CLASS DESCRIPTIONS—MATHCAMP 2015

Classes

Abelian Groups. (Don)

I love group theory, but I don't love when elements of groups don't commute with each other; that's not good for the environment!

Even with all of our groups being abelian, though, algebraists are still far from a complete understanding. What we can do is look at a variety of different extra restrictions, and classify the abelian groups meeting those restrictions; say, the finitely generated abelian groups, or the divisible abelian groups.

Prerequisites: Group Theory.

Abel's Theorem. (Alfonso + Mira + Julian)

How do you solve a polynomial equation of degree n?

- The linear ($n = 1$) case is easy: the solution to $ax + c = 0$ is just $x = \frac{-c}{a}$.
- The quadratic formula ($n = 2$) was known to the ancient Babylonians.
- Gerolamo Cardano discovered the cubic formula ($n = 3$) in the 1540's.
- Very soon afterwards, Cardano's student Ludovico Ferrari figured out the quartic formula ($n = 4$).

Then everyone started looking for the quintic formula ($n = 5$). They looked and looked, for almost three centuries — until in 1824, the great Norwegian mathematician Niels Abel proved that there is *no general formula* for solving a polynomial equation of degree 5. This is the theorem for which our class is named and which you will prove over the course of the summer (although not by the same method that Abel used).

Notice how we said "*you* will prove"? That's because the course will be taught by the Moore Method. This means that we will not lecture and you are not allowed to use books, the internet or any other resources (though you are encouraged to collaborate with each other). We will be giving you definitions and asking many questions. You will be doing all the work to answer them in the form of daily homework. Class time will be spent presenting and discussing your solutions. So you will be the ones proving all the theorems!

This is a five-week course, which, by Mathcamp standards, is a big commitment. You also have to be prepared to commit significant time daily to working on homework. In return, by the end of the course, you will have gained an in-depth understanding of at least two important fields of mathematics (group theory and Riemann Surfaces); a lot of experience writing and presenting proofs; and the deep satisfaction of discovering new mathematics for yourself.

Because of the nature of this class, there is homework due on the first day of classes (Tuesday). Read through and prepare your answers to the first few questions. You can download the notes from `http://www.math.toronto.edu/~alfonso/MC/abel.html` and we will also have physical copies with us during opening assembly.

Prerequisites: Group theory up through the First Isomorphism Theorem. The Week 1 Group Theory course is sufficient and it is okay to take it at the same time as this course; we'll need more group theory, but we'll develop it as part of the class.

Absolute Values: All the Other Ones. (J-Lo)

Can you cut up a square into an odd number of triangles, all of which have equal area? Or find a set of at least two consecutive unit fractions (e.g. 1/4, 1/5, 1/6, and 1/7) that add up to a whole number? These questions turn out to be related to a strange way of measuring the size of numbers. A way which shows that all triangles are isosceles, allows $1 + 2 + 4 + 8 + 16 + \dots$ to converge, and gives us a new way to talk about the viral Gangnam Style phenomenon.

In this class, we will discuss what it means for a function to count as an absolute value, classify all possible absolute values you can use on the rational numbers, and study a few of them for their applications to number theory, geometry, computer science, and more. We'll also get a sneak preview of the p-adic numbers, an alternate and very different world from the real numbers you know and love.

Prerequisites: None.

Abstract Nonsense. (Chris)

So you've learned some category theory but you still feel like that's too concrete for you? Then this class is the answer! Abstract Nonsense is the semi-official title of categorical proving-techniques that on first sight feel completely non-sensical, but apply to a wide range of situations due to their abstract nature.

One aspect of this are so-called simplicial sets — the most abstract and weird way to describe topological spaces. In this class we will be studying these simplicial sets and maybe in the end use them to define the classifying space of a category. Yes, that's right – we will turn a category into a space. Why? Because we can! And also because we then can define K-Theory (maybe).

Prerequisites: Category Theory (Categories, Functors).

A Card Trick, and a Set. (Don)

The Gilbreath principle says that when you shuffle a deck a certain way, much of its structure is preserved; you can use the similarity of the shuffled deck to the original to perform a number of magic tricks.

The Mandelbrot set is a marvelously complicated set in $\mathbb{C}$ that is infinitely self-similar yet not regular.

These two notions of self similarity are, in a certain sense, the same! We'll see why in this class.

Prerequisites: None.

Advanced Linear Algebra. (Asilata)

Here's a puzzle! Suppose you have a set of thirteen weights with the following property: if you remove any one of them, you can divide the rest into two sets of equal weight. What are all the possibilities?

Wait a minute, this was supposed to be a blurb about advanced linear algebra. So, here's a question about eigenvalues! If someone specifies a (multi-)set of eigenvalues for you, how many essentially different matrices can you come up with that have the given eigenvalues?

In this class, we'll explore questions like the one above, and discover some beautiful theorems of linear algebra that you might not have seen in a first course. We'll also see a neat algebra trick that may or may not lead us to a pretty solution to the first puzzle.

Prerequisites: Linear algebra, ring theory recommended.

A Dynkin diagram miscellany. (Asilata)

Dynkin diagrams are a few innocuous graphs that show up in a whole bunch of mathematical classification problems.

In this class, we'll take a stab at a couple of problems that are not at all about graphs, but for which (spoiler alert!) Dynkin diagrams ultimately come along and save the day.

There is no actual overlap between this class and Lie Algebras or Reflection Groups, so you can show up whether or not you took those classes!

Prerequisites: Linear algebra. Group theory recommended but not required.

A game you can't play (but would win if you could). (*Stefan Banach + Alfred Tarski*)

Once upon an infinity, in the Kingdom of Aleph, King Alephonso decided to put his 100 advisors to a test. He had 100 identical rooms constructed in his palace. In each room the king placed an infinite sequence of boxes; in each box he put a real number. The sequence of numbers was exactly the same in each room, but otherwise completely arbitrary.

The king told his advisors that when they were ready, each of them would be locked in one of the 100 rooms. Each of them would be allowed to open all but one of the boxes in the room. (This, of course, would take an infinite amount of time, but in the Kingdom of Aleph, they're pretty cavalier about infinity.) Finally, each advisor would be required to name the number in the box that he or she did not open. If more than one advisor names the wrong number, they would lose their jobs and their lives.

There is no reason for anyone to hurry in the Kingdom of Aleph, and the king gives his advisors an infinite amount of time to work out a strategy. Do they have any hope of making it through his cruel test alive? What should they do?

Prerequisites: None.

Algorithms. (*Michelle Bodnar*)

What is the fastest possible algorithm to sort a collection of objects? I have a list of things to do, each with a deadline, and each with a penalty for not finishing. What's the best order in which to carry out these tasks? An employment agency sends you a list of candidates to interview for a job, but interviewing and hiring temporary employees is costly. What strategy can you use to minimize these costs? What strategy can a malicious agency use to maximize them? In this class, you'll learn some of the main ideas underlying algorithms that solve these problems and many more. In addition, we'll talk about how to prove correctness and optimality of algorithms, and what we can prove about an algorithm that doesn't always give the right answer.

Prerequisites: None.

Algorithms in Number Theory. (Misha)

Modular arithmetic is a key tool in number theory that can be used to find answers to all sorts of problems. Are there integer solutions to $13x + 17y = 1$? Is $23^{43} + 43^{23}$ divisible by 66? Is the number 47 147 113 459 prime? We will learn how to answer all of these questions, and more.

In this class, we will learn about all those things from an algorithmic point of view. Unsatisfied with merely knowing that solutions exist, we will find recipes for finding these solutions.

With the aid of computers, we will go even further. You probably don't want to check if a 100-digit number is prime by hand, but you will learn to teach your computer to do it. (No previous programming knowledge is required.)

Prerequisites: None.

Almost All You Want. (*mmmmmike Hall*)

"It might be terrible, but at least it fits in a breadbox." – I made this quote up

Most functions you meet in your first calculus class are quite nicely behaved, but in general a function can do some nasty things. Like manage to be increasing but with a discontinuity at every rational number. Or be continuous everywhere but differentiable nowhere (!). Is there a compromise somewhere among the good, the bad, and the ugly?

Many classic results in real analysis revolve around cleverly partitioning space up into some pieces where a function is nice, and others where it's allowed to misbehave, but which limit the extent of its malfeasance. Our goal will be to show, when possible, that we can shove all the bad behavior into a tiny, tiny region – a set of "measure 0". The title of this class is a reference to properties that hold "almost everywhere" – that is, everywhere except on a set of measure 0.

As we'll see, these sets still leave enough room to go wild – but just a little.

Prerequisites: Familiarity with limits, infinite sums, derivatives, integrals.

A Mathematician Goes to Vegas. (Don)

Many would say that someone who knows math would never gamble.

Some would say that someone who knows math would only gamble if there's a positive expected value.

I would say that simply by existing, we have no choice but to constantly gamble, and so you might as well learn how to do it right.

Learn how to do it right. Come to this class.

Prerequisites: Calculus.

Aperiodic Tiling. (Steve)

Don't you wish the floor went off to infinity?

A *tiling* of the plane is just a way of covering the plane with shapes which don't overlap (except at their boundaries). For example, floor tiles usually describe a tiling of the plane by squares, except for the part where you run out of floor.

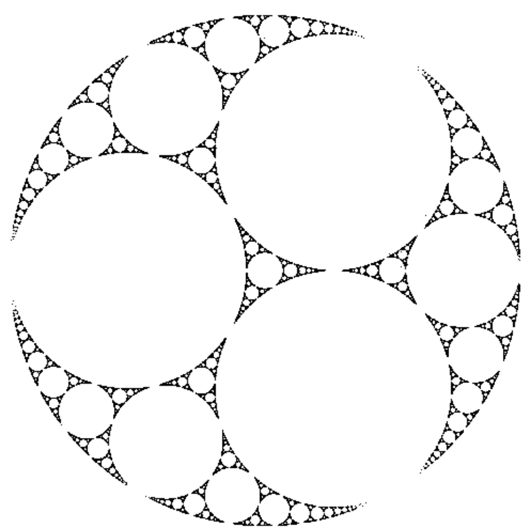
For other examples, it's easy to tile the plane using triangles, parallelograms, or regular hexagons. With slightly more work, you can tile the plane using regular pentagons and appropriately-shaped triangles together, and it's a fun exercise to show that you can't tile the plane using pentagons alone. All the tilings I've mentioned above are kind of boring, though — they repeat themselves.

Don't you wish the floor was surprising and new?

In this class we'll look at two specific examples of *aperiodic* tilings: a hierarchical tiling, built out of L-shapes, and the *Penrose tiling*, made with two different types of rhombus. We'll learn how we can *prove* that a certain tiling cannot repeat itself, and see where (mathematically) aperiodic tilings come from.

Prerequisites: None.

Apollonian Circle Packings. (*Sunny Xiao*)



Given any three touching circles, do there exist other circles that simultaneously touch all three? The answer was discovered by a renowned Greek geometer, Apollonius of Perga, around 200 BC. It turns out that there are, and there are always exactly two choices!

Grab a pen and try it now. Can you find both of them? Awesome! Now you have five circles. Pick another set of three touching ones and iterate the process. What kind of picture do you get in the end? What if you change the radii of the initial three circles? Do you get a different packing? How many possible configurations are there? Can we classify them?

Come to this class to find out! The family of beautiful, intricate fractals generated by this procedure is known as the Apollonian circle packing (ACP), an object with fascinating properties—for example, the amount of space filled up by an ACP has a "fractional dimension" of about 1.3057, meaning that it is "fuller" than a line but "skinnier" than the whole plane! In this class, we will first derive some classical geometric results about the ACP. Then we will study more of its interesting properties in connection with other fields of modern mathematics (number theory, group theory, dynamics, and topology). Time permitting, we will look at some open questions in this area and give them a try ourselves.

Prerequisites: None.

A (re)Introduction to Polynomials. (*Adam Marcus*)

This course will be a (re)introduction to polynomials. Why a course on boring polynomials, you ask? Because polynomials appear in a lot of places. Places like convex optimization, algebraic geometry, complex analysis, combinatorics, as well as a whole theory built around, well, just polynomials. This makes them *very* versatile creatures, and as a result, some really famous unsolved problems have been solved (or re-solved in an absurdly easy way) recently using them. Problems involving things like distinct distances in the plane (geometry/combinatorics), matrix paving (functional analysis), matchings (graph theory), permutations (combinatorics), just to name a few.

OK, so maybe not *so* boring. But they require very little background and can be extremely powerful. And a lot of it is pretty new (like developed in the last 10 years) with some of the most powerful stuff being *extremely* new (like developed in the last 2 years). That makes for lots of open questions that would make interesting research projects (if, you know, you were looking for such things).

Prerequisites: These will be useful, but not necessary: derivatives, generating functions, induction, basic linear algebra, complex numbers.

Automated Proofs in Geometry. (Misha)

Many proofs in geometry rely on cleverly spotting the right similar triangle or cyclic quadrilateral. On the other hand, even a computer can convert a geometry problem into a system of equations and then provide a coordinate proof. Such a proof is often unsatisfying: it will be too long to read, and having read it, you will gain no intuition for why it works.

As a compromise, we will develop a systematic method for solving geometry problems (once again, one that a computer can implement) that manipulates natural geometric quantities such as areas. This method provides solutions to a wide class of geometry problems, and often (since we avoid coordinates) they turn out to be elegant ones that do not require a computer to read.

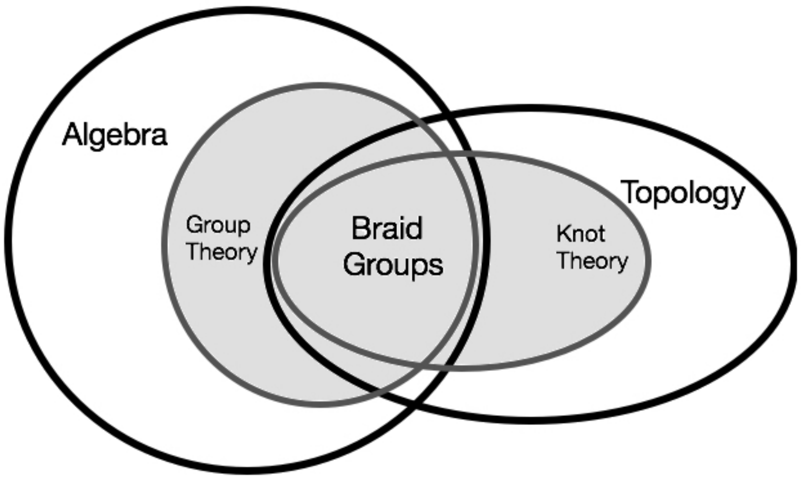
Prerequisites: None.

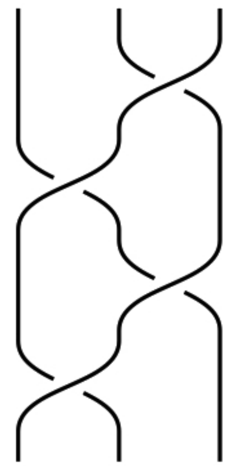
Bad History: A Crash Course in Historiography and how not to do the History of Math. (Sam)

It turns out that, when mathematicians try to do research in the history of mathematics, they sometimes do awful things. We'll talk about some of the common pitfalls (i.e., whig history) and how to avoid them. That way, if you want to ever think seriously about the history of math, you don't accidentally garner the scorn of the history of mathematics community! Yay!!

Prerequisites: None.

Braid Groups. (Nancy)





example braid

A braid group is an infinite group generated by stacking tangled diagrams like the one shown above. For every natural number n, there is a braid group on n strands. Braid groups arise in many different topological settings including knot theory and much much more!

Prerequisites: Group theory (need to know definition of group, homomorphism, quotient group, group presentation, symmetric group, and understand statement of cayley's theorem).

Building a machine that learns like a human. (*Josh Tenenbaum*)

People can learn a new concept almost perfectly from just a single example, yet machine learning algorithms typically require tens or hundreds of examples to perform similarly. People can also use what they learn in richer ways than most machines do — to guide their action, imagination, and explanation. I will talk about how we can build machine learning algorithms that better capture these human learning abilities. Our approach represents concepts as simple programs that can generate observed examples, and learning can be described as a search for the program that best explains what you see. I will show results from a series of "visual Turing test" experiments probing this model's ability to classify as well as to create new instances of concepts, showing that in many cases it is indistinguishable from the performance of humans. Time permitting, I will also talk about other exciting recent work in AI that aims to build machines with human-level learning abilities, such as the Atari Video Game player from Google DeepMind.

Prerequisites: Previous classes on probability, statistics, machine learning, Bayesian inference, or monte carlo would be helpful but are not necessary. .

Burnside's Lemma. (*William Burnside*)

How many different necklaces can you build with 6 pebbles, if you have a large number of black and white pebbles? Notice that you won't be able to tell apart two necklaces that are the same up to rotation or reflection. You probably can answer the above question by counting carefully, but what if we are building necklaces with 20 pebbles and we have pebbles of 8 different colours?

There is a nifty little result in group theory that allows us to solve this type of problem very quickly and without a calculator. Come and learn it!

Prerequisites: Basic group theory.

Category Theory in Sets. (Don)

Category Theory is a relatively young field of mathematics, founded in the mid-1900's. Originally, it was used to work with intimidating topics like cohomology functors and homological algebra. Fortunately, in this class, we won't need anything that complicated. When doing category theory in the category of sets, the thing we get out is much easier: arithmetic.

The arithmetic we'll be doing will have an unusual character to it; instead of taking our numbers and operations as given, we'll describe what they ought to look like, and then define them using categorical language. From this, we can prove the ordinary rules of arithmetic; but having defined things categorically, we'll be able to apply it not just to arithmetic, but to any kind of mathematical object you might encounter.

Prerequisites: None.

Classifying Spaces. (Chris)

Imagine there are mathematical objects that you really cherish and care about. Let's call them vector bundles over a space M. It is really easy for you to come up with many examples of those vector bundles, but unfortunately it is very hard to decide if two vector bundles are essentially the same or not. Wouldn't it be great if there was one space — let's call it the classifying space $BGl(n)$ — and one bundle — let's call it the universal bundle $EGl(n)$ — such that *every* vector bundle over *any* space can be constructed out of a (essentially unique) map from X to $BGl(n)$ and $EGl(n)$. This way you could reduce your study of those cool and complicated objects to the study of maps from X to a fixed space which is relatively easy to understand.

All of this is indeed possible and in this course I will explain what vector bundles are, why you should care about them, and how you can classify them.

Prerequisites: Linear Algebra (at least the definitions of vector spaces, linear maps, injective maps.) Topology (understand this statement: "A map is continuous if preimages of open subsets are open.").

Classifying Symmetry. (*Frank Farris*)

This course uses mathematical art produced by Frank Farris to study the concept of symmetry, with an emphasis on classifying patterns. The main mathematical vocabulary is the concept of a group. If this is your first time studying groups, the patterns in this class provide the best first examples; if you are familiar with the concept, you'll still learn new things about it. Each day will include a presentation about a new topic and then a workshop session where you put the concepts into action by classifying patterns. You should emerge from this course seeing the world differently. (Personally, I find myself pausing movies to classify the wallpaper patterns; very distracting.)

On day one, we establish basic vocabulary about symmetry and explain why the symmetries of a pattern necessarily form a group. The pattern types for the first day are rosette and frieze patterns. On the second day, we study wallpaper patterns. Although we will not prove the wonderful result that there are exactly 17 (isomorphism classes of) wallpaper groups, the result will be highlighted by vivid examples. On the last day, we study color-reversing symmetries, which naturally introduce the concept of normal subgroups. Time permitting, we will also glimpse polyhedral and non-Euclidean patterns.

The homework will involve classification of patterns; for those who wish, there will be several puzzles each day in the area of transformation geometry.

Prerequisites: None.

Coloring in Space. (*Moon Duchin*)

The Four Color Theorem is very famous, and it has a well-known statement in terms of graph theory: *for any planar graph, there is a way to assign each vertex one of four colors so that no two neighboring vertices share a color.* So coloring graphs is already kinda interesting. But graphs are just a tiny little corner of the world of metric spaces. How about coloring all points of a metric space as follows: pick your favorite number d and require that any two points whose distance is exactly d should not be the same color. Then it gets interesting… even the chromatic number of the Euclidean plane is unknown. Not content with a problem that is merely hard enough that nobody can solve it, we will then make it harder: the culmination of this three-day course will be coloring problems in negative curvature.

Prerequisites: Metric spaces.

Coloring Maps. (Jeff + Marisa)

In a "properly" colored map, states that share a border are painted different colors. During an election year, when we paint the states in the US red and blue, no matter how carefully we rig the elections, we'll never get a proper coloring – because Washington, Oregon, and Idaho will need three different colors. Even if the Green Party were to win many states, we would need more colors: consider Nevada and its neighbors! So, what's the minimum number of colors you need to properly color the US?

In this class, we will introduce graphs, a combinatorial structure underlying maps, and prove that any configuration of countries on the Earth can be properly colored with five colors.[1] Then we'll leave the Earth behind and consider what happens to our colorings when the planet is shaped like a donut.

[1]You can do four, actually, but that's much harder to prove.

Prerequisites: None.

Combinatorial Game Theory. (Alfonso)

We have a plate with blueberries, blackberries, and raspberries. We take turns eating them. In your turn, you may eat as many berries as you want, at least one, but they all have to be of the same type. Then it is my turn. Unless you are Marisa, the goal is not to eat a lot of berries: the winner is the person who eats the last one. Will you beat me?

The above is an example of an impartial combinatorial game. There are tons where it came from, and you have probably encountered some. To solve them all, there are basically only two skills that you need to learn. If you want enlightenment, my young grasshopper, avoid spoilers and come to this class. I will motivate the two skills and I will guide you all to

figure them out. The beauty of this topic is as much in the final results as it is in the journey, and I do not want to deprive you of the pleasure of discovering it slowly. We will also attack many examples, from easy ones to actual open problems.

Prerequisites: None.

Combinatorial Topology. (Jeff)

So, you want to be a topologist. But, you've never taken point-set topology[2]. How much can you prove about topology?

Turns out, quite a bit. In this class, we'll be developing simplicial complexes, which give combinatorial representations of topological spaces. Then, we'll look at discrete Morse theory, a combinatorial representation of a construction from differential topology. Along the way, we'll draw lots of pictures and diagrams, and get a feel for what topology *should* do, without messing around with all of those icky open sets.

Prerequisites: None.

Continued Fractions. (Susan)

Suppose you wanted to find all of the integer solutions to the equation $x^2 - 18y^2 = 1$. Actually, let's make it easier: suppose you wanted to find just *one* integer solution to this equation. What would you do? Well, I'll tell you what I would do: I would calculate the continued fraction expansion of $\sqrt{18}$. Its expansion is given by

$$\sqrt{18} = 4 + \cfrac{1}{4 + \cfrac{1}{8 + \frac{1}{4+\frac{1}{8+\frac{1}{4+\frac{1}{8+\frac{1}{4+\dots}}}}}}}$$

Because continued fractions are awesome, this tells me that one solution to the equation $x^2 - 18y^2 = 1$ is given by $x = 17$ and $y = 4$. And because Pell equations are awesome, this tells me that another solution is given by $x = 407$ and $y = 136$. In this class, I'll show you exactly how I figured that out. But more importantly, we'll explore why it works. We'll see how to find all of the solutions to any Pell equation—an equation of the form $x^2 - Dy^2 = 1$. And in the process we'll get a tour of the world of continued fractions. We'll see what the expression above means, and find out why it repeats itself in an infinite alternating pattern.

Prerequisites: None.

Cosine Waves on Musical Staves. (J-Lo)

J-Lo's first class on the connections between math and music. The two days are independent; you can come to this one even if you don't go to the second day.

You will learn how to hear a trigonometric identity (I used this one to tune the piano in the main lounge), why musical harmony should be thought of as a branch of number theory, and how the continued fraction expansion of $\log_2 3$ contains a summary of the development of musical scales.

Prerequisites: Exposure to trig functions and logarithms.

Counting the Faces of Cut-Up Spaces. (*Matt Stamps*)

Suppose you cut a pizza into 10 pieces using 4 straight cuts so that each pair of cuts intersect somewhere in the interior of the pie, and without seeing the pizza, your friend Jane says "Hmm... three of the cuts must have gone through a single point." How did she arrive at this conclusion? Come to this class and find out! We'll study a whole collection of problems based on spaces that have been cut up by others.

Prerequisites: None.

Cryptography. (Pesto)

Does cryptography exist, and why is it a thriving branch of mathematics if we're not sure?

Suppose you want to {send a secret message on the internet, prove that you are who you say are to someone on the internet, run a long-distance poker tournament in which no one trusts any one else or any third party to draw cards fairly}. But, there's an evilly eavesdropping enemy Eve who can see everything you say, knows how you're trying to communicate, and has {infinitely much computational power, a few supercomputers' worth of computational power} available. Can you do it safely despite Eve {no matter what, iff P $\neq$ NP, iff you can do any cryptography at all}?

We'll start with questions as abstract as "is there an easily-computable function that is indistinguishable from random to anyone without lots of computer power?" (no one knows, so we'll assume so) and work our way up to protocols to solve some of the above.

Prerequisites: None, but familiarity with computer science reductions helpful. Intro complexity theory useful.

Cryptography: What is it and Why Do we Care? (*Kristin Lauter*)

Cryptography is the science of keeping secrets. This is useful for e-commerce and in general for secure and authentic communication and transactions. For example, how do you buy something on the internet without an eavesdropper learning your credit card number? How do you know the that "Amazon" is really Amazon? How do you send private email over the internet? The security of many cryptosystems is linked to hard problems in mathematics. This class will explore some of those hard problems and explain what research relevant to implementing and attacking these cryptosystems looks like.

Prerequisites: None.

Cyclotomic Polynomials and Extensions. (Milica)

Definition: The n-th cyclotomic polynomial is the polynomial whose roots are the primitive n-th roots of unity.

Theorem: The regular n-gon can be constructed by straightedge and compass if and only if n is the product of a power of 2 and distinct Fermat primes.

Find out how these two things are related!

Prerequisites: Familiarity with field extensions and with complex roots of unity.

Development of Probability. (Sam)

In the middle of the 17th century, the mathematical field of probability emerged almost entirely out of the blue. People had been gambling for millennia, so you might wonder why it could have taken so long for a mathematical study of games of chance to emerge. You might also wonder what it even means for a field of mathematics to emerge, or how I can feel even remotely confident about when it actually emerged. These are some of the questions we'll explore on day 1.

From then on, the course will focus on the mathematical development of probability during the first 50–100 years of the field's existence. We'll look at the early problems that motivated the field, the clever (and sometimes convoluted) methods that were used to solve these problems, and the cultural context that drove these problems/methods. We'll see brilliant mathematicians propose "solutions" that just sound crazy (like distinguishing between moral and mathematical probabilities to resolve the St. Petersburg Paradox), and we'll see the conflation of ideas required for a new field of mathematics to emerge.

Bonus: Along the way, we'll see how society and mathematics influence each other's development. And we'll see lots of cool and brilliant mathematics. Sadly, still no free set of steak knives…

Prerequisites: None! If you know lots of probability, you'll get to see the context in which it developed; if you don't know too much, then you'll get to better empathize with the people who developed the field!

Differentials and higher differentials. (*Anti Shulman*)

A differential is a thing like $\mathrm{d}x$ or $\mathrm{d}y$. You've probably seen them appearing in pairs in Leibniz' notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the derivative, but appearing singly they are rarely given much attention in calculus classes. However, a highfalutin' generalization of them called "exterior differential forms" plays an important role in differential geometry.

A *higher* differential is a thing like d^2y or $\mathrm{d}x^2$. You may have seen them appearing pairwise in Leibniz' notation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for the second derivative, but for the most part they have been banished from appearing singly in modern mathematics. However, this prejudice is unjustified! I'll explain about how to make sense of them, and why they make life much easier when computing higher derivatives and integrals. In particular, we'll see that Leibniz' notation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is actually *wrong*!

Prerequisites: Calculus. Multivariable calculus recommended, but not required.

Differentiation under the Integral Sign. (Kevin)

Richard Feynman, the eccentric and entertaining physicist, attributes his "great reputation for doing integrals" primarily to one trick – differentiation under the integral sign. In this class, we'll introduce and play with this technique, along with some other useful bits and pieces like half-angle and hyperbolic substitutions.

Prerequisites: Single-variable calculus.

DIY Hyperbolic Geometry. (*Katie Mann*)

If you've ever tried to do geometry on the surface of a sphere, you'll know that it's a wacky place: what should be "parallel" lines always eventually intersect, the area of a disc is less than πr^2, and when you try to flatten a piece of sphere (perhaps you've tried this with a piece of orange peel), you end up tearing it.

In this class we'll explore an even wilder place to do geometry, the hyperbolic plane. This space is the opposite of the sphere: you can draw both parallel and non-parallel lines that never intersect, the area of a disc is way larger than you expect, and when you try to flatten a piece of it, you are forced to wrinkle it up.

This class is called "DIY" for a reason: expect to make and experiment with paper models, attempt to build a hyperbolic soccer ball, discover the laws of the hyperbolic universe for yourself, and re-invent some of M. C. Escher's tessellations. (Artistic talent is definitely not a requirement though!)

Prerequisites: None.

Elliptic Curves. (Ruthi)

Elliptic curves are some of the important objects in number theory, not to mention deeply beautiful. They are integral to solving many problems in number theory, including the solution to Fermat's Last Theorem.

The days of this class will get progressively more difficult:

- Day 1: We will learn what the definition of an elliptic curve over the rationals is and how to add points on one.
- Day 2: We'll generalize our definition of elliptic curves, and discuss what their abelian group structure looks like. (In particular, we'll state Mordell-Weil Theorem.)
- Day 3: We'll discuss heights on curves and prove Mordell's Theorem.
- Day 4: I'll try to talk about some of modern questions of interest about elliptic curves, and recent research and results.

Prerequisites: Know what an abelian group is.

Error-Correcting Codes. (Tim!)

Don and I are secretly planning to take over the camp. Shhh, don't tell anyone; it's a secret! Of course, we have a secret code. Don sends me messages by knocking or tapping four times on my door. For instance, knock-tap-knock-tap means "I've hacked the class schedule and replaced every class with Category Theory", tap-tap-tap-knock means "Tonight is the night to steal Susan's idol of power", and so on.

One night, Don knock-knock-knock-knocks on my door, but I mishear it as knock-tap-knock-knock. So, instead of the message "Let me in", I respond to the message "May Day! Burn down the dorms!". This is a setback.

The problem is that if I mishear even one of the knocks, I get the wrong message-phrase. But there is a solution! There are codes that are *error-detecting* — if I mishear one of the knocks/taps, I'll know just from what I heard that something has gone wrong. Even more amazingly, there are codes that are *error-correcting* — if I mishear one of the knocks/taps, then the knocks/taps I do hear will tell me exactly what I misheard and what the correct message was supposed be. It seems too good to be true, but the simplest error-correcting codes are easy to construct, and the best ones are used in real-life computers and computer systems all over the world (and all over the solar system — the New Horizons spacecraft that just sped by Pluto really wants to make sure its photos and data get transmitted back to earth correctly).

Come check out the power and magic of these codes!

Prerequisites: Linear algebra.

Exploring Equality via Homotopy and Proof Assistants. (*Jason Gross*)

What does it mean for two things to be equal? What if the things are themselves proofs of equality? Enter homotopy type theory, an exciting new branch of mathematics, which gives us a new way to think about mathematical objects. When proofs of equality are fundamentally paths between points in a space, we can use ideas about shapes (topology!) to study them. In this class we will explore the nature of mathematical object-hood and of equality using Coq, an interactive theorem prover.

Prerequisites: Proof by induction, basic exposure to formal logic (comfort with modus ponens, and the difference between axioms and theorems). Helpful but not required: programming. .

Extensions. (Asilata)

Take any two abelian groups, for instance $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Then the product group $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has the property that $\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is an injective homomorphism, and $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ is a surjective homomorphism, and the kernel of the second map is exactly the image of the first map.

Based on this example, we define an *extension* of H by G to be any new abelian group E for which there is an injective homomorphism $G \hookrightarrow E$ and a surjective homomorphism $E \twoheadrightarrow H$, and the kernel of the second homomorphism is precisely the image of the first.

Are there any other extensions than the "product extension" shown above? What does "other" mean? How many are there, and what do they all look like?

We'll think about all of this. But first, hang on tight, it's about to get meta: not only are there (sometimes) lots of different extensions, but the *set of extensions* itself forms an abelian group! Doesn't that sound pretty? Let's learn about it!

Prerequisites: None.

Finite Fields. (Mark)

You may well know that the integers modulo p form a field (basically, a set in which all four arithmetic operations are possible, including division limited only by Rule 4) if and only if p is prime. However, there are other finite fields whose sizes are not prime. Finite fields have applications in areas such as coding theory and cryptography as well as in more abstract mathematics, and they are elegant algebraic objects that are well worth exploring. In this class we'll see how to construct finite fields, and if time permits, show that we have found them all.

Prerequisites: Some ring theory (in particular, polynomials, and rings modulo ideals) and a bit of linear algebra (the idea of dimension).

Fractal dimensions. (*Hermann Minkowski*)

A line has dimension 1, a plane has dimension 2, and the space we live in has dimension 3. Can you think of something of dimension 1.5? What does it mean to have dimension 1.5? Actually, what does it even mean to have dimension 2?

In this class, I will give you one possible definition of dimension and we will compute the dimension of a few objects, including some with non-integer dimensions.

Prerequisites: You need to be comfortable with logarithms and limits.

From Counting to a Theorem of Fermat. (Mark)

A standard theorem stated by Fermat (it's actually uncertain whether he had a proof) states that every prime p congruent to 1 modulo 4 is the sum of two squares. (On the other hand, if p is 3 modulo 4, it has *no* hope of being the sum of two squares.) There are many proofs of this theorem, but perhaps the weirdest one, due to Heath-Brown and simplified by Zagier, uses just counting — no "number theory" at all! In this class we'll see at least that proof, and maybe some others and/or related proofs of other things.

Prerequisites: None!

From Geometric Optics to High Speed Underwater Photography. (*Allan Adams*)

Lecture: Why does squinting help you see better? Why does Architeuthis dux have an eye the size of a dinner plate? (And no, squid can't squint, which is an interesting hint.) Why don't most cell phone cameras bother with an adjustable focus? Geometric optics answers all these questions and explains almost everything there is to say about almost everything that can see. In case you think that's too strong a statement, one famous mathematician used geometric optics to construct a proof of the existence of god! His proof proved flawed, but along the way he developed the principle of least action, so I'll call that a victory.
Lab: Armed with this knowledge — plus an underwater high-speed videocamera from my lab at MIT — we'll run some experiments to answer more questions, like what happens when you pop a balloon underwater? (Hint: It's awesome.)

Prerequisites: Euclidean geometry, trig helpful but not strictly necessary, calculus helpful but by no means necessary.

Functions of a Complex Variable. (Mark)

Spectacular (and unexpected) things happen in calculus when you allow the variable (now to be called $z = x + iy$ instead of x) to take on complex values. For example, functions that are "differentiable" in a region of the complex plane now automatically have power series expansions. If you know what the values of such a function are everywhere along a closed curve, then you can deduce its value anywhere inside the curve! Not only is this quite beautiful math, it also has important applications, both inside and outside math. For example, complex analysis was used by Dirichlet to prove his famous theorem about primes in arithmetic progressions, which states that if a and b are positive integers with $\gcd(a, b) = 1$, then the sequence $a, a + b, a + 2b, a + 3b, \dots$ contains infinitely many primes. This was probably the first major result in analytic number theory, the branch of number theory that uses complex analysis as a fundamental tool and that includes such key questions as the Riemann Hypothesis. Meanwhile, in an entirely different direction, complex variables can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow. Dirichlet's theorem is certainly beyond the scope of this class and heat conduction probably is too, but we should see a proof of the so-called "Fundamental Theorem of Algebra", which states that any nonconstant polynomial (with real or even complex coefficients) has a root in the complex numbers. We should also see how to compute some impossible-looking improper integrals by leaving the real axis that we're supposed to integrate over and venturing boldly forth into the complex plane! This class runs for two weeks, but it should definitely be worth it. (If you can take only the first week, you'll still get to see one or two of the things mentioned above; check with me for details.)

Prerequisites: Multivariable calculus (including Green's Theorem; if the MV crash course doesn't get to Green's Theorem, it will be covered near the beginning of this class).

Fundamental Group. (Sachi)

"Before functoriality people lived in caves." — Brian Conrad

If you look at the equator of a sphere living in n-dimensional space, then the equator is itself a sphere of one dimension lower. It seems intuitive, at least in the only dimensions we can visualize, that you cannot retract the bigger sphere to the smaller sphere without cutting holes in the sphere. But, how do topologists prove something like that? One way is by using the powerful tool of functors. Functors give us a way of assigning groups or other algebraic objects to topological spaces. In this class we will look in particular at the functor called the fundamental group, which is the group of loops that one can draw on a space.

Prerequisites: Point Set Topology, Group Theory.

Fundamental Theorem of Calculus in Dimension n. (Jeff)

Legend has it that if you take the ancient tome "*On Certain Differential Expressions and the Pfaff Problem, 1899*" and hold the binding over a fire, you will see the following script written by Dark Lord Cartan from when he forged the book:

Fashion Four Derivatives for Dimension Three
Called Total, Partial, Curl and Grad;
In the realms in Dim Two, Only Two Shall be;
On the Domain of Dim One, only $\frac{d}{dx}$ be had.

One Derivative to rule them all, One Derivative intertwines them
One Integral to take that One, and in the Darkness bind it.

In this class, we will explore the exterior derivative, the One True Derivative of Calculus, which combines all of the above derivative operations into a single object that extends to any dimension. On our journey, we will:

- Create a language to describe higher dimensional analogues of vector fields.
- Prove a Fundamental Theorem of Calculus For Dimension n that can be stated in just 9 characters.
- Glimpse into the future on how we can do calculus on manifolds.

Prerequisites: Multivariable Calculus, Linear Algebra. You should be able to prove that V is isomorphic to the space of linear maps from $V \to \mathbb{R}$. .

Fun with Compactness in Logic. (*Matt Wright*)

Mathematical proofs are finite—and that's a good thing, because if they weren't, we wouldn't be able to write them down or present them in (finitely long) Mathcamp classes! But that simple observation leads to a surprising theorem of logic called the compactness theorem, which lets us take finite structures and in some cases use them to say things about infinite ones. This has some strange consequences! We'll use the compactness theorem to show that there are structures that satisfy all of the axioms of the natural numbers and yet look pretty wildly different, and we'll look at some simple things we can't express in logic.

Prerequisites: None.

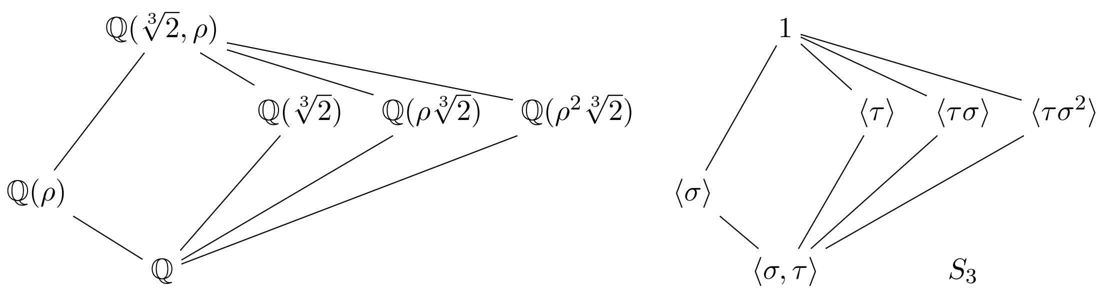
Galois Cohomology. (Ruthi)

A lot of math is about figuring out when two things are the same — or rather, isomorphic. This turns out to sometimes be very difficult, so we do what mathematicians always do: we try to solve an easier question. From such an attempt comes group cohomology, and specifically Galois cohomology, which deals with profinite groups — bizarre groups that are almost finite… but not quite, and carry a weirdly unintuitive topological structure.

Galois cohomology turns out to be the right abstract tool to study certain aspects of advanced number theory, and if you put the content of this class together with other topics (algebraic number theory, quadratic forms, elliptic curves, to name only a few), you turn out to be able to solve interesting problems in beautiful, succinct ways.

Prerequisites: Group theory, Point-set topology.

Galois Theory. (Nancy)



Do you think it is a coincidence that the subgroup lattice of S_3 looks so similar to the subfield lattice of $\mathbb{Q}(\sqrt[3]{2},\rho)$, where ρ is a primitive cube root of unity? It's not!! This beautiful relationship is called the Galois Correspondence.

Here is roughly how it works:

(1) Start with a field F and a polynomial $f(x)$ that has no roots in F
(2) Adjoin all the roots of f to F to get a new field K
(3) Hey look! When we permute all the roots of f in K we get a group! Lets call it the Galois group of K over F
(4) Hey look again! The subgroups of the Galois group look a lot like the subfields of K but just flipped upside-down! Ta-da, the Galois correspondence.

Galois theory is considered one of the most beautiful subjects in mathematics. Everything that you want to to work just fantastically works!

Let's climb Mt. Évariste!

Prerequisites: Group theory, ring theory. Understand the statement "A ring mod out by a maximal ideal is a field".

Generating Functions, Catalan Numbers, and Partitions. (Mark)

Generating functions provide a powerful technique, used by Euler and many later mathematicians, to analyze sequences of numbers; often, they also provide the pleasure of working with infinite series without having to worry about convergence.

The sequence of Catalan numbers, which starts off $1, 2, 5, 14, 42, \ldots$, comes up in the solution of many counting problems, involving, among other things, voting, lattice paths, and polygon dissection. We'll use a generating function to come up with an explicit formula for the Catalan numbers.

A *partition* of a positive integer n is a way to write n as a sum of one or more positive integers, say in nonincreasing order; for example, the seven partitions of 5 are $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1,$ and $1+1+1+1+1$. The number of such partitions is given by the partition function $p(n)$; for example, $p(5) = 7$. Although an "explicit" formula for $p(n)$ is known and we may even look at it, it's quite complicated. In our class, we'll combine generating functions and a famous combinatorial argument due to Franklin to find a beautiful recurrence relation for the (rapidly growing) partition function. This formula was used by MacMahon to make a table of values for $p(n)$ through $p(200) = 3972999029388$, well before the advent of computers!

Prerequisites: Summation notation; geometric series. Some experience with more general power series may help, but is not really needed. A bit of calculus may be useful.

Graphs on Surfaces. (Marisa)

You've heard the joke about the topologist at breakfast, who can't tell the difference between a coffee cup and a donut. Maybe you've seen some topological spaces before, like the sphere and the torus and the double-torus (a donut with two holes); and maybe some non-orientable ones, too, like the Klein bottle. Sure, a Klein bottle looks weird. But how differently does it behave from the sphere? After all, if you were an ant walking around any of these surfaces, everything around you would look just like the plane.

One way to see just how weird a surface can be is to draw graphs on it, and that's what we'll do in this class. We'll be able to fully classify the surfaces on which some families of graphs can live, and quickly get to the frontier of what graph theorists don't know about the behavior of surfaces as simple as the double-torus.

Prerequisites: Basic graph theory or Coloring Maps. We'll cover the relevant topology in class.

Group operations' harmonic implications. (J-Lo)

J-Lo's second class on the connections between math and music. The two days are independent; you can come to this one even if you don't go to the first day.

Discover how the four-chord harmonic progression you hear in almost every pop song can be interpreted via the symmetries of a regular dodecagon, why minor really is just negative major, and how Beethoven's Ninth Symphony almost traces out one of the generators of the fundamental group of a torus.

Prerequisites: Introduction to Groups.

History of Math. (*Moon Duchin*)

Notation: What we write and what we write on and why it matters. Symbols, pictures, and diagrams. Special attention to the history of paper and the curious story of the complex plane.

Parts of the whole: Ways of dealing with what's less than one. Special attention to the surprising complexity of Egyptian fractions.

Competition: Duels, flame wars, and contests. Hobbes and Wallis. Tartaglia the stutterer. Special attention to the singular culture of the Cambridge Tripos exam.

Prerequisites: None.

Homotopy Theory. (Chris)

In topology, two spaces are (homotopy) equivalent if they can be "continuously transformed into each other": a cube is the same as a sphere, but not the same as a torus (donut). Deciding if two spaces are homotopy equivalent is very hard and we often have to turn to invariants. The first such invariants are the homotopy groups of a space X : For $i \in \mathbb{N}_0$ the set $\pi_i(X)$ are the (homotopy) equivalence classes of maps from the i dimensional sphere S^i to X. If $i \geq 1$ this is a group and if $i \geq 2$ it's even abelian. It is easy to see that two homotopy equivalent spaces have the same homotopy groups, so calculating those groups is a good way of deciding if spaces are not homotopy equivalent. There also is a huge class of spaces (so called CW-complexes) for which having matching homotopy groups immediately implies homotopy equivalence.

So these homotopy groups are easy to define and tell us a whole lot about the space, but there is one big problem: they are incredibly hard to calculate. So hard that after over 80 years we still have not been able to compute all the homotopy groups of the 2-sphere S^2! Calculating those homotopy groups of spheres has been driving modern mathematics for over half a century and these efforts have produced a large amount of new ideas, techniques and theories, and have shaped the way we think about mathematics in many areas such as topology, algebra and geometry. In this course we will introduce the homotopy groups and make our way to the long exact sequence of the homotopy groups of a fibration, the first (and most important) tool used to calculate them.

Prerequisites: Understanding factor groups G/H and direct sums of groups; (some, not necessarily strict) definition of topological spaces and continuity, some idea why the fundamental group is a group (I can explain this to you during TAU if you want to).

Induced Matchings from Szemeredi's Regularity Lemma. (*Po-Shen Loh*)

Start with a set of N vertices. Successively lay down induced matchings $M_1, M_2, \ldots, M_N$, where at each turn, M_k is a (not necessarily perfect) matching within this vertex set, which does not include any edge whose endpoints are in the same M_j for an earlier $j < k$. Let G be the resulting graph. Note that for each k, the subgraph of G induced by the vertices of M_k is precisely M_k, and nothing more. What's the maximum number of edges G can have at the end?

Prerequisites: None.

Infinitesimals. (Don)

The early days of calculus made liberal use of a controversial tool: infinitesimal values. There's no need to define limits when you can work directly with a notion of "incredibly close." However, leading minds and religious authorities alike put forth bans on the use of infinitesimals, and ultimately, the debate over infinitesimals was won by these opponents. The debate lay dormant for centuries, until a few mathematicians realized that, even though $\mathbb{R}$ contains no infinitesimals, there are reasonable systems of mathematics that do.

In this class, we'll be looking at a couple of ways to do math in which infinitesimals create no contradictions. In one, as long as we have powerful enough axioms, we can just add infinitesimals to $\mathbb{R}$. In another, by weakening logic itself, we'll get amazing results, like every function being smooth — but that is quite a high price to pay.

Prerequisites: None (Calculus suggested).

Introduction to Complexity. (Pesto)

This is a class about what computers that don't exist can do. If you had a computer that could run forever (certainly much more than the lifetime of the universe), but only had a fixed finite memory, what could it do? What if you had a computer that could guess the answer to any question you ask it, but wasn't sure whether it had guessed correctly? What about a computer that could ask *those* computers for advice and get answers immediately?

We'll sort those computers by their power and classify problems according to which of those computers can solve them: for instance, we'll show that designing an optimal route for a traveling salesman (or even getting very close to it) is exactly as hard as solving a jigsaw puzzle, in that the same set of those computers can do so, but determining who wins a game of (generalized) chess is harder.

This is a class not only too pure (like the week 1 algorithms class) to look at actual code, but also too pure to look at actual algorithms. No coding experience relevant. Algorithms experience only relevant for a bit of flavor.

Prerequisites: None.

Introduction to Groups. (Mira)

In how many ways can you paint the vertices of a cube with one of three colors? (We consider two colorings the same if they can be obtained from each other by rotating the cube – so the answer is not just 2^8.) How can you analyze the symmetries of geometric figures, or the workings of a Rubik's cube? How do physicists predict the existence of certain elementary particles before setting up expensive experiments to test those predictions? Why can't fifth-degree polynomial equations, like $x^5+3x+17=0$, be solved using anything like the quadratic formula, although fourth-degree equations can?

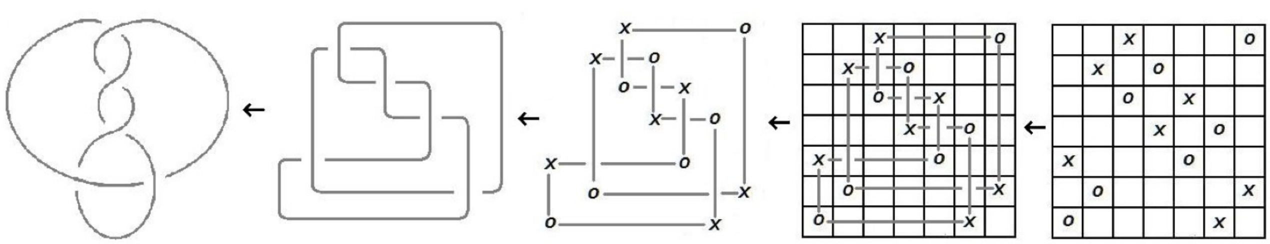
The answers to all these questions depend on group theory (although most of them are beyond the scope of this class). Knowledge of some group theory is at least helpful, and often crucial, for many other parts of mathematics. This course will cover the basics of group theory through the First Isomorphism Theorem, and also hopefully have time for some fun applications and examples.

(If you can state and prove the First Isomorphism Theorem in group theory, then you do not need to take this course.)

Prerequisites: None.

Intro Knot Theory. (Nancy)

Knot Theory!



Knot theory is the study of distinguishing when two knots are the same, or not. If this looks cool, come take my class!

If you've seen some knot theory before but are wondering whether you should take this class, here are some topics we'll be covering: Reidemeister's Theorem for classifying knots, tri-colorability, Seifert Surfaces, grid diagrams, Alexander Polynomial, braid group, and maybe more.

Prerequisites: Understand what a matrix is and how to take a determinant.

Knight's Tours. (Misha)

A well-known puzzle asks if a knight on an 8×8 chessboard can make 64 moves in a way that visits each square of the chessboard exactly once and then returns to the starting square.

You probably have already seen the answer, so here's a different question to think about: can you do the same in 16 moves on a 4×4 chessboard?

Here's a third question which the mathematician Sir William Hamilton tried to market as a puzzle: can you find a path along the edges of a dodecahedron that visits each vertex exactly once and returns to the starting point? (The puzzle did not sell: it was too easy.)

Here is a fourth question (resolved only a few years ago): can you make 43 252 003 274 489 856 000 twists of a Rubik's cube that put it in each possible state exactly once before returning to the solved position? I won't tell you why the answer is what it is, but I'll explain why we should have seen it coming.

Prerequisites: None.

LARGE cardinals! (Steve)

Some numbers are so big we need special notation to describe them. Well, it turns out that some *sets* are so big, you need special *axioms* to even show they exist! These are called *large cardinals* and are a major focus of current research in set theory.

Large cardinals come in two flavors: the smaller(!) ones tend to arise from taking nice theorems in combinatorics and going "But what if INFINITY?!?!" The larger ones appear when we take the entire universe, stretch it, and put it inside itself for no obvious reason. Yay science!

Over time, set theorists have discovered many species of large cardinals; in this class, we'll tour the zoo.

Prerequisites: Familiarity with basic set theory: ordinals, cardinals, and the axiom of choice.

Latin Squares. (Marisa)

Do you remember Sudoku, the little popular logic puzzle? It went like this: take a 9×9 grid and fill it with the numbers 1 through 9 so that each digit appears once in every row and every column. (And there was some constraint about subsquares, which I will ignore.) If I give you two different Sudokus, with different fills of the 9×9 grid, you can superimpose one on top of the other and look at the pairs of symbols you've produced from the matching squares. Here's a deeper puzzle: can you find two Sudoku fills that are *so* different from each other that when you superimpose them, you get all 9^2 possible pairs? Great. Now can you find *eight* different Sudoku grids, with the property that *every* pair of grids produces all 9^2 possible pairs? No need to do it by hand — we can hit this with the hammer of modular arithmetic. The result is a set of *mutually orthogonal Latin Squares* (MOLS), a combinatorial object that turns out to be related to finite geometries, designing schedules, building codes, and all kinds of other combinatorial objects.

Prerequisites: None.

Laurent Phenomenon. (Kevin)

Consider the recurrence

$$a_n = \frac{a_{n-1}a_{n-3} + a_{n-2}^2}{a_{n-4}}.$$

If we begin with $a_0 = a_1 = a_2 = a_3 = 1$, then we find that the sequence always consists of integers, despite dividing by a_{n-4} in the recurrence. Why?? How can we prove this?

It turns out that this recurrence can be viewed as a special example of a process called mutation. In the early 2000s, Sergey Fomin and Andrei Zelevinsky introduced this process and proved that all values yielded by mutation can be written as Laurent polynomials in the initial values. This so-called Laurent phenomenon provides a unified proof for integrality of a large number of rational recurrences. This led directly into their work on cluster algebras, one of the most exciting new fields in modern mathematics.

In this class, I'll give several examples of mutation (from triangulations to rational recurrences and more!) and prove the Laurent phenomenon. Time permitting, I will also sketch out the definition and basic properties of cluster algebras and the myriad places they've been showing up in mathematics in the last decade or so.

Prerequisites: None.

Learn to code! (Asilata)

This class will be a gentle introduction to programming and basic principles of coding, in one of my favourite programming languages: Haskell. If you know how to code in another language, this class is probably not for you. (You can feel free to show up to learn the basics of Haskell, but remember that it will be gentle.)

If you haven't coded much or never coded at all before, this class is definitely for you! We'll start with some basic general principles of coding and do hands-on examples before moving on to Haskell-specific features. Haskell is a language that is particularly satisfying if you are mathematically inclined, and we'll see why. We'll also discover that Haskell is lazy, and how we can exploit that to our advantage.

Here is some bonus pretty code taken straight from the Haskell homepage. The following lines generate an *infinite list* (yes you read that right!) called "primes". This is exactly a list of all the prime numbers, obtained by implementing Eratosthenes' sieve!

```
primes = filterPrime [2..]
  where filterPrime (p:xs) = p : filterPrime [x | x <- xs, x `mod` p /= 0]
```

Prerequisites: None.

Lebesgue Measure. (Alfonso + Steve)

In your calculus class you probably learned to integrate using Riemann sums. This is wrong. There are functions that can't be integrated this way, but should have integrals: for example, the Dirichlet function (the characteristic function of the rationals). Since there are only countably many rational numbers at all, we might intuitively say that its integral should be zero. This is backed up by the fact that (exercise!) we can write the Dirichlet function as the pointwise limit of a sequence of continuous functions whose integrals tend towards zero. However, any attempt to integrate the Dirichlet function in the usual way — i.e., via Riemann sums — fails completely.

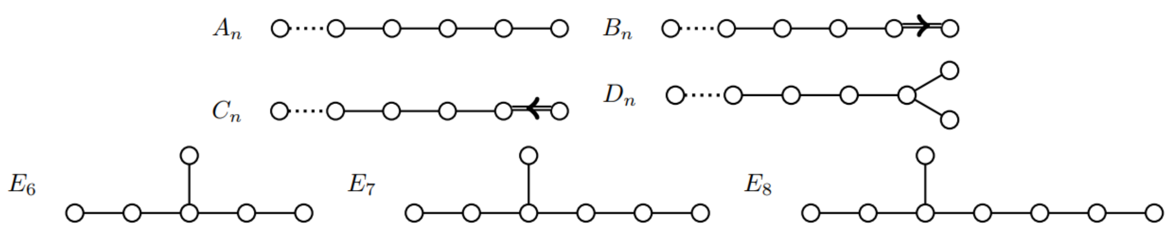
In this class we will present the *right* way to integrate things: Lebesgue integration.

Prerequisites: Basic real-line topology or metric spaces; calculus.

Lie Algebras. (Asilata + Kevin)

The product of two matrices with trace 0 does not necessarily have trace 0. But there is another operation that actually preserves the set of $n \times n$ matrices with trace 0 — just send $[A, B]$ to $AB - BA$! We call this the *bracket* of A and B, and it makes the set of $n \times n$ traceless matrices into a *Lie algebra*. The bracket has many properties that we are not used to. For example, it is anti-commutative: $[A, B] = -[B, A]$. In general, a Lie algebra is a vector space that has a "bracket" operation with similar properties to the one we just saw.

In this class, we will start out with some fundamental examples which will then lead us into the rigid and beautiful structure of the world of Lie algebras. We'll see how the seemingly complicated algebra implied by this mysterious bracket can be distilled down to an elegant description given by the strange pictures drawn below.



Prerequisites: Group theory, ring theory.

Linear Algebra. (Mark)

You may have heard that linear algebra involves computations with matrices and vectors, and there is some truth to that — but it sounds much less interesting than the subject really is; what's exciting about linear algebra is not those computations themselves, but

(1) the conceptual ideas behind them, which are elegant and which crop up throughout mathematics, and
(2) the many applications inside and outside mathematics.

In this class we'll deal with questions such as:

- How can you talk about geometric concepts (such as lengths and angles) if you're not in the plane or 3-space, but in higher dimensions?
- What does "dimension" even mean, and if you're inside a space, how can you tell what its dimension is?
- What does rotating a vector, say around the origin, have in common with taking the derivative of a function?
- If after a sunny day the next day has an 80% probability of being sunny and a 20% probability of being rainy, while after a rainy day the next day has a 60% probability of being sunny and a 40% probability of being rainy, and if today is sunny, how can you quickly find the probability that it will be sunny exactly one year from now?

Do come join me in exploring this material. Despite having taught at Mathcamp for many years, I've never taught this class here, so I'm especially excited to find out just how far we can reasonably get in a week!

Prerequisites: None beyond the Mathcamp crash course (even though the blurb refers to taking a derivative, you'll get by if you have no idea what that is).

Many Facets of Optimization. (Tim!)

Odie and Evan play a game. Simultaneously, they each hold out one or two fingers. If the total number of fingers is odd, Odie wins, and if it's even, Evan wins. The winner must pay the loser a number of dollars equal to the total number of fingers up. What's the best strategy for each player?

Games like this will lead us to another kind of problem. Suppose you're starting a delivery service, and you want to buy reindeer and moose to help you with delivery. Each animal has a certain cost, requires a certain amount of food, and can make a certain number of deliveries per day, and you want to maximize the amount you can deliver given your budget.

Luckily, computers can solve this sort of problem very efficiently, as long as the number of animals doesn't need to be an integer. When we've set up the probleWait what was that about an integer number of animals? That seems kind of important. . .

As soon as we want all our variables to be integers, the problem becomes way harder! But there's a strategy that often works well: Think of your set of possible solutions as a polygon, lift it up to a higher dimension, cut off some of its corners, then project it back down. We'll see what this means and how to do things with it.

This class is a tour through zero-sum matrix games, linear programming, and lift-and-project hierarchies.

Prerequisites: Linear algebra.

Markov Chains to Support your Probabilistic Exploits. (*Nina White*)

Which properties in Monopoly are most often landed on? How do you decide when to attack in Risk? How long does a game of Chutes and Ladders typically last? All of these questions can be answered using the foundational probabilistic tool of Markov chains (and the help of a computer). In this course we'll progress from some basic examples of "game analysis" questions to more complex ones. We'll develop the tools of Markov chains along the way and use Matlab to help us. There may be opportunities to turn classwork into projects.

Prerequisites: Basic linear algebra, basic experience with counting and discrete probability.

Martin's Axiom and Ramsey Ultrafilters. (Steve + Susan)

Let's play with the complete graph on $\mathbb{N}$![3] We like colors, so we're going to color the edges — each edge gets colored either "red" or "blue"[4] It turns out that no matter how we do this, we can find an infinite set of vertices H (called a "homogeneous set") such that all the edges between vertices in H have the same color. For instance, maybe we color the edge between a and b red if and only if a and b have the same parity (even or odd); then we can take H to be the set of odd numbers.

But that's graph theory, and we want to do logic. Also, we're lazy; we want a machine that will magically find homogeneous sets for every coloring. Also also, we like ultrafilters. Can has useful ultrafilter please?

It turns out that, if we assume Martin's axiom[5], there is an ultrafilter $\mathcal{U}$ such that — for *any* coloring at all — there is a homogeneous set H for that coloring such that $H \in \mathcal{U}$. Time remaining, we'll show that *every* ultrafilter has an ultrafilter like $\mathcal{U}$ living inside it.[6]

Prerequisites: Ultrafilters, Martin's Axiom.

Mathcamp Crash Course. (Alfonso)

This course covers fundamental mathematical concepts and tools that all other Mathcamp courses assume you already know: basic logic, basic set theory, notation, some proof techniques, how to define and write carefully and rigorously, and a few other tidbits. If you are new to advanced mathematics or just want to make sure that you have a firm foundation for the rest of your Mathcamp courses, then this course is *highly* recommended.

Here are some problems to test your knowledge:

(1) Negate the following sentence without using any negative words ("no", "not", etc.): *"If a book in my library has a page with fewer than 30 words, then every word on that page starts with a vowel."*
(2) Given two sets of real numbers A and B, we say that A *dominates* B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two, disjoint, non-empty sets A and B such that A dominates B and B dominates A.
(3) Prove that there are infinitely many prime numbers.
(4) Let $f : A \to B$ and $g : B \to C$ be maps of sets. Prove that if $g \circ f$ is injective then f is injective. (This may be obvious, but do you know how to write down the proof concisely and rigorously?)
(5) Define rigorously what it means for a function to be increasing.
(6) Prove that addition modulo 2013 is well-defined.
(7) What is wrong with the following argument (aside from the fact that the claim is false)?

    Claim: On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

    Proof: We proceed by induction on n. The claim is clearly true for $n = 1$. Now suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k+1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k+1$, so by induction it holds for all n. QED.
(8) Explain what it means to say that the real numbers are uncountable. Then prove it.

If you are not 100% comfortable with most of these questions, then you can probably benefit from this crash course. If you found this list of questions intimidating, then you should *definitely* take this class. It will make the rest of your Mathcamp experience much more enjoyable and productive. And the class itself will be fun too!

Prerequisites: None.

Mathematical Magic. (Don)

> It is the unspoken ethic of all magicians to not reveal the secrets.
>
> *David Copperfield*

> The secret impresses no one. The trick you use it for is everything.
>
> *"The Prestige" (2006)*

> For centuries, magicians have intuitively taken advantage of the inner workings of our brains.
>
> *Neil Degrasse Tyson*

This will be a course in which you learn magic.

The problem is, I'm a magician. I cannot tell you my secrets.

What I can do is ask you questions. Answer enough questions, and you'll have a sufficient understanding of the inner workings of a deck of cards to create your own tricks. In class, you will present both your answers to these questions, and the magic tricks based on the answers.

A mathematical magic trick is a trick that requires no sleight of hand or misdirection; rather, it is based on potentially unintuitive mathematical principles. Because all of your tricks will be mathematical in nature, you won't need to explain them; your classmates can, and will, solve them.

Because the class will be Moore Method, you'll need to work on a few problems (and try to think of at least one trick based on them) before the first day of classes. You can pick up the homework due on the first day from me or in the office, starting on Sunday.

Prerequisites: None.

Measure and Martin's Axiom. (Susan)

If we want to develop a notion of "length" on the real line, there are some properties we know it ought to satisfy. For instance, the length of the interval (a, b) ought to be $b - a$. And the length of a single-point set ought to be zero.

In this class, we will be interested in the behavior of the sets with size zero. Any countable union of measure-zero sets has measure zero. However, we can take the union of continuum-many measure-zero sets and obtain a set that does not have measure zero. (For example, the union of all single-point sets $\{x\}$ with $0 \leq x \leq 1$ is the unit interval, which has measure 1.)

So what happens if we take the union of an uncountable—but not continuum-sized—collection of measure-zero sets? This question turns out to be independent from standard ZFC set theory. We'll explore one possible answer that arises in a universe where we've added an extra-set-theoretic axiom called Martin's Axiom to the mix.

This class should be fun for campers who want to mess around with really big numbers ($\aleph_1$) and really small numbers ($\epsilon > 0$) at the same time.

Prerequisites: Understanding of basic facts about cardinality .

Metric Spaces. (Steve)

A *metric space* is just a set X of "points" together with a *distance function*, d, which behaves the way distance should: the distance between two points is zero iff they are actually the same point; the distance between x and y is the distance between y and x; and it is never more efficient to go from x to y to z, than to just go from x to z (this is the *triangle inequality*: $d(x, z) \leq d(x, y) + d(y, z)$). For example:

- The set $C_0[0, 1]$ of continuous functions from $[0, 1]$ to $[0, 1]$ forms a metric space, with the distance function $d(f, g) = \max\{|f(x) - g(x)| : x \in [0, 1]\}$.
- Another metric we can put on $C_0[0, 1]$: $d(f, g) = \int_0^1 |f(x) - g(x)| dx$. Therés a bunch more . . .

- Given a combinatorial graph G — that is, a set of vertices V together with a set of edges E between them — as long as G is connected, we get a metric on V given by the ”shortest path” function.
- Words can be turned into a metric space, where the distance between two words is the number of typos you would need to make to transform one into another.

And so on.

Because they are so general, metric spaces appear everywhere throughout mathematics. In this class, we will begin by studying the general properties metric spaces may have — including compactness, completeness, and connectedness — and how these properties are used. We will then turn to an example of a truly strange metric space: the metric space of compact metric spaces! Time permitting, we will consider another truly bizarre metric space — a line which cannot be cut into two smaller lines ...

Prerequisites: None.

Multiplicative Functions. (Mark)

Many number-theoretic functions, including the Euler phi-function and the sum of divisors function, have the useful property that $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. There is an interesting operation, related to multiplication of series, on the set of all such multiplicative functions, which makes that set (except for one silly function) into a group. If you'd like to find out about this, or if you'd like to know how to compute the sum of the tenth powers of all the divisors of 686000000000 by hand in a minute or so, you should consider this class.

Prerequisites: No fear of summation notation; a little bit of number theory. (Group theory is *not* needed.).

Multivariable Calculus. (Mark)

In real life, interesting quantities usually depend on several variables (such as the coordinates of a point, the time, the temperature, the number of campers in the room, the real and imaginary parts of a complex number, ...). Because of this, “ordinary” (single-variable) calculus often isn't enough to solve practical problems. In this class, we'll quickly go through the basics of calculus for functions of several variables. As time permits, we'll look at some cool applications, such as: If you're in the desert and you want to cool off as quickly as possible, how do you decide what direction to go in? What is the total area under a bell curve? What force fields are consistent with conservation of energy?

Prerequisites: Single-variable calculus (differentiation and integration).

Non-classical Constructions. (Chris)

Compasses are for the weak! I never liked using them. The ancient Greeks loved doing geometric constructions with straight-edge and compass. I, on the other hand, can get by pretty well with just a straight-edge and so can you!

Straight-edge and compass constructions were first studied around 600 BC. Astonishingly, we can construct many things with those tools, such as 36° degree angles, regular 17-gons, the golden ratio ϕ and much more. It is equally surprising that there are things that cannot be constructed by straight-edge and compass, such as the regular 7-gon or the trisection of an angle. It took over 2000 years, but by the early 19th century all this was well studied and understood.

Now I ask the question: But what if I don't have a compass? We will explore these non-classical constructions using only a straight-edge and there is still a surprising amount of things that can be constructed. Indeed, if we give ourselves just one circle in the plane, we can do everything only using a straight-edge that the ancient Greeks still needed a compass for. In this class we will not only give a proof for this fact, but we will also understand algebraically why just a straight-edge alone is not enough. In the homework you will discover many non-classical constructions (such as finding parallels, perpendiculars, ...) yourself, some of which are vital to the proofs done in class.

Prerequisites: Basic (high school) geometry .

Normal Numbers. (Steve + Susan)

A real number between 0 and 1 is *normal* if, whenever we consider its base-b representation, every finite pattern of digits occurs with the expected frequency. So, for example, if x is normal, then when we write x in binary, "half" the digits are "0."

It is not at all clear that normal numbers exist. Perhaps surprisingly, it turns out that *most* numbers are normal! In fact, it's conjectured that lots of interesting real numbers — π, e, $\sqrt{2}$ — are all normal. However, no individual natural examples are known, and for a long time nobody knew a nice way to produce an example at all.

Come let us introduce you to these marvelous objects! We'll prove they exist, and show you how to build them. Time permitting, we'll discuss some recent work by one of Steve's professors (and others) on quickly computing normal numbers.

Prerequisites: None.

Optimization Problems on Graphs. (Sam)

Suppose you were in charge of running the first chess-sprinting world championship. Because the sport is so popular, you have a ton of money to run the championship and, because you like the beach, choose to host the tournament in Bora Bora. Nike has also sponsored the tournament, and has given you funding to fly competitors to Bora Bora (so long as everyone in the tournament wears Nike's chess-sprinting-shoe-line). All that remains is to book the flights.

Now suppose also that you like a challenge (that is, after all, how you got into the chess-sprinting business in the first place). You thus try to book flights without using any online tools: all you let yourself have is a map that shows, for each pair of airports, the cost of the cheapest flight between those airports (if no such flight is available, the cost is infinite). If you just had this map, how could you figure out what flights to purchase?

It turns out that this type of problem is ubiquitous, and actually occurs in significantly less contrived scenarios. More formally, suppose you have a graph G, and for each pair of vertices u, v, the edge between u and v is labeled with some sort of cost (say, corresponding to the cost to get from u to v). If there is no edge between u and v, we can represent that cost with a really large number. In the shortest path problem, we're interested in finding the path from u to v of minimum cost, where the cost of the path is defined as the sum of the costs along all edges in that path.

In this class, we'll introduce the shortest path problem then talk about a few algorithms for solving that problem (like Bellman-Ford and Dijkstra's). We'll also talk about a few related problems! We will not, alas, get to go to Bora Bora.

NOTE: this is going to be a course with a relatively practical flavor. My goal will be to introduce certain types of optimization problems that occur on graphs and then to show clever ways of solving those problems. I'll justify some of the techniques, but we won't necessarily be rigorously proving all of them. But, the techniques we will talk about are all pretty awesome, and they're great techniques to have in your toolkit!

Prerequisites: A smattering of basic graph theory: know what a graph is and what a path between two vertices is.

Ordinal Arithmetic. (Jalex)

If you were here on the first day of camp, you know that ordinals are the things you get when you count by *shvoomp*ing. For the purposes of counting, ordinals uniquely extend the natural numbers — in other words, infinite counting works just the way we want it to. Exponentiation is iterated multiplication is iterated iterated addition, so as long as we come up with an appropriate notion of iterating iteration past the finite numbers, we can do all sorts of arithmetic with ordinals.

In this class, we'll look at several different notions of "iterating iteration". For multiplication and addition, we'll discover that we can't get nice algebraic properties unless we sacrifice topological ones. For exponentiation, we can't have the nice algebraic properties at all — a result proven just this March! Along the way, we'll make a little detour to prove this fun fact:
There is a set of points in $\mathbb{R}^2$ which intersects each line exactly twice. (If that doesn't weird you out, think about it a little harder.)

Prerequisites: Counting.

pNumber Theory. (*Noah Snyder*)

Number theory is the study of integers, while pnumber theory is the study of polynomials. pNumber theory has a lot in common with number theory, for example pnumbers also have unique factorization into prime pnumbers. Fortunately, pnumbers are easier to study than numbers for several reasons, most notably that you can take derivatives of polynomials and that polynomials are related to geometry. In this class we will prove the pABC conjecture, pFermat's Last Theorem, and the pRiemann hypothesis.

Prerequisites: Elementary number theory, the derivative rules for polynomials.

Point-set Topology. (Nancy)

Topology lurks behind the scenes in many subjects of mathematics including analysis, geometry and algebra. While these subjects can be learned on their own, having a knowledge of topology gives a deeper understanding of the underlying concepts and themes.

In a sense, a topology is a mathematical generalization of "closeness". For example, a space with a notion of distance (aka metric) is an example of a topological space. But we use the word "close" to mean all sorts of things. For example, my mother and I are very close, but we live hundreds of miles apart. To describe this type of closeness with a topology, we would say that my mom and I are elements of an open set.

Why do we care about "closeness" anyway? Believe it or not, we can use information about closeness of points to determine the shape of the space. Is my space connected (in one piece)? Is my space compact (will it fit in your suitcase)?

Maybe you thought you knew what the world around you looks like, but by the end of this class you will finally be able to understand why a donut and a coffee mug really are the same!

Prerequisites: None.

Posets. (Kevin)

What do the principle of inclusion-exclusion, divisors, graph colorings, and polytopes have in common? We can study and prove results about these things using partially ordered sets! And the list goes on and on: subgroup structures, partitions, permutations… so many things come naturally with a partial ordering!

Unlike a total ordering, we can't always compare two elements in a poset. For example, it's natural to order sets based on containment, but then the sets $\{1, 2\}$ and $\{1, 3\}$ are incomparable—neither contains the other. Even though it might seem like a partial order doesn't provide much structure, we'll see that we can do quite a lot of magic with what seems to be very little!

Prerequisites: None.

Primitive roots. (Mark)

Suppose you start with 1 and keep multiplying by a modulo n, where a and n are relatively prime positive integers. As it turns out, you will always get back to 1. But will you have seen all the integers k with $\gcd(k, n) = 1$ by then, as part of the "number wheel" you just made? In this class we'll explore when (that is, for what values of n) you can find an a such that every integer modulo n that's relatively prime to n shows up on that single wheel (such an a is called a *primitive root* mod n). We may not get much beyond the case that n is prime, but even in that case the analysis is interesting. In particular, we'll be able to show that a exists in that case without having any idea of how to find a, other than the flat-footed method of trying $2, 3, \ldots, n-1$ until we find a number that works.

Prerequisites: A little elementary number theory.

Problem Solving: Combinatorics. (Misha)

Here are examples of the kinds of problems we will solve in this class:

- A fair coin is flipped 10 times. What is the probability that no outcome (heads or tails) comes up 3 times in a row?
- A convex polyhedron has 32 faces, each of which is either a pentagon or a triangle. At each of its V vertices, T triangular faces and P pentagonal faces meet. Find V, T, and P.
- Find an eight-letter word in the English language for which the probability is as small as possible that, in a random permutation of its letters, all the vowels are adjacent.

More generally, this class covers the application of combinatorics and graph theory to olympiad problem solving. We will spend most of our time solving olympiad problems by collective brainstorming, with some guidance from me.

Prerequisites: None.

Problem Solving: Inequalities. (Pesto)

High-school olympiads usually try to choose problems relying on as little prior knowledge as possible. In inequalities problems, they usually fail completely; training is necessary to solve most and sufficient to solve many of them. We'll go over the common olympiad-style inequalities, and solve problems like the following:

(1) Prove that if a, b, and c are positive and $ab + bc + cd + da = 1$, then $\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}$.

(2) [USAMO 2004] Prove that if a, b, and c are positive, then $(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$

This is a problem-solving class: I'll present a few techniques, but most of the time will be spent having you present solutions to olympiad-style problems you'll've solved as homework the previous day.

Prerequisites: None.

Problem Solving: Linear Algebra. (Misha)

Most high school math contests (the IMO included) do not use any topic considered to be too advanced for high school, such as linear algebra. This is a shame, because there have been many beautiful problems about linear algebra in undergraduate contests such as the Putnam Math Competition.

In this class, we will look at linear algebra from a new perspective and use it to solve olympiad problems.

Prerequisites: Linear algebra.

Problem Solving: Tetrahedra. (Misha)

In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

Prerequisites: None.

P vs NP. (Pesto + Jalex)

Are there problems whose answers we can check easily (in NP) but not find easily (in P)?

It certainly seems so. We can check whether a filled-in Sudoku grid is correct easily, but none of the millions of people who do Sudokus because they're nontrivial knows how to fill one in as easily. We can check whether a claimed proof of a theorem is correct much more easily than we can find proofs (otherwise, mathematicians'd be out of their jobs!).

But those could be false, because no one can prove that $P \neq NP$. In fact, we'll prove that none of the methods that people have tried to use to prove that $P \neq NP$ can possibly work, and talk about other results related to this most famous open problem in computer science.

Prerequisites: Understand the statement (not necessarily the proof) "NPSPACE $\subseteq$ PSPACE". Have seen a "proof by diagonalization" (e.g. of the time hierarchy theorem). (Intro complexity theory suffices.).

Quadratic Reciprocity. (Mark)

Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

Q1: "Is q a square modulo p?"

Q2: "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square mod 9973, you'll be able to answer a lot more quickly, whether or not you use technology!

Prerequisites: Number theory through Fermat's little theorem.

Qualifying Quiz Problem 6. (Jalex)

Problem 6 on the Qualifying Quiz this year asked you to find an optimal strategy for an n-player game when n was a power of 2. It turns out that the induction part of the intended solution was nonessential — many campers found that more-or-less the same strategy works for any even n. More surprisingly, somebody submitted a nonconstructive proof that there is an optimal strategy for all n. The proof is clean — it's just a few key lemmas about directed multigraphs.

Prerequisites: Know what a directed graph is.

Quantum factoring. (Pesto)

With the best currently known algorithms, all the world's supercomputers working together for a year couldn't factor a 4000-digit integer into prime factors. Even a small[7] computer that takes advantage of quantum mechanical phenomena that aren't well approximated by what classical computers do could factor such numbers using "Shor's algorithm". We'll race through a description of how we model quantum computation (without assuming any knowledge of physics), then see Shor's algorithm.

Prerequisites: Linear algebra (in some sense, almost everything we do in this class will be multiplication of matrices), Number Theory (Fermat's Little Theorem).

Quaternions and rotations. (Chris + Alfonso)

See the other blurb

Prerequisites: None.

Quaternions and rotations. (Alfonso + Chris)

The composition of any number of rotations in $\mathbb{R}^3$ is a rotation. This is not trivial to prove! (For comparison, this is false in four dimensions, and the composition of reflections in $\mathbb{R}^3$ is not a reflection.) In this class you will prove the result and you will use and learn a lot about quaternions to do so.

The quaternions are "a set of numbers" similar to the complex numbers, but where we have i, j, and k, instead of having only i. Some know them simply as a mathematical toy, but they have many applications. For example, they are the best way to represent rotations in $\mathbb{R}^3$, and the best way to do calculations with them.

You will be doing all the work yourselves during class time. Ask any student who took our Banach-Tarski class in Week 2 if you want to know more about the format.

Prerequisites: Linear algebra: you need to know how matrix multiplication works and how to write the matrix of a rotation.

Rapid Fire Problem Solving. (Misha)

"Compute the ordered pair of integers (a, b) that minimizes $|a^4 + a^b - 2015|$."

This problem comes from the ARML 2015 Tiebreaker round. You can probably solve it. But can you solve it in less than a minute?

This class is about solving math problems very quickly. We will practice solving such problems and discuss tricks to speed up problem solving.

Prerequisites: None.

Reflection Groups. (Don)

A reflection group is a group generated, unsurprisingly, by reflections of $\mathbb{R}^n$. Many interesting families of groups are reflection groups, including dihedral groups, S_n, and the symmetry groups of polyhedra.

In this class, we'll study the basic properties of reflection groups, and do our best to classify the finite reflection groups. In particular, we'll see how to classify the crystallographic coxeter groups - groups which fix a lattice in the same $\mathbb{R}^n$ the group is acting on. These groups have deep and important applications, ranging from chemistry, to virology, to my own research - and yet their classification uses just a few families of (very simple) finite graphs.

Prerequisites: Linear algebra, group theory.

Representation Theory of Finite Groups. (Mark)

It turns out that you can learn a lot about a group by studying homomorphisms from it to groups of linear transformations (if you prefer, groups of matrices). Such a homomorphism is called a *representation* of the group; representations of groups have been used widely in areas ranging from quantum chemistry and particle physics to the famous classification of all finite simple groups. For example, Burnside, who was one of the pioneers in this area along with Frobenius and Schur, used representation theory to show that the order of any finite simple group that is not cyclic must have at least three distinct prime factors. (The smallest example of such a group, the alternating group A_5 of order $60 = 2^2 \cdot 3 \cdot 5$, is important in understanding the unsolvability of quintic equations by radicals.) We may not get that far, but you'll definitely see some unexpected, beautiful, and important facts about finite groups in this class, along with proofs of most or all of them. The first week of the class should get you to the point of understanding character tables, which are relatively small, square tables of numbers that encode *all* the information about the representations of particular finite groups; these results are quite elegant and very worthwhile, even if you go no further. In the second week, the chili level will ramp up a bit (from about π to 4) as we start introducing techniques from elsewhere in algebra (such as algebraic integers, tensor products, and possibly modules) to get more sophisticated information.

Prerequisites: Linear algebra, group theory, and general comfort with abstraction.

Ring Theory. (Sachi)

When pressed for a one sentence definition, many mathematicians will describe a ring as "a set of things that you can add and multiply together". However, this hardly scratches the surface of what rings do and why they're everywhere. In this class, you'll learn not just how rings are places where you can add and multiply, but how we use rings to describe analogues of familiar objects like the integers, functions, matrices, and even geometric shapes like the parabola and hyperbola. We will study prime numbers and their cousins, understand how to build square roots and imaginary numbers from polynomials, and see why points on a line have an alter-ego as maximal ideals of a ring. Come learn why rings are such a fundamental part of modern mathematics!

Prerequisites: None.

Shortest Distance. (Jeff)

How do we find the shortest distance between two points? Here's a strategy that Euler used (with a bit of help from calculus):

> Let's pick two points $p, q \in \mathbb{R}^2$. If r is any third point, then the distance from p to r plus the distance from r to q is longer than the distance from p to q. Therefore, the line is the shortest distance. If one would like to show the same thing about curves, approximate the curve by line segments and use the tools of calculus to show that the line minimizes distances.

See any problems?

We defined the distance between two points by saying that it *is the length of the line between them!* It seems that we've somehow used real world intuition[8] that the line minimizes distances to come up with our definition of distance... probably not the best idea for our proof!
Here is a different approach. Let $\mathcal{P}_{ab}$ be the space of all paths between a and b. There is a function from $\mathcal{P}_{ab}$ to $\mathbb{R}$, called the length. Now, use calculus to find the minimum of this function. That was easy!
Surprisingly, this method (basically) works, and will allow us to find geodesics without referring to straight lines. We will use these tools to find length minimizing curves on hyperbolic and spherical spaces (which have no straight lines), look at brachistochrone curves, and minimize other quantities over spaces of paths.

Prerequisites: Can you define what a vector field is? Can you define what $\frac{\partial f}{\partial x}$ is? .

Special Relativity. (*Nic Ford*)

Around the beginning of the twentieth century, physics was undergoing some drastic changes. The brand-new theory of electromagnetism made very accurate predictions, but it forced physicists to come to grips with a strange new truth: there is no such thing as absolute space, and there is no such thing as absolute time. Depending on their relative velocities, different observers can disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

[8] In fact, we have not used real world intuition, as straight lines are not geodesics in the theory of general relativity.

In this class, we'll talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics has to be rebuilt to accommodate these observations. We will see how, as Minkowski said, "space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." Along the way, we will also have to revise the classical notions of momentum and energy, allowing us to derive the famous relation $E = mc^2$. If there's time at the end, we might discuss some other topics in modern physics.

Prerequisites: High school physics.

Statistical Modelling. (Sam)

Note: this is a course where you'll get to do actual data analysis; during the course you'll need to get your hands dirty working with real data!

We'll start by introducing a framework for statistical modelling, and we'll illustrate that framework with one of the most powerful tools in the statistician's toolkit: the linear model. We'll cover fundamental statistical principles, including: fitting, analyzing, and interpreting linear models, hypothesis testing, and extensions of the linear model. By the end of the week, my hope is that you'll be comfortable with these statistical methods from a mathematical and intuitive perspective, and that you'll also be comfortable with your own ability to analyze data you care about.

Prerequisites: You should feel good about basic matrix/vector notation and operations (inversion and transposition) and a few ideas from probability (if you're OK with the sentences "X and Y are independent" and "X is normally distributed with mean zero and variance σ^2," you should be set. If you aren't quite sure you have the prereqs, but you're not too put off by those sentences, talk to me and we should be able to work something out!). .

Stupid Games on Uncountable Sets. (Susan)

Let's play a game. You name a countable ordinal number. And then I name a bigger countable ordinal number. We keep doing this forever. When we're done, we'll see who wins. In this class we'll be discussing strategies for winning an infinite game played on ω_1. In particular, we'll talk about how to set up the game so that at any point, *neither* player has a winning strategy.

Prerequisites: None.

Summing Series. (Kevin)

Combinatorial sums like $\sum_{i=0}^{n} \binom{n}{i}^2$ often crop up. Some, like this one, are wonderful – there are elegant combinatorial interpretations that yield a closed-form solution, and failing that key combinatorial insight, several neat strategies exist to work your way towards an answer.

In this course, we'll develop some of these amazingly effective strategies that allow us to sum all sorts of beautiful series. And by summing some difficult series, we can (and will, if you do all the homework!) prove some famous results in mathematics, like the fact that the circumference of the unit circle is 2π! This is *not* the most sensible approach, but we'll also use our tools for more appropriate (and more difficult!) tasks, like proving the celebrated hook length formula for tableaux.

And time permitting, we might even tackle some monsters like $\sum_{i=0}^{k-1} \frac{(i+2)(i+3)(i+9)(i+10)}{(i+13)(i+14)} 2^i$. Good luck finding a combinatorial interpretation of this disaster, but we'll learn how to handle them systematically!

Prerequisites: None.

Sylow Theorems. (Nancy)

Sylow Theorems $\subset$ Group theory $\cap$ Number theory

The Sylow Theorems form a fundamental part of finite group theory and have very important applications in the classification of finite simple groups.

Basically, here's what the Sylow theorems do:

Suppose you have a finite group G of size m. Depending on the prime decomposition of m, you can determine how many subgroups G will have of various sizes.

Prerequisites: Group theory.

The Banach–Tarski Paradox. (Alfonso + Chris)

You have heard of the Banach–Tarski paradox: take a ball, break it into a few pieces, shuffle those pieces, glue them back in a different way, and now we have two balls of the same size as the original one! Nifty trick, but how does it work? In this course you get to develop the whole mathematical theory behind this construction and prove that it actually works!

This is a superclass that meets for two class hours a day (and possibly the first hour of TAU if we need it). You will be doing most of the work: we will provide worksheets with the right definitions and questions; you will spent a big chunk of the time working, alone or in groups, sometimes with our help, on all the steps of the construction. Some of the class time will be spent on presentation and discussion of your proofs.

This course is time-consuming, but all the work (homework included) is contained in the three daily hours.

Prerequisites: Basic group theory, linear algebra (matrix multiplication, and understand how a matrix represents a linear transformation).

The Cake is a Lie. (Sachi)

It's my birthday, and I want to share cake with my friends.

Susan and Kevin are attending my party, but they're still having a feud, so I need to make sure I give them the same amount of cake. Unfortunately, Susan really like the pink frosted flowers, and Kevin likes the gel icing writing. So, cutting the cake in thirds will not look fair to them, depending on the distribution of the icing.

How can I cut the cake so that Kevin, Susan, and I each believe that we have at least 1/3 of the cake and would not like to trade with anyone else?

Prerequisites: None.

The Factorial Function. (Sachi)

Factorials show up everywhere in combinatorics, whether you're counting the number of combinations of LN_2 ice cream flavors or the number of ways to form TPS teams. What you might not realize is that the factorial function shows up many places in number theory, too. For example, the number of polynomial functions from the integers $\mathbb{Z}$ to the integers modulo n, $\mathbb{Z}/n\mathbb{Z}$ is

$$\prod_{k=0}^{n-1} \frac{n}{\gcd(n, k!)}.$$

Fields medalist Manjul Bhargava published a paper in 1997 which generalized the factorial function to subsets of the integers. With this super-powered factorial, we can expand our original theorems to cover arbitrary subsets of the integers.

Prerequisites: Basic number theory (modular arithmetic).

The Fast More Four Do Over. (Yüv)

If I give you a real a and one more real b, how fast can you find out $a \cdot b$? We need to do this all the time, so we want to be able to do it fast. And if we try to do it the way we were told when we were kids, it will be too slow. Can we make it more fast?

The More Four Do Over is a math idea that does take some data and does turn it into a set of data in such a way that the new data has a lot of info on how the old data does vary. It's a tool that is very full of use, and we love it. Also, it lets us find out $a \cdot b$, and that is very good.

But wait! If we try to do the More Four Do Over, it is also too slow. We want to do it fast! Very very fast! What do we do? Well, luck is on our side. It does turn out that we can find a sly way to do the More Four Do Over that is way more fast than what we did in the past. And if we know how to do the More Four Do Over fast, then we can find out $a \cdot b$ fast too, and it all just gets more fast.

In this talk, we will find out how to do the Fast More Four Do Over and how to use that to find out $a \cdot b$. And we will do it all in the Game of Four.[9]

Prerequisites: None.

The Ham Sandwich Theorem and Friends. (Yuval)

The Ham Sandwich Theorem says that if you have a ham sandwich (consisting of two pieces of bread and a piece of ham), then you can cut it into two pieces with one knife cut in such a way that each piece gets exactly half the ham and half of each slice of bread.

This is a great theorem, but there's a problem: a ham sandwich consisting of just bread and ham is a really boring ham sandwich. What if you want your ham sandwich to have cheese, vegetables, mayonnaise, mustard, or other things? Well, you can't divide each of these equally if you only allow yourself a straight knife. But if you have a polynomial-shaped knife, you can!

[9] If I give you two numbers a and b, how fast can you calculate their product $a \cdot b$? Since this is such an important operation, we want to be able to do it quickly, and the algorithm that we learned in school is actually too slow. Can we make it faster? The (Discrete) Fourier Transform is an operation that converts a sequence of numbers into another sequence that remembers the structure of the original sequence. It's an extremely useful tool; in particular, it lets us calculate $a \cdot b$.
However, there's a problem: if we try to calculate the Discrete Fourier Transform naively, then it's also too slow. But luckily, there is a fast algorithm for computing it, which also means that we can multiply quickly.
In this class, we'll learn about the Discrete Fourier Transform and about the Fast Fourier Transform algorithm, and we'll see how we can use this to multiply efficiently. And we will do it all in the Game of Four.

In this class, we'll learn about the Ham Sandwich Theorem and some of its friends, and we'll even learn to cut some things that aren't sandwiches. We'll eventually get so good at cutting things that we'll be able to prove famous results, like the Szemerédi-Trotter theorem.

Prerequisites: None.

The Hidden Dance of Partial Differential Equations. (*Adam Larios + Jared Whitehead*)

At the heart of almost all science lie partial differential equations. They govern diverse areas, such as the flow of blood in the heart, the turbulent patterns of the weather, the spread of diseases, the shape of sounds waves and light waves, the flickering motion of flames, and even the formation of traffic jams. Moreover, the mathematics involved is strikingly beautiful, full of chaos, order, and mystery.

We will begin by learning one of the key ingredients to understanding partial differential equations: Fourier series. Fourier series are wonderful mathematical tools in their own right. To get a feel for them, we'll see how we can pull apart images using Fourier transforms. We will then discuss some strange and remarkable partial differential equations, and use the machinery of Fourier series to begin to untangle them and see the hidden dance that makes them work. With these ideas in hand, we will be able to discuss one of the most difficult problems in all of mathematics: the $1,000,000 Millennium Prize Problem for the Navier-Stokes equations, posed by the Clay Mathematics Institute in the year 2000.

Prerequisites: Differential calculus and integral calculus. Infinite series may be useful, but are not strictly necessary.

The History of Calculus. (Sam)

Have you heard that Newton and Leibniz invented calculus? Have you heard anyone categorically reject that statement?

In this class we'll do a quick survey of the history of calculus. We'll talk about some of conceptual ideas that had occurred to the Ancient Greeks, we'll talk about what was known about calculus before Newton and Leibniz, we'll talk about what Newton and Leibniz actually did, and then we'll talk about why that wasn't enough. We'll also hear what Bishop Berkeley had to say, and how he used reason to try to shut down calculus*.

NOTE: This class can also be replaced with "The History of X," where X is another part of mathematics if there is sufficient demand.

BONUS: You'll learn (some of) why Cauchy is my favorite mathematician!

* We'll also qualify this sentence!

Prerequisites: Feeling good about calculus is necessary.

The Hoffmann Singleton Theorem. (Pesto)

"I'm thinking of a set of four integers. Four of them are 2, 3, and 7. What's the other one?"

"How should I know? You could have picked any other integer."

"No, these are the answers to a perfectly natural mathematical question: as natural as "If you can get from any other vertex of a graph to any other vertex in at most two steps, but there are no triangles or squares, and every vertex has the same degree, what's that degree?" "

"Seems like a natural enough question. Maybe 5, like the first four primes?"

"Nope, 57."

"No way! How could 2, 3, 7, and 57 be the answer to *anything* that natural?"

Prerequisites: Linear algebra (enough to understand the statement "if A is a matrix and $A^2 = I$, then every eigenvalue of A is 1 or -1" and either know its proof or be willing to accept it without proof).

The Löwenheim-Skolem Theorem. (Susan + Steve)

Oh no! A ninja has snuck into the Museum of Real Numbers and stolen all but countably many of them! You, the curator, have a huge exhibition tomorrow. What are you going to do? Why, it's simple! You'll use the Löwenheim-Skolem theorem to build a countable model of set theory, complete with the real numbers. From inside the museum, no one will be able to tell that it's countable. To keep real number ninjas from interfering in your life, come to this class!

Prerequisites: None.

The mathematics of polygamy (and bankruptcy). (*Judah the Prince*)

Here is a passage from the *Mishnah*, the 2nd century codex of Jewish law:

A man has three wives; he dies owing one of them 100 [silver pieces], one of them 200, and one of them 300.

If his total estate is 100, they split it equally.

If the estate is 200, then the first wife gets 50 and the other two get 75 each.

If the estate is 300, then the first wife gets 50, the second one 100, and the third one 150.

Similarly, any joint investment with three unequal initial contributions should be divided up in the same way.

For 1800 years, this passage had baffled scholars: what could possibly be the logic behind the Mishnah's totally different ways of distributing the estate in the three cases? Then, in 1985, a pair of mathematical economists produced a beautifully simple explanation based on ideas from game theory. They showed that for any number of creditors and for any estate size, there is a unique distribution that satisfies certain criteria, and it turns out to be exactly the distribution proposed in the Mishnah. The proof is very cool, based on an analogy with a simple physical system! See if you can figure out this ancient puzzle for yourself, or come to class and find out.

Prerequisites: None.

The Projective Plane. (Sachi)

We can construct an object called the "projective plane" by taking a sphere and allowing teleportation between two antipodal points. Alternatively, we could wrap the normal plane $\mathbb{R}^2$ in a line "at infinity" and require that all parallel lines meet at one point. Or, we could take a Möbius strip and glue a disc to it. If we happened to already have a projective plane lying around, we could switch all its points with all its lines and get a projective plane back.

We'll talk about what kinds of geometry we can do with this weird shape, and why all these notions are the same.

Prerequisites: None.

The Pruefer Correspondence. (Mark)

Suppose you have n points around a circle, with every pair of points connected by a line segment. (If you like, you have the complete graph K_n). Now you're going to erase some of those line segments so you end up with a tree, that is, so that you can still get from each point to each other point along the remaining line segments (without changing direction except at the points on the circle), but in only one way. (This tree will be a spanning tree for K_n.) How many different trees can you end up with? The answer is a surprisingly simple expression in n, and we'll go through a combinatorial proof that is especially cool.

Prerequisites: None!

The Robinson-Schensted Correspondence. (Asilata)

The Schensted algorithm is a simple and beautiful recipe that turns permutations into pictures, also known as standard Young tableaux.

The algorithm is tricky, but simple to describe. Amazingly, it can be turned into a reversible procedure that sends any permutation to a *pair* of Young tableaux and vice-versa. This is known as the Robinson–Schensted correspondence, which has applications into some unexpectedly deep mathematics.

Expect a fun hour with delightful combinatorial surprises and lots of examples. It'll be a treat.

Prerequisites: None. Group theory helpful but not required.

The tale of summation is in order! (*Paweł Pi.*, camper teaching project)

Don't you agree?

Let p be Zach's favorite prime (or yours). Take an integer $x \in \{0, 1, \ldots, p-1\}$ and start computing its powers $x, x^2, x^3, \ldots \bmod p$. The minimal exponent d for which $x^d \equiv 1 \bmod p$ is called the *order* of $x \bmod p$ (If there is no d with $x^d \equiv 1 \bmod p$, then you took $x = 0$; unlucky for you.)

Now suppose you *fix* d and add up (mod d) all the $x \in \{0, 1, \ldots, p-1\}$ which have that order d. What will you get? And what does this computation mod p have to do with complex roots of unity? To find out, come to this class!

Prerequisites: Some experience with polynomials; Fermat's Little Theorem (or Lagrange's Theorem from group theory).

Tiling Problems. (Sachi)

If you take a chessboard, and remove two opposite corners, then there is no tiling of the chessboard by dominos (each of which covers two adjacent chess squares.) Every domino covers a black and a white square, but two opposite corners have the same color.

Suppose instead you take a tiling of the plane by equilateral triangles. Then, if someone outlines a region on this triangular lattice, is it possible to tell whether one can tile it with tiles which cover two adjacent triangles at a time (a shape which is called a lozenge)? It would take a long time to check all the possible tilings, and such a brute force attack would be hard to keep track of and replicate. Instead, we will borrow some tools from group theory: we will see that groups have graphs associated to them, and how words in these graphs actually constitute tiles which we can use to simplify elements of our group.

Prerequisites: Group theory.

Time-Frequency Analysis. (Jeff)

When you put your ear up to a shell, you hear the tumbling of the ocean around you. The sound that you hear is not caused by a miniature ocean confined to a shell, rather, the shell's shape and composition distort the white noise into the sound of churning water.

One of the goals of signals analysis is to understand how systems like the shell distort input signals. For instance, if we know how the shell sounds when we strike it gently with a hammer, could we figure out what an actual ocean would sound like if it was in the shell? How would a sine wave sound like if it were playing in the shell? What about a one thousandth scale version of the Beatles singing *Love me Do?*

In this class, we will draw from our human and physical intuition to develop mathematical tools including Fourier transforms, distributions, and time-frequency representations. This will allow us to take derivatives of things like $y = |x|$ at $x = 0$, and solve nonhomogeneous ordinary differential equations. We will then turn this machinery the other way around to study how we perceive reality by building audio filters, exploring the Uncertainty Principle, and studying lossy AV compression.

Prerequisites: Calculus, Linear algebra.

Tower of Hanoi. (Julian)

> In the great temple at Benares, beneath the dome which marks the centre of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four discs of pure gold, the largest disc resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah. Day and night unceasingly the priests transfer the discs from one diamond needle to another according to the fixed and immutable laws of Bramah, which require that the priest must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been thus transferred from the needle on which at the creation God placed them to

> one of the other needles, tower, temple and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish.
>
> — *Henri de Parville (1884), translated by W. W. Rouse Ball*

So goes the legend. But what of the mathematics of the problem?

- What is the optimal solution (that is, the one which requires fewest moves) if we have n discs?
- What is the optimal way to get between two given (but arbitrary) legal states? (This is more subtle than it seems: many incorrect solutions to this question have been published! 'Legal' means that no disc lies on top of a smaller disc.)
- What is the optimal solution for moving n discs from one disc to another if there are 4 pegs instead of 3? Or 5 pegs? Or p pegs?
- What other interesting variants and corresponding problems can we suggest?

As we explore these questions and others, we will run into connections with fractals (!) and other pretty mathematics. We will see open problems and conjectures along the way; you might wish to explore one of these for a project.

Before class begins, you might wish to think about the following problem:

For a Tower of Hanoi with n discs (and 3 pegs) draw a graph showing all legal positions and moves between them. Can you find a nice way to present this graph?

Prerequisites: None.

Trail Mix. (Mark)

Is your mathematical hike getting a little too hardcore? Would you like to relax a bit with a class that offers an unrelated topic every day, so you can pick and choose which days to attend, and that does not expect you to do homework? If so, how about some Trail Mix? Individual descriptions of the four topics follow. There is basically no homework, although there will certainly be some things you could look into if you like.

Trail Mix Day 1: Perfect Numbers. Do you love 6 and 28? The ancient Greeks did, because each of these numbers is the sum of its own divisors, not counting itself. Such integers are called perfect, and while a lot is known about them, other things are not: Are there infinitely many? Are there any odd ones? Come hear about what is known, and what perfect numbers have to do with the ongoing search for primes of a particular form, the so-called Mersenne primes — a search that has largely been carried out, with considerable success, by a far-flung network of individual "volunteer" computers.

Prerequisites: None.

Trail Mix Day 2: Integration by Parts and the Wallis Product. Integration by parts is one of only two truly general techniques known for finding antiderivatives (the other is integration by substitution). In this class you'll see (or review) this method, and two of its applications: How to extend the factorial function, so that there is actually something like (1/2)! (although the commonly used notation and terminology is a bit different), and how to derive the famous product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots,$$

which was first stated by John Wallis in 1655.

Prerequisites: Basic single-variable calculus.

Trail Mix Day 3: Intersection Madness. When you intersect two ellipses, you can get four points, right? So why can't you get four points when you intersect two circles? Well, you can, and two of the four points are always *in the same place*! If this seems interesting and/or paradoxical, wait until we start intersecting two cubic curves (given by degree 3 polynomial equations). The configuration of their intersection points has a "magic" property (known as the Cayley-Bacharach theorem) that leads to proofs of various other cool results, such as Pascal's hexagon theorem and the existence of a group law on a cubic curve. I can't promise that we'll have time for all these things, but we'll do some of them.

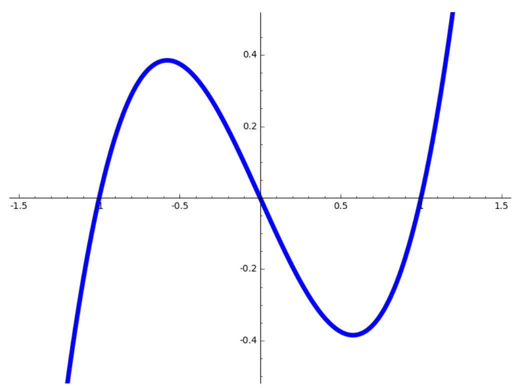
Prerequisites: None, although a little bit of linear algebra might show up.

Trail Mix Day 4: The Jacobian Determinant and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. How do you change variables in a multiple integral? In the "crash course" in week 1 we saw that when you change to polar coordinates, a somewhat mysterious factor r is needed. This is a special case of an important general fact involving a determinant of partial derivatives. We'll see how and roughly why this works; then we'll use it to evaluate the famous sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (You may well know the answer, but do you know a proof? If so, do you know a proof that doesn't require Fourier series or complex analysis?)

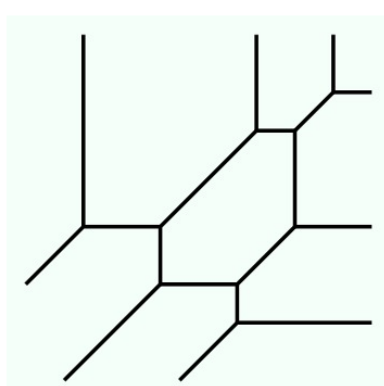
Prerequisites: Multivariable calculus (the crash course is plenty); some experience with determinants.

Tropical Curves. (Ruthi)

In the tropical world, everything is different. Instead of cubics looking like this:



They look like something like this:



You may think that addition and multiplication work like this:

$$1+1=2 \quad 1\times 1=1$$

But in tropical world, they work like this:

$$1+1=1 \quad 1\times 1=2$$

We can make rigorous what I mean by this, and these worlds are not unrelated: tropical world makes it easier to solve problems in our world (and vice versa!). Tropical geometry is all about curves and surfaces are related to polytopes – how algebra and geometry relate to combinatorics. In this class we'll try to understand where these things come from and why we might study this weird world.

Prerequisites: None.

Turing and his Work. (Sam)

Were you sad to hear that Turing was not visiting Mathcamp this year? Disappointed that you couldn't hear him give a colloquium on some of his work, or that you wouldn't be able to have a conversation with him? Then come to this class, where we'll do the next best thing!

Our goal in this course will be to, as much as possible, try to understand who Turing actually was, and to learn about some of his work in AI. Day 1 will put his work into the context of his life; even though Benedict Cumberbatch is awesome, his portrayal of Alan Turing was more "Hollywood" than "accurate." After that, you guys will get a chance to drive the course: you'll be asked to read one or two of Turing's more accessible papers, and in class we'll chat about both the papers and the related mathematics (largely within the subject of computability). At a bare minimum, you'll get to read the paper where Turing gave his most complete treatment of The Imitation Game! We may also discuss Turing Machines and his algorithm for a computer to play chess. Note: "Homework Required" means that you will be asked to read one (maybe 1.5) accessible paper by Turing over the course of the week.

Free Bonuses: you'll get to read actual papers by Turing, learn from him vicariously, and feel awesome about having read an actual paper. (Unfortunately, I'm not allowed to throw in a free set of steak knives.)

Prerequisites: None!

Ultrafilters. (Steve)

You might have played the game "20 questions" before, where I think of some object and you get 20 yes/no questions to figure out what it is. Well, there's a related game called "infinity questions" — I think of a natural number x, and you get to ask me infinitely many yes-no questions about x: e.g., "Is x prime?" or "Is x the sum of two cubes?" Note that each question can be phrased as "Is $x \in A$?" for some set A.

Now, certainly you can win this game — just have your nth question be "Is $x = n$?" But what if I cheat, and don't actually have a number in mind? Well, as long as my pattern of answers is consistent, it will be as if I'm describing something that's like a number, but not quite: it's not a natural number, but it is either even or odd, either prime or composite, etc.

These "generalized numbers" are *ultrafilters* on the set of natural numbers. In this class, we'll talk about what ultrafilters are and what you can do with them. We'll see that there is a natural "space" of ultrafilters, and that this space has properties which make it useful for combinatorics(!); and we'll show how ultrafilters can let us take the "average" of an infinite collection of mathematical structures (groups, rings, etc.).

Prerequisites: None.

Unlikely Maths. (Misha)

A popular way to construct a combinatorial object we want is to choose it at random. This works great if the properties we desire hold for almost any object we could choose. But sometimes we are greedy and want so much from our construction that a randomly chosen object has, at best, an exponentially small chance of making us happy.

This class is about the Lovász Local Lemma: one of the ways to prove that this exponentially small chance is still positive. We will use the LLL to show the existence of a few unlikely objects, with applications to graph theory and computer science. We will also see a few approaches to taking the final step: actually finding such an unlikely object.

Prerequisites: Graph theory.

Unsolved Problems in Cosmology. (*Charles Steinhardt*)

One of the wonderful things about astronomy is that modern astronomy is a very young field, with many important unsolved problems. In the past couple of decades, we have come to realize the vast extent of our fundamental ignorance about the nature of the universe: as it turns out, all of the physics that we have produced throughout history seems to be a good description of about 4% of the stuff that makes up the universe, with the rest composed of mysterious "dark matter" and "dark energy".

We'll go through some of the highlights of modern cosmology, focusing on the things we still don't understand and building up to trying to understand what happened in the first few instants after the Big Bang, and why it poses such a fundamental challenge to our view of the universe.

Prerequisites: Some high school physics/chemistry (feel free to ask if you're not sure). Knowing differential equations will help for part of the course, but is not necessary.

Voting Theory. (Alfonso)

When a large group of people have to make a decision together, bad things can happen. For example, suppose that a group of 10 campers is trying to decide which game they want to play tonight. Suppose further that 3 of them want to play Dominion, and the remaining 7 would prefer to play any game they can possibly think of other than Dominion. If the remaining 7 are divided between 5 or 6 different games, a strict plurality election system will force them to play Dominion, even though a majority of the 10 campers would be thoroughly unsatisfied. It seems, then, that the plurality election system is unfair. What could we do to make it fair? Which election system is the most fair? What does "fair" mean, anyway?

In this class we will try to formalize the question of "what is a fair voting system" mathematically, and we will analyze actual voting systems used in the world accordingly. We will use real data from recent elections in various countries!
Warning: Your faith in democracy may vanish.

Prerequisites: None.

Wagner's Theorem. (Pesto)

Draw five points on a piece of paper. Try to connect every pair of them without any edges crossing. You can't do it.

Three utilities each need to connect to each of three houses. Then some two of the utility lines must cross.

Stated graph-theoretically, these results say that K_5 and $K_{3,3}$ aren't planar. (We'll prove so, quickly.)

What's really surprising is that these are in some sense the *only* non-planar graphs: you can find one of them in every nonplanar graph if your vision's fuzzy enough that you can make any connected set of vertices blur into one. This is Wagner's Theorem, which we'll state and prove.

Prerequisites: Graph theory: if G is a connected planar graph drawn with f faces, and G has v vertices and e edges, then $v - e + f = 2$. .

What's Up With e? (Susan)

The continued fraction expansion of e is

$$1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots$$

What's up with that? Come find out!

Prerequisites: Calculus (derivatives and integrals).

Colloquia

An Intro to "Data Science". (*Adam Marcus*)

Data science is all over the media these days. But what is it? And what do "data scientists" do? I will give an introduction to data science (a.k.a. machine learning) and discuss some of the more interesting mathematical problems. Topics will include what a career in data science might entail and how to prepare for such a career. No background knowledge will be required, and I hope to leave plenty of time for questions.

Are We Special? (*Charles Steinhardt*)

Unlike in mathematics, in science we don't rigorously prove things according to a set of axioms, but learn from experiments and observations about the world around us. In order for this to be a fruitful approach, it is necessary to assume that our experience is somehow typical, and that we can use an experiment in a laboratory to learn about the forces responsible for, say, assembling galaxies. But bizarrely, the leading theories for the fundamental nature of our universe and of its origin seem to require that we live in a very special time and place, without any hope of testing that assertion. We'll explore the idea of what it means to be a "random" observer, what it might mean to be "special", a question that is currently the subject of a significant split in the scientific community, and what it tells us about the future of the scientific method.

Can You Hear the Shape of a Drum? (*Moon Duchin*)

For every geometric object, there is an infinite list of numbers called its spectrum that describes the frequencies at which it can vibrate– let's interpret this as the sounds it can make when you strike it. Suppose there are some differently shaped drumheads that are hidden from view and you get your friends to smack them melodiously in an effort to help you tell them apart. Is it possible that two differently shaped drumheads are isospectral, or sonically indistinguishable?

A closely related question asks you to consider a surface and study the length spectrum– the list of all possible lengths of curves that are pulled tight on the surface. Can two differently shaped surfaces support all of the same curve-lengths? What geometric features of a surface can you "read off" of the length spectrum?

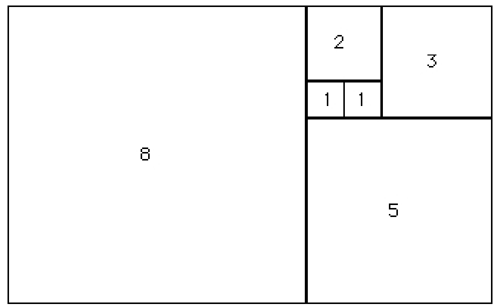
I'll explain the connection between these, describe some fun related problems, and tell you about some of my own research in shape-hearing. (I proved that, for a suitable way of making all the words precise, you can hear the shape of a billiard table!)

Cauchy's functional equation. (*Katie Mann*)

In 1820 or so, Augustin-Louis Cauchy wondered: "Which real-valued functions satisfy the property $f(x + y) = f(x) + f(y)$?" (for all real numbers x and y), and probably thought they were all pretty nice functions.

We'll see what the nice ones look like, then discover some terrible monsters. Expect to meet:

- A function whose graph you can't possibly draw
- A vector space of shocking proportions, and
- A proof that (unlike the famous example below) *most* rectangles can't possibly be subdivided into little squares.



(Amazingly, Cauchy's question and its friends have made an appearance recently in my research… we might not get to this in the colloquium, but if you're curious, ask!)

Computers: Friends or foes of the modern mathematician? (*Adam Larios*)

As computers gain a stronger hold in mathematics, should mathematicians start to worry? Computers obviously depend on mathematics, but more and more areas of mathematics are starting to depend on computers. Does this mixture violate the purity of mathematics? Will the beauty of mathematics be forgotten, reduced to cold hard computation?

Fortunately, the situation may be not so scary. We will see many examples of mathematics and computers living happily together. Moreover, pairing these disciplines can be beneficial to both sides. I will argue that computers do not spell doom for mathematics; indeed, they make mathematics more exciting than ever.

Finding my Place in {Teacher Educators} ∩ {Mathematicians}: Bringing a Mathematical Eye to Teacher Education. (*Nina White*)

When we think about mathematicians' contributions to K-12 math education, many people first think of curriculum development. But this is just the tip of the iceberg; curricula are not enacted without quality teachers and one very central role mathematicians play in improving K-12 education is educating teachers. In this colloquium I'll share why teacher education is not only important, but interesting. We'll focus on two questions that have been asked over the ages: "what mathematics do teachers need to know?" and "who should teach it?" I'll include current and historical examples of how mathematicians contribute to teacher education and describe different professional trajectories, including my own, from the world of mathematics to the world of mathematics education.

Making Cantor Super Proud. (Steve)

How big is infinity? In 1874, Georg Cantor discovered something awesome: some infinities are bigger than others! In particular, there are more real numbers than natural numbers - there is no injection from $\mathbb{N}$ to $\mathbb{R}$.

New question: how bigger is infinity? We know $\mathbb{R}$ is bigger than $\mathbb{N}$, but how much bigger? Cantor thought that the answer was, "barely": he conjectured that there is no set of reals larger than $\mathbb{N}$ but smaller than $\mathbb{R}$. This became known as the "Continuum Hypothesis".

Cantor — and others — spent a long time trying to prove or disprove CH. SPOILER: that's impossible! In 1940, Kurt Goedel showed that we cannot *disprove* CH from the usual set-theoretic axioms; in 1963, Paul Cohen showed that we cannot *prove* CH from the usual set-theoretic axioms.

OOPS.

It turns out, though, that Cantor had the right idea: every ”reasonably describable” set of real numbers is either countable, or has the same size as $\mathbb{R}$. In fact, ”reasonably describable” sets of real numbers are well-behaved in lots of ways. This was the beginning of the field known as *descriptive set theory*, which seeks to answer the following questions:

- What properties do ”reasonably describable” sets have?
- What does ”reasonably describable” mean?
- I like games!
- Something something large cardinals?

We will look at different aspects of descriptive set theory and the continuum hypothesis; time permitting, we'll talk about some current directions in set theory.

My Favorite Prime. (*Zach Abel*)

Meet my favorite prime number! All twelve thousand(ish) digits of it. (See appendix.) *Wait, that's a massive number. How do you know it's actually prime?* My computer said so. *But how does the computer know?* It asked the Fibonacci numbers. *And how did you* find *a prime so large, anyway?* By setting Pascal's triangle on fire. *OK, but why is* this *prime your favorite?* Because I'm its favorite! It told me so. *What?!* Seriously, this prime is mine and no one else's, and it'll tell you the same. We'll discuss the above questions in more detail, and all the prime-number mischief they unlock.

Neural Codes. (*Mo Omar*)

Neurons in the brain represent external stimuli via neural codes: binary strings that encode which neurons are on/off at any given time or location. These codes often arise from stimulus-response maps, associating to each neuron a convex set where it fires. An important problem confronted by the brain is to infer properties of the regions in which a neuron fires, using only information intrinsic to the neural code. How is this construction problem approached? With some convex geometry, some algebra, and a whole lot of fun!

Pad Thai with Electrons. (*Allan Adams*)

The world is so deeply weird, so staggeringly strange, that people studying it one hundred years ago invented a whole new language to describe its lemony ways: Quantum Mechanics. In this colloquium, we'll introduce a couple of the most counterintuitive aspects of reality.

Polyhedral Symmetry in the Plane? (*Frank Farris*)

Can you see the tetrahedron in the image provided? To do so, you have to break some conventions about symmetry. Under traditional definitions, the only thing you can do to that image to leave it completely unchanged is turn it upside down—a 180 degree rotation. In this talk, we expand the definition of symmetry to go beyond the usual rosettes, friezes, and wallpaper patterns to find polyhedral symmetry in the plane. We combine a little group theory, a little complex analysis, and several other mathematical ingredients in the service of mathematics and art.

Putting the Algebra in Algebraic Topology. (*Noah Snyder*)

Algebra is the study of operations like multiplication, addition, and composition. Topology is the study of rubbery bendy spaces. Algebraic topology is usually thought of as a way to use algebraic structures to study topology, but there's another way to think about it which is that topological spaces themselves can be thought of as algebraic structures. For example, paths can be "multiplied" by concatenating them end-to-end. The goal of this talk will be to introduce paths, loops, 2-paths, and 2-loops, and to explain why 1-loops can only be multiplied in one way, but 2-loops can be multiplied in a tremendous number of ways. No background in topology will be assumed, but some familiarity with the definition of a group would be helpful.

Secure Genomic Computation: a Contest. (*Kristin Lauter*)

Over the last 10 years, the cost of sequencing the human genome has come down to around \$1,000 per person. Human genomic data is a gold-mine of information, potentially unlocking the secrets to human health and longevity. As a society, we face ethical and privacy questions related to how to handle human genomic data. Should it be aggregated and made available for medical research? What are the risks to individual's privacy? This talk will describe a mathematical solution for securely handling computation on genomic data, and highlight the results of a recent international contest in this area.

Triples. (*Po-Shen Loh*)

What's the longest sequence of triples $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$ that satisfies the following properties?

- Each number is an integer between 1 and N inclusive.
- For every $j < k$, if we compare the triples (x_j, y_j, z_j) and (x_k, y_k, z_k), there are at least two coordinates in which the latter triple strictly exceeds the former triple.

It turns out that this simple-sounding problem is equivalent to a question from Ramsey Theory, inspired by a question from k-majority tournaments, and related to deep question involving induced matchings and Szemeredi's Regularity Lemma.

No knowledge of Ramsey Theory, k-majority tournaments, induced matchings, or Szemeredi's Regularity Lemma is required for this colloquium. The talk will be a tour of Combinatorics which introduces many of these concepts along the way.

Appendix: Zach's Favorite Prime

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