🌶🌶🌶🌶, Viv, Tuesday–Saturday)
The convex hull of a set $X$ of points is the smallest set $C$ containing $X$ and all lines between points in $C$. Given four points in the plane, I can *always* partition them into two sets whose convex hulls intersect. And if I live in any Euclidean space and I'm given enough points, I can do the same thing. And if, before starting my partitioning process, I draw my convex hulls however I like, rather than through such silly procedures as "following the definition", I can *still* do the same thing. And if I'd like to split my set into arbitrarily many subsets instead of two, I can still do the same thing. . . provided I'm partitioning into a prime power number of subsets.

If this seems really weird, that's because it's really weird. We'll prove these statements and discuss the recent astonishing counterexample to the composite case.

*Homework:* Recommended

*Prerequisites:* Linear Algebra

## 1:10 Classes

**Chaotic? Good!** (🌶🌶🌶, Ben, Tuesday–Saturday)
Say we take a number, double it a few hundred times, and want to predict the first digit after the decimal place when we've done this. How hard can that be? It can get quite hard, it turns out, because we'd be trying to predict the behavior of a chaotic dynamical system!

Starting from the goal of predicting how certain dynamical systems evolve, we'll see how "chaos" in the colloquial sense sneaks into the picture. Then, we'll discuss a more precise definition of chaos and how to see that a system is chaotic. As we go, we'll run into a couple kinds of bifurcations and meet some Cantor sets. We'll also introduce the method of symbolic dynamics, which will greatly simplify our analysis of some dynamical systems.

*Homework:* Recommended

*Prerequisites:* Know what fixed points and periodic points are, and how to classify fixed points as attracting or repelling. Talk to me if you're interested in taking this and haven't seen these!

*Related to (but not required for):* Crash Course in Complex Dynamics (W1)

**Combinatorial Poetry.** (🌶🌶, *Matt Stamps*, Tuesday–Saturday)
Have you ever noticed how various creative writing techniques in literature are used to express different modes of thinking and feeling? Poetry, for instance, makes use of linguistic attributes, such as rhyme or rhythm, to elevate the aesthetic of a piece, but also to convey additional meaning that leads to a deeper appreciation for its subject. Is there an analogous notion in mathematics? Certainly there are proofs that are clearer and more concise than others — mathematicians like to call these elegant — but are there elegant proofs that elevate the reader's understanding and insight so substantially they fall into the category of poetry? In this class, we will explore some mathematical "poems" that not only establish the truth of their respective propositions, but transform the way we think about a given subject. Class meetings will consist of team problem solving and practice writing mathematical poetry about familiar combinatorial objects, including the Fibonacci numbers, Pascal's Triangle, and (if time permits) the Catalan numbers. Particular emphasis will be placed on finding different ways to enumerate and construct one-to-one correspondences between finite sets so as to reveal their underlying combinatorial structure.

*Homework:* Recommended

*Prerequisites:* None (though it would be helpful if they have seen some basic discrete math and/or enumeration before)

**Guess Who?** (🌶, Tim!, Tuesday–Saturday)
I've become a bit obsessed with the board game *Guess Who?*. It's a simple children's guessing game — you ask yes/no questions to try to determine your opponent's secret character from among 24 possibilities, and whoever guesses correctly with fewer questions wins.

A YouTube star (and former NASA scientist) claimed to find the "best strategy," and you can probably think of this strategy too. But actually this strategy is not best. It can be totally destroyed, as we will see.

But then, what really is the "best strategy." There's an answer to this, but this is where things start getting weird. We'll explore it all together. Expect to see terms like "decision trees," "entropy," "matrix games," "convex optimization," and numbers like $\frac{127}{255}$. This class will also involve a lot of *relaxation*, but in this case that's a technical term. . . .

*Homework:* Optional

*Prerequisites:* None

*Related to (but not required for):* Calculus on Graphs (W1); Linear Programs & Convex Optimization (W3)

**Modular Forms.** (🌶🌶🌶🌶, Shiyue, Tuesday–Saturday)
A modular form is a function on the complex upper half plane satisfying certain symmetries. The theory of modular forms is really part of complex analysis, but has ubiquitous applications in number theory. It became hugely popular in the 1990s, when it was used to prove Fermat's Last Theorem.

We won't get to the proof of FLT in this course – we'll only have time to scratch of the surface of the theory, but it will give you a sense of how complex analysis (calculus over complex numbers) and number theory interact. We will talk about Eisenstein series, the Ramanujan cusp form, the valence formula (a super cool geometric result), the structure theorem for level 1 modular forms (which was totally amazing to me when I first saw it), the Miller basis, and hopefully touch on Hecke operators.

*Homework:* Recommended

*Prerequisites:* linear algebra; calculus; a very solid familiarity with complex numbers; the basic definitions of group theory

*Related to (but not required for):* Algebraic Number Theory (W1); DIY Hyperbolic Geometry (W1); Commutative Algebra and Algebraic Geometry (2/2) (W4); Cohomology via Sheaves (W4)

**The Outer Automorphism of $S_6$.** (🌶🌶🌶, Aaron, Tuesday–Saturday)
What do mystic pentagons, platonic solids, pentads, nobbly wobblies (yes, the dog toys), and linear fractional transformations have to do with each other? They all describe the outer automorphism of $S_6$!

In this IBL class, you'll work through problems together to discover how these are all examples of the same phenomena, and even prove that $S_6$ is the only symmetric group with an outer automorphism.

*Homework:* Required

*Prerequisites:* Group theory

*Related to (but not required for):* Symmetries and Polynomials (W1); Commutative Algebra and Algebraic Geometry(1/2) (W3); Representation Theory (1/2) (W3); Representation Theory (2/2) (W4); Commutative Algebra and Algebraic Geometry (2/2) (W4)

## COLLOQUIUM

**Project Fair.** (Tuesday)

**Accidental Mathematics.** (*Matt Stamps*, Wednesday)
Growing up, I always loved learning about world-changing scientific breakthroughs that were discovered by accident. Penicillin, artificial sweeteners, X-rays, and synthetic dyes (to name just a few) were all stumbled upon by scientists with other goals in mind. More recently, I have come to wonder why anecdotes about accidental discoveries in mathematics are not as commonplace. Is it because mathematicians like to inflate their egos by attributing all of their theorems to their superior intellects? Is it because society prefers to maintain the popular perception of mathematics as a series of dry logical deductions speckled with astonishing strokes of genius? Or is it is because of something else entirely? Regardless of the answer, I argue that mathematics happens accidentally all the time. In this talk, I will share some accidental discoveries from my own work involving Penrose tilings, circle packings, chordal graphs, lecture hall partitions, lattice polytopes, and polynomial rings.

***Not* the Continuum Hypothesis.** (Steve, Thursday)
After Cantor showed that the reals are uncountable, a natural question which he asked becomes: "Is there anything 'between' the reals and the naturals?" That is, could there be a set of real numbers which is uncountable but still not as big as $\mathbb{R}$ itself? (In symbols: is $2^{\aleph_0} > \aleph_1$?)

The claim that there is no such "intermediate cardinality" became known as the **continuum hypothesis** (CH), and a lot of early effort went into proving or refuting it (by Cantor and others). The eventual solution, by Paul Cohen in 1963 (following work of Kurt Godel in 1951), was incredibly surprising, beautiful, and fundamentally changed set theory and indeed all of logic.

But that sounds hard, so let's make Susan do it. Instead, I want to talk about something else. Long before the problem itself was solved, an important development occurred when people realized that (in a precise sense) any counterexample to CH would have to be *really ugly*. This marked the birth of **descriptive set theory** the study of shapes that aren't quite as bad as they could be. Descriptive set theory provides techniques for showing that "reasonable definable" sets of real numbers have nice properties, such as *measurability* (a technical property which prevents Banach-Tarski-paradox-y stuff from happening) and *not being a counterexample to CH.* The key tool here is the notion of a *determinacy principle*: that "niceness properties" correspond to certain games having winning strategies for one player or the other (= being determined), and so we can show that a set of reals has nice properties by showing that various games related to it are determined.

I'll present the initial descriptive set theory of the continuum hypothesis, via the **perfect set game**.

**The Lost Mentor and the Sleepy JC.** (Misha, Friday)
Suppose that a mentor is lost on a hike in the woods. If they call you on the phone, can you give them directions, even if you don't know where in the woods they are? (And even if you don't have a map?)

## Visitor Bios

**Matt Stamps.** Matt's research is at the intersection of algebra, combinatorics, geometry, and topology. He is especially fond of applying ideas from topology to solve challenging problems in combinatorics. At Yale-NUS College, Matt teaches a historical immersion course called *Geometry & The Emergence of Perspective* that explores the interactions between geometry and art during the Italian Renaissance. Outside of work, he loves to travel and tell outlandish stories, some of which are even true.

**Fedya Manin.** Fedya was a camper in '04 and '05 and a mentor in '10. These days, he mostly studies how topology affects geometry: that is, what information about distances and other measurements we can recover after we've ostensibly forgotten all about them. He also plays clarinet and in '16 and '17 he once again got to be a camper – at klezmer camp.