

CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2007

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IMPORTANT NOTES

As usual, continuing classes are only listed if they are specifically inviting new students to enter. Many continuing classes are changing their meeting time and/or room; check the schedule. Rooms may be switched in the first few days of classes; watch out for announcements at assembly and on the schedule board.

NEW CLASSES AT 9AM

Matrix game theory. (*; Ari; week 1 of 1)

In this course, we will examine simple 2-person games, both zero-sum and nonzero-sum. We'll find that the most interesting questions about strategy arise from nonzero-sum games, where the other player is not necessarily an enemy. We'll investigate several examples of such games, including the famous Prisoner's Dilemma. The class will end with a tournament, where every student will submit a strategy for a certain game, and we'll toss them all in a computer and see how well they do against each other!

Matrix game theory is a branch of game theory completely different from Alfonso's "Combinatorial Game Theory" class.

Prerequisites: Not having attended last year's Matrix Game Theory class

Homework: submit a strategy for the tournament at the end of the class.

Related to: Mathematics of elections (Week 3)

Nonstandard Analysis. (**; Dave; week 1 of 1)

“There are good reasons to believe that nonstandard analysis, in some version or another, will be the analysis of the future.” – Kurt Gödel

When Newton and Leibniz first started doing calculus in the seventeenth century, they did not use the concept of a limit. Instead, they created a system in which, if you want to understand the behavior a function $f(x)$ in arbitrarily small neighborhoods of some real number a , then you just evaluate $f(b)$, where b is some real number that is infinitely close to a .

Of course, this idea is silly. Two distinct real numbers a and b cannot be infinitely close to each other—there is some fixed distance between them. In this class, we will attempt to make this notion un-silly. We will create a number system where two numbers can be infinitely close to each other, where there are numbers that are larger than every real number, and where there are positive numbers smaller than every positive real number. Along the way, we will completely circumvent the traditional real analysis so that you will never have to do another epsilon-delta proof again.

Prerequisites: None.

Homework: Optional.

Related to: Real analysis (Weeks 1–3)

Zero Knowledge Proofs. (**; Marisa; Wed–Fri)

A theoretical computer scientist’s use of the word “proof” is a little more expansive than the meaning we usually encounter at mathcamp: in my class, I’ll use “prove” to mean “convince beyond reasonable doubt.” And when I say convince, I mean it: a “proof”, or more specifically an *interactive proof system*, is a two player game between a prover (“you”) and a verifier (“me”). In a zero knowledge proof, your goal is to convince me that something is true without leaking any information about how it works. We’re going to encounter zero knowledge proofs from philosophy, graph theory, and cryptography (among others). The course will be self-contained, but I’ll be defining lots of words from TCS (Theoretical Computer Science) as we go. And of course we’ll talk about what it means to prove something in the first place.

Prerequisites: Ask me if you don’t already know what a permutation is.

Homework: Unlikely.

Related to: Cryptography (Weeks 2–4); P vs. NP (Week 1); Computable functions (Weeks 3–4); Information and coding theory (Weeks 1–2, 4)

Visualizing complex maps. (**; Moon Duchin; Saturday)

The complex numbers are usually geometrized as a flat plane. But then to graph a complex function, like $f(z) = z^2$, you’d seem to need a picture in four dimensions. We’ll find sneaky ways to save the day by visualizing higher dimensions rigorously.

Prerequisites: Complex numbers

Homework: None.

Enumeration Celebration. (**; M@; week 1 of 1)

In this course we will explore powerful techniques for counting. We will discuss Cauchy-Frobenius-Burnside counting under symmetry, the Hook-Length formula, and the Matrix Tree Theorem, and perhaps other methods.

Sample problems:

- (1) How many different ways can you color an icosahedron with one of three colors on each face? Which colors would you choose? (From the Google Labs Aptitude Test.)
- (2) How many walks are there along the 4-dimensional lattice from $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ to $(4, 3, 2, 1)$ if you can only take steps which increment exactly one coordinate, and at each step you must ensure $x_1 \geq x_2 \geq x_3 \geq x_4$?
- (3) Show that the number of spanning trees on a k -dimensional cube is

$$2^{2^n - n - 1} \prod_{k=1}^n k^{\binom{n}{k}}.$$

A direct combinatorial proof of this formula is not known.

Prerequisites: Basic combinatorics.

Homework: Recommended.

Related to: Intro Combinatorics (Week 1); Catalan Numbers (Week 1); Group Actions (Week 3)

Group Actions. (**; Leigh; week 1 of 1)

Groups aren't just pretty sets with nice properties. They move! To be more precise, groups *act* on other sets (even themselves) by permuting the elements of those sets. For example, the group generated by the moves “Up”, “Down”, “Left”, “Right”, “Front”, and “Back” acts on a certain vividly colored cube Mathcampers like to dazzle other summer camps with in the cafeteria. You might also have noticed that matrices are consummate actors, moving around vectors in \mathbb{R}^n (for instance, SO_3 is just the rotations of a sphere).

So in this class we'll define and give examples of group actions. We'll prove some really nice results about them. One of these results leads to Burnside's Lemma (aka Polya's Theorem), which we'll play with because it is a lot of fun (this will overlap with “Enumeration Celebration”). This lemma lets us answer questions such as “How many different ways, up to symmetry, can we paint 10 equal segments, using 3 colors, on a Mobius strip?” (The answer is 3210). And next we might take a closer look at matrix group actions. Or we might find something else to act upon.

Prerequisites: You should be comfortable with the following statements:

- (1) S_n is the group of permutations of the set $\{1, 2, \dots, n\}$.
- (2) \mathbb{Z}_n is the group of equivalence classes with respect to congruence mod n .
- (3) The order of a group G is the product of the order of a subgroup H with the number of cosets of H .

Homework: Optional.

Related to: Enumeration Celebration (Week 1); From Greece to Galois (Weeks 1–4); The Shape of Groups (Week 3); Groups and Symmetry (Weeks 1–2); Rubik's Cube and Other Games (Week 4); Wallpaper Patterns (Week 4)

The Hairy Ball Theorem. (*Emily Landes*; **–***; week 1 of 1)

As a field topology may initially seem a bit squishy. Squishy does not mean less rigorous. It just means a different way of thinking; one in which many people don't actually realize is pure mathematical thought. Topology is all about first building intuition through pictures and then providing a rigorous proof to describe an intuitive occurrence.

Have you ever noticed that every head of hair has a cowlick, i.e. a spot where the hair just never seems to grow in the same way as all the rest? This phenomenon, otherwise known as the hairy ball theorem, is the intuitive occurrence whose proof is the goal of this class.

The beauty of the hairy ball theorem is the way so many fundamental concepts come together to compose a concise proof. Hence, this theorem provides a great context to learn about 2-manifolds, simplicial decompositions, the Euler characteristic, connect sums, vector fields, flows, and the global Lefschetz number. With the motivating goal of proving the hairy ball theorem we spend the class understanding and putting to good use all of these concepts.

Prerequisites: None.

Homework: Recommended (as good problems to think about but nothing heavy)

Related to: Low-dimensional topology (Week 2); Orientable and non-orientable spaces (Week 2)

Representation theory of finite groups. (****; Mark; week 1 of 1)

It turns out that you can learn a great deal about a group by studying homomorphisms from it to groups of linear transformations (if you prefer, groups of matrices). Such a homomorphism is called a representation, and representations of groups have been widely used in areas ranging from the physics of elementary particles and quantum chemistry to the famous classification of all finite simple groups. For example, Burnside, one of the pioneers of the subject—along with Frobenius and Schur—used representation theory to show that the order of every finite simple group that is not cyclic must have at least three distinct prime factors. (The smallest example of such a group, the alternating group A_5 , is important in understanding the unsolvability of quintic equations by radicals.) We won't get quite that far, but you should definitely get to see some unexpected, beautiful, and important facts about finite groups, and proofs of most if not all of them. In particular, for those of you who went to the talk on “Monstrous Moonshine”, this class should help to further demystify character tables.

Prerequisites: Group theory and linear algebra; no fear of abstraction.

Homework: Recommended.

NEW CLASSES AT 11:10AM

Mathematics of Elections. (*; Dave; Tue–Thu)

When a large group of people have to make a decision together, bad things can happen. For example, suppose that a group of 10 campers is trying to decide which game to play tonight. Suppose further that 3 of them want to play Mao, and the remaining 7 would prefer to play any game they can possibly think of other than Mao. If the remaining 7 are divided between 5 or 6 different games, a strict plurality election system will force them to play Mao, even though a majority of the 10 campers would prefer any other candidate to the winner.

It seems, then, that the plurality election system is unfair. What could we do to make it fair? Which election systems are the most fair? What does “fair” mean, anyway? Come to this class and find out.

Prerequisites: None.

Homework: None.

Related to: Matrix game theory (Week 3)

Mathematics of Juggling. (*; Dave; Fri–Sat)

In addition to being fun and beautiful, juggling is full of abstract patterns. In this class, we will explore these patterns through a mathematical perspective, helping us to develop a language for talking about juggling. We will use this language to discover even more patterns—patterns that were unknown prior to this mathematical perspective.

Also, if you come to this class, you will see a juggling pattern that no person has performed before in the history of humankind.

Prerequisites: None.

Homework: None.

p -adic numbers and p -adic calculus. (***; *Fernando Gouvea*; week 1 of 1)

There are more numbers in mathematics, Horatio, than are dreamt of in your calculus class. You probably know how to do calculus with the real numbers. We'll introduce a whole new family of numbers, the p -adics; actually, it's one family of p -adics for each prime number p . We'll explore this strange new world, and its new mathematics, boldly going where lots of number theorists have gone before. The p -adic version of calculus is like normal calculus viewed in a distorting mirror: similar, and yet quite strange. And, surprisingly, useful too.

Prerequisites: Calculus, modular arithmetic.

Homework: Recommended.

Related to: Intro number theory (Weeks 1–2); other number theory; Real analysis (Weeks 1–3); Metric spaces (Week 3)

Computable Functions. (****; Anti; week 1 of 2)

What is a computation, and what can we compute? Intuitively, a computation is anything that can be done by a computer, but to prove theorems about computability we need a mathematical model. Instead of the more well-known theory of Turing machines, in this class our model of computation will be the *lambda calculus*, which models the behavior of computer *programs* rather than that of the computer itself. This has the advantage that we can easily translate between mathematical “computations” and real-world computer programs, especially if we use a “functional” programming language such as Lisp or Haskell.

We'll start by introducing typed lambda calculus as a theory of computation, including the existence of uncomputable functions and the behavior of computable sets. Since Haskell is essentially identical to typed lambda calculus, we'll use it to hone our lambda-fu with some optional homework problems and TAU sessions in the computer lab. Then we'll get into domain semantics and cartesian closed categories, which give a way to model the actual behavior of computer programs, rather than just the functions they compute. Finally, we'll see how the notion of *monad* from category theory can encapsulate different types of computation—a fact which forms the basis of the input/output system in Haskell!

Homework: Recommended exercises to be done on paper, and optional exercises to be done in Haskell. No programming is necessary to understand the class, but for those who are interested, the optional exercises and TAU sessions will provide an opportunity to learn to program in Haskell.

Prerequisites: None.

Related to: P vs. NP (Week 1); Mathematical logic (Week 2)

NEW CLASSES AT 1:10PM

The Stable Marriage Algorithm. (*; Alfonso; Tuesday)

N single men and N single women want to pair up and get married. These are their names and preferences:

- Alfonso: Nina, Sarah, Marisa, Yvonne, Leigh.
- Dan: Leigh, Nina, Yvonne, Sarah, Marisa.
- fMatt: Leigh, Yvonne, Sarah, Nina, Marisa.
- Greg: Marisa, Sarah, Yvonne, Leigh, Nina.
- Ryan: Marisa, Leigh, Nina, Yvonne, Sarah.
- Leigh: Alfonso, Greg, Ryan, fMatt, Dan.
- Marisa: fMatt, Dan, Alfonso, Ryan, Greg.
- Nina: fMatt, Greg, Ryan, Dan, Alfonso.
- Sarah: fMatt, Ryan, Alfonso, Dan, Greg.
- Yvonne: fMatt, Greg, Dan, Ryan, Alfonso.

Is it possible to make everybody happy? Obviously not since everybody wants to marry fMatt. But is it possible to at least create a *stable* situation? For instance, it is a bad idea for fMatt to marry Sarah and for Yvonne to marry Greg, because then fMatt and Yvonne would prefer each other rather than staying with their partners, so they will run away together. How can we at least avoid having a run-away couple? Is there more than one way to do it? What is the *best* way to do it? What if we move to Spain where Alfonso and Greg can marry each other?

Prerequisites: None.

Homework: None.

The life and legend of Galois. (*; Moon Duchin; Wednesday)

A swashbuckling good time.

Prerequisites: None.

Homework: None.

Rational Trigonometry and Universal Geometry. (**; Julian; Thu–Fri)

The intersecting chords theorem(*). Nice result. At least on the Euclidean plane (\mathbb{R}^2). But what would happen if we were to use to a different field, working on say \mathbb{C}^2 or \mathbb{F}_7^2 (here, \mathbb{F}_7 is the integers mod 7)? What could the statement of the theorem mean? And could it still, in some sense, be true?

And is there a way of doing some trigonometry (triangle measuring questions), getting exact answers without needing a calculator or tables? And how could we do trigonometry in \mathbb{F}_7^2 ??!

Come and learn about some very recent developments in the fields of trigonometry and (elementary) geometry in these exotic settings!

(*) The intersecting chords theorem: if AB and CD are two chords of a circle which intersect at P, then $AP \cdot PB = CP \cdot PD$.

Prerequisites: Modular arithmetic, complex numbers useful but not essential.

Homework: None.

Discrete Calculus. (*-**-; Mathieu; Saturday)

Wouldn't it be great to take derivatives and integrals of functions on the integers instead of on the real line? After all, it's so useful!

When we study single-variable calculus, we look at the derivative, which represents the instantaneous rate of change of a function, and at the integral, which represents the area under the graph of a function. Surprisingly, these two concepts turn out to be inverses of each other—a result known as the Fundamental Theorem of Calculus.

Unfortunately, this only works for functions on the real line, because it involves limits. However, if we look at the non-instantaneous rate of change, we can get a lot of the theory for functions on the integers (i.e. sequences of numbers), and we don't have to take any limits!

With this new theory, we'll be able to painlessly prove that $\sum_{k=0}^n k^2 = \frac{n(n+1)(2n+1)}{6}$ and easily compute sums of higher powers, among other exciting possibilities.

Prerequisites: None, but regular calculus would be good.

Homework: None.

Probability and hailstone numbers. (**-***; Moon Duchin; Thursday)

The Collatz function takes a natural number input—if it's even, cut it in half; if odd, take $3x + 1$. Do this a lot. What happens in the long term? Numbers bounce around a lot, but do they always eventually come crashing to the ground? Though the hailstone process is completely deterministic (no choices to make), there is a precise sense in which it is also as random as possible. All kinds of math come into the mix.

Prerequisites: None.

Homework: None.

Summing Divergent Series. (**-***; Ryan; Tue–Wed)

Nope, not a typo! All those years that you've been told that you can't add up a series of numbers, just because the partial sums don't "converge" to a nice "limit", you've been badly misinformed! Ever gotten into an argument in which you asserted that surely

$$1 - 1 + 1 - 1 + 1 - 1 + \dots = \frac{1}{2}$$

by the formula for geometric series? Ever find yourself wanting to add up a series of numbers that don't even have the common courtesy to be bounded? Despair no more! Come to this class and learn new ways to add up things that you really oughtn't be adding up. You'll also learn about a handful of very sensible properties that any reasonable summation algorithm ought to have, and how you can use those properties to justify your own mathematical machinations. A must-see for the student who *really* wants to make her pre-calculus or calculus teacher squirm! (Or for the student who has any interest in becoming a quantum field theorist, because they pull these kinds of stunts all the time. Shameless!)

Prerequisites: Knowing about Laplace transforms ahead of time might come in handy; otherwise, just some basic familiarity with sequences and series, and a willingness to stare into the unblinking eye of infinity.

Homework: None.

Related to: Real analysis (Weeks 1–3); Complex analysis (Weeks 2–3)

The shape of groups. (**-***; *Moon Duchin* and Yvonne; Fri-Sat)

A group is a set where you have structured ways to combine elements, like addition or multiplication on appropriate sets of numbers. But these structures also endow the groups with a shape—some are round and some are flat and some are branchy and bushy. We'll tour the zoo.

Prerequisites: It would help to have seen the definition of a group, but it's not required.

Homework: Recommended.

Intro to Cryptography. (**; Marisa; week 2 of 3)

Last week in Intro to Cryptography, we introduced the setup for symmetric key and public key cryptosystems, came to a reasonable definition of security, and started working to build secure encryption schemes. If one-way functions and hardcore bits sound familiar to you, then there might be a nice entry point into this course at the beginning of its second week. We're working towards ElGamal encryption. If you'd like to join this class, talk to me about your background.

Prerequisites: Last week's class or Marisa's permission.

Homework: Recommended.

Hyperplane Arrangements. (****; David, M@; week 1 of 1)

Consider a field K and the vector space $V = K^n$. A hyperplane in V is a dimension $n - 1$ (affine) subspace of V , and a hyperplane arrangement in V is a finite set of hyperplanes in V . In this class we will consider various combinatorial questions about hyperplane arrangements when $K = \mathbb{R}$. One of the more interesting questions is: what can one say about the number of regions defined by these hyperplanes? In the process of answering this question, we will explore a fascinating theory that is still under active development.

In particular, we will define partially ordered sets (posets) and lattices. We will associate a partially ordered set to every hyperplane arrangement, and using this poset define a powerful invariant of the hyperplane arrangement called the characteristic polynomial. Knowledge of the characteristic polynomial allows one to easily read off the number of regions formed by the hyperplanes in an arrangement. We will then proceed to relate hyperplane arrangement to graphs, and the characteristic polynomial to the chromatic polynomial of a graph. By doing so, one can rephrase the four color theorem in terms of hyperplane arrangements, as well as compute the characteristic polynomial of various hyperplane arrangements using graph theory. Finally, we will relate hyperplane arrangements over the reals to hyperplane arrangements over finite fields, and use this relationship to calculate the characteristic polynomial of previously intractable arrangements.

Prerequisites: Linear algebra.

Homework: Recommended.

Related to: Graph theory.

Olympiad Problem Solving with Calculus. (****; Bogdan; week 1 of 1)

There are many math competitions around the world these days. Almost every country has its national olympiad and there are also regional and international ones.

In several such competitions calculus problems are often proposed to the students (I mention only two: William Lowell Putnam Mathematical Competition and the International Mathematical Competition).

I will show some nice calculus problems and some tools and tricks to deal with them. To see what that's all about, here are some examples of problems I'm going to discuss during this class.

- (1) Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.
 - a) Prove that $\lim_{n \rightarrow \infty} x_n = \infty$;
 - b) Prove that the sequence $y_n = \{x_n\}$ diverges.
- (2) Let $x > 1$ be a real number such that $\lim_{n \rightarrow \infty} x^n \{x^n\} = 0$. Prove that x is an integer number.
- (3) Let $f : R \rightarrow R$ be an onto function with the following property: if $(x_n)_n$ is a divergent sequence, then $(f(x_n))_n$ is a divergent sequence as well. Prove that f is continuous.
- (4) Prove or disprove: there exists a continuous function $f : R \rightarrow R$ such that $f(x)$ is a rational number if and only if $f(x+1)$ is an irrational number.
- (5) Find a such that $2^x + a^x \geq 3^x + 4^x$, for all real x .

Come and see the solutions!

Prerequisites: Knowledge of basic calculus.

COLLOQUIA (4-5 PM)

The Mathematics of Lights-Out. (Leigh; Tuesday)

Maybe you've seen an old electronic game called "Lights-Out" or "Merlin's Magic Square". The game begins with a $n \times n$ grid of lights, each of which is "on" or "off". Pushing any button will toggle the lights adjacent to it, switching their state (on to off or off to on). The goal of the game is to get from the initial configuration to some set winning configuration, like "all lights off".

It turns out that the mathematics of solving this game is linear algebra; in this colloquium we'll see why. No linear algebra background is required, though it won't hurt.

Qualifying quiz presentations. (Students; Wed, Fri)

Come see your fellow campers present solutions to this year's qualifying quiz problems!

The shape of infinity. (*Moon Duchin*; Thursday)

By a historical accident, the notion of a space where you can measure distance (a metric space) came well after lots of work on spaces where the distance comes from calculus (manifolds). I'll talk about purging the calculus out of your geometry. Armed with only our measuring tape, we'll study the geometry of geodesics and various "boundaries at infinity."

DIGESTIF (2:10–2:40 SATURDAY)

Numbers of the form $x^3 + y^3$ modulo p . (Noah)

Which numbers are cubes modulo 7? Well, we can easily compute that $1^3 = 2^3 = 4^3 = 1$, that $3^3 = 5^3 = 6^3 = 6$, and that $0^3 = 0$. So which numbers are the sum of two cubes modulo 7? Well, only 0, 1, 2, 5, 6 can be, 3 and 4 cannot. It turns out that 7 is very special this way. For any other prime p , every number can be written as a sum of two cubes modulo p . There is a very cute proof of this fact due to Dan Shapiro that I'll be explaining in this talk.

MARATHON (MOST OF THE DAY, EVERY DAY)

Metric Spaces, the Real Numbers, and the Cantor Set. (***) (Nina and Dan; week 1 of 1)

A metric space is a generalization of a "space with distance." By selecting out the essential properties that any kind of "distance" will have, we can create all kinds of new spaces that also satisfy these properties, but which also do strange and wacky things.

When you study metric spaces carefully, you suddenly have access to many different areas of math: topology and analysis are just two examples. (For example, any metric space yields a topological space, and metric space techniques are essential to proving the existence and uniqueness of solutions to differential equations.) In this class, we'll use metric spaces as a gateway to topology, and we'll also use them to gain a deeper understanding of some of spaces we already know about.

We'll start out by constructing the real numbers and then proceed to metric spaces. We'll build up a topological theory in addition to a more specific sequence from the spaces themselves. Finally, we'll turn our attention to the Cantor set, study its properties, give topological ways to identify the Cantor set, and use its universal property among metric spaces to construct a space-filling curve.

This class will be an excellent chance to study a topic at Mathcamp in greater depth, as well as giving your work a consistent focus. We'll leave time in the class to do any work we assign. In order to attend, you should come to the organizational meeting (which is posted on the schedule board), or else speak to either Nina or Dan.

Prerequisites: None.

Homework: Required, but it's all in class.

Related to: Point-set topology (Weeks 1–2); Real analysis (Weeks 1–3); p -adic numbers and p -adic calculus (Week 3); Algebraic topology (Week 4 marathon)

VISITOR BIOS

Dave Savitt. (University of Arizona)

Dave is actually the Deputy Director of Mathcamp, aka Ze Top Blueberry. Originally from Vancouver, Canada, David Savitt was the first-ever counselor at Mathcamp, and this year will be his eleventh Mathcamp. David received his PhD at Harvard University in 2001 (where his work focused on an extension of the results which led to the proof of Fermat's Last Theorem) and did his postdoctoral research at McGill University (Montreal) and Institute des Hautes Etudes Scientifiques (Paris). His current research interests include number theory and arithmetic geometry.

Moon Duchin. (UC Davis)

Moon Duchin is interested in geometry, topology, and dynamics, in lots of different combinations. Lately she's got geometric group theory on her mind. She also thinks about philosophy, cultural studies, gender theory, what they have to say about math, and what math has to say back!

Fernando Gouvea. (Colby College)

Fernando Gouvea started his mathematical career as a number theorist, but is slowly morphing into a historian of mathematics. He is the author (or co-author) of four (or five, depending on how you count) books, and he was co-editor of a fifth (or sixth). He is also the editor of FOCUS, the news magazine of the Mathematical Association of America (MAA), and of MAA Reviews, the MAA's online book review service. He has a wide range of interests, and is fond of describing himself as "Christian, orthodox, Brazilian, American, conservative, husband, father, member of a Lutheran church, Sunday School teacher, choir director, editor, author, dog owner, bibliophile, science fiction fan, wine geek, adoptive Mainer, historian wannabe, and the proud possessor of a graying scraggly beard."

Emily Landes. Emily Landes is a graduate mathematics student at the University of Texas, Austin. She studies topology, in particular character varieties.