Crocheting the Lorenz manifold

Supervised by Julian and Yvonne

Description The Lorenz dynamical system was an early model of weather systems, and was the first in which chaotic behaviour was identified. Recently, two researchers investigating the “stable manifold” of this beautiful system discovered that their algorithm for generating it could be used to make a crochet model of it.

Expected project output A crocheted Lorenz manifold!

Difficulty level $*$ (or $**$ if you need to learn to crochet as well!)

Mathematical prerequisites None.

Math you'll get to learn along the way Dynamical systems, a little about how to generate stable manifolds.

Computer use None.

Preferred number of project participants As many as want to participate.

Degree of supervision Low—once you know what you're doing, you just do it!

The finite group game

Supervised by Alfonso and Noah.

Description Are you hooked on abstract algebra? Can't you get enough on your daily fix and you want a really challenging problem?

Let's play the finite group game. We will take turn describing all the finite groups of order n up to isomorphism. I will start with $n = 1$. Heh.

Expected project output

1. A list of all groups of order less than or equal to 60, up to isomorphism.
2. A list of all simple groups of order less than or equal to 100, up to isomorphism.
3. The three smallest non-solvable groups.

Difficulty level $****$

Mathematical prerequisites Group theory.

Math you'll get to learn along the way A *very* solid understanding of the Sylow theorems, of semidirect products, and other finite group theory concepts. This is the ultimate goal of the project.

Computer use Probably not.

Preferred number of project participants One or a few.

Degree of supervision Medium.

π_1(**Dorms**)

Supervised by Alfonso.

Description I stand at the MC office holding one end of a very long rope. You take the other end and run through the corridors and lounges of Taylor, Sturtevant, and Leonard, then come back to me and give me the other end. At this point, I pull both ends and try to recover as much rope as possible. If there were no walls (or floors or ceilings), I would be able to recover all of the rope. But since there are walls, the rope will stay describing a *path* through the dorms. Can you describe all the possible paths constructed in this way? In other words, can you describe the fundamental group of the space of corridors and lounges?

Expected project outpout A description of the above-mentioned fundamental group.

Difficulty level $***$

Mathematical prerequisites Some group theory. Some topology.

Math you'll get to learn along the way You will come up with and prove a theorem on algebraic topology, but I should not spoil that part.

Computer use No.

Preferred number of project participants 2 or 3.

Degree of supervision Depending on your background, more at the beginning to give you the initial tools and concepts. Less later, while you run along the corridors.

Projects in Combinatorial Game Theory

Supervised by Alfonso.

Description Here are various projects of different difficulty. If you sign up for this project, please specify which of the following you want to do! If you would like to do a project on CGT but none of these sounds interesting, let me know.

1. **Periodicity of substraction games.** $****$

   Let S be a finite set of natural numbers. We play a game with various piles of beans. On your turn you are allowed to remove any number in the set S from a pile. As usual, the first player who cannot move loses. In order to play this game with various piles of beans, we need to calculate the nim value of a pile with n beans. As we mentioned in class, this sequence of nim values is eventually periodic (the first few terms may be an anomaly). The question is: can we find a formula for the period as a function of the set S? We have an answer when S has just 1 (heh) or 2 elements, but not in general.

   We do not know if it has a nice solution. Programming knowledge would be useful.

2. **Heads turn!** (name suggested by Conway a few day ago) $***$

   This is a variation on problem #9 in the Qualifying Quiz. We start with a row of coins, where some are heads-up and some are tails-up. Two players take turns in playing. In her turn, a player can remove any coin which is heads-up. Its immediate neighbors (if any, and if they are still in the game) get them flipped over. The first player who cannot move (because there aren't any heads-up coins left) loses. To analyze this game, we need to calculate the nim-value of every position, even if we want to play with only one row, because rows split in various pieces, and we need to add them!

   I have some partial results, and so does Conway, but we do not know what the general result is. What I know hints towards a general formula, but there may not be one. Programming knowledge may be useful.

3. **Almonds** $**$

   This is an ideal game to play at a party with snacks. We start with various piles of almonds. A turn in this game consists of

   - either spliting one pile into various piles of equal size,
   - or merging two piles together that had a different number of almonds.

   This game is not strictly like the rest of impartial games that we have tried, because loops are possible! Can you find a way to attack this game anyway?

4. **The princess and the roses** $**$–$****$

Princess Avi has two suitors: Dave and David. There are n rose bushes in the palace garden, and each bush has roses of a different colour (with a certain initial number that may be different for each bush). Dave and David alternate days in bringing roses to the princess. Each day, one of them goes to the bushes and picks either one rose, or two roses of different colours for Avi. When all the roses are gone, the suitor who has no flowers to offer leaves in embarrassment. Which strategy will allow them to win?

This problem cannot be attacked by calculating the nim-value of one bush, and adding them. (Why?) We have a pattern for up to five bushes ($**$), but we do not know what to do with six of them ($****$).

5. **Wyt Queens** $**$–$****$

   I explained this game in class. We have a very nice formula that gives us the P-positions ($**$). If you want a real challenge, try to find a formula for the nim-value of any position ($****$). We have no idea about it! Or, what would happen in a 3D board?

6. **Wyt Knights** $**$

   A Wyt Knight can move NNE, NNW, NWW, or SWW. Here we have a complete pattern for the nim-value of every position, so we know how to play with various knights.

7. **Euclid game** $**$

   In this two player game, we start with two natural numbers a and b. In her turn, a player may replace the largest number (say a) by any strictly positive number c of the form $c = a - bn$ for some positive n. The first player who cannot move (because he has been handed a pair of two identical numbers) loses. What is the winning strategy?

   This was problem #10 in the Qualifying Quiz in 2005, but in subsequent years problem #10 has been much more challenging. The solution is nice, though.

8. **Tic-tac-toe** $*$–$**$

   Imagine that you play tic-tac-toe on an board of size $m \times n$, and that you win if you can get k in a line. This is called (m, n, k)-tic-tac-toe. The game is very different depending on the values of m, n, and k, of course. (What is $(3, 3, 3)$-tic-tac-toe?) A particularly interesting case is $m = n = \infty$. In this case the game could go on for ever (a draw), or be a first-player win, or a second-player win. Can you find some examples of each one of them?

9. **Variations on Nim** $***$–$****$

   There are various interesting variations on Nim that have been proposed:

   - **Antonim**. *Antipathetic Nim* is Nim in which no two heaps are allowed to have the same number of beans. We know of a pattern for 3-pile Antonim ($***$), but we believe there is not a simple rule for 4-pile Antonim ($****$).

- **Synonim**. *Sympathetic Nim* is Nim in which all heaps of the same size must be treated alike—if you reduce one heap of a given size, you must reduce all heaps of that size by the same amount (no move may affect heaps of different sizes). Synonim and Antonim are highly related.
- **Simonim**. *SImilar MOve NIM* is just nim with the additional feature that a player may make any number of moves provided that these are all exactly similar, i.e. that they all reduce all the heaps of a given size. We have a general rule for 4-heap Simonim ($***$–$****$), but it is quite messy to capture.

Expected project output See above.

Difficulty level See above.

Mathematical prerequisites Some Combinatorial Game Theory (in particular, nim-sum and the mex rule) is needed for most of them, but I could put you up to speed if you did not take my class.

Math you'll get to learn along the way More CGT!

Computer use For some of them.

Preferred number of project participants 1 or small group per project.

Degree of supervision Not much. I expect *you* to solve the problem. I am available to check your conjectures and proofs and to discuss your ideas, as well as to provide hints if you get stuck, but do not expect me to provide a hint on the open problems!

Circuits Over Sets of Natural Numbers

Supervised by Dan

Description A circuit is a computational object that takes inputs and passes them through *gates* that perform operations on the inputs. For example, many circuits in common use take in boolean values (true or false) and put them through AND gates, OR gates, and NOT gates. The circuit outputs a boolean value that is a result of passing the inputs through the gates. Computation can actually be defined with circuits instead of Turing machines.

Now suppose a circuit takes as input sets of natural numbers, and has operations

$$\cup, \cap, \text{ complement}, +, \times.$$

This kind of circuit will output a set of natural numbers. Here's a question: is the outputted set decidable by some algorithm, given the circuit C and input to the circuit w?

This problem is still open, although there are many intermediate results. In particular, simpler versions of this problem are (variously) NP-complete, PSPACE-complete, and NEXPTIME-complete (nondeterministic exponential time complete). We'll learn about attempts to solve this problem and then attempt to solve it ourselves.

By the way, if this turns out to be decidable, then you can use this to decide Golbach's conjecture! I personally suspect the problem isn't decidable, but if it turns out to be decidable and an algorithm appears, we can run a Turing machine/computer that will output the solution to Goldbach's conjecture!

Expected project output Maybe a theorem, at least a poster.

Difficulty level $***$–$****$

Mathematical prerequisites Theoretical CS. Number theorists are also quite welcome.

Math you'll get to learn along the way Complexity theory, maybe some number theory.

Computer use None.

Preferred number of project participants 3. (Note that one student is already doing the project.)

Degree of supervision Some, but I'm also busy!

Juggling Braids

Supervised by Dave

Description Tape one end of a piece of string to the wall, and the other to a juggling ball. If you do this to all of your juggling balls and then juggle them, the result will be a braid. In Ron Graham's book on the mathematics of juggling, he shows that every braid can be juggled—provided the juggler has arbitrarily many hands! I'd like to know if every braid can be juggled by a two-handed juggler. I'd also like to see efficient algorithms for juggling the juggleable braids in the fewest number of throws.

Expected project output Answers to the above questions.

Difficulty level $***$

Mathematical prerequisites Some knowledge of groups.

Math you'll get to learn along the way The braid group.

Computer use None.

Preferred number of project participants 1 to 4.

Degree of supervision Medium

Sliding Block Problems

Supervised by Leigh and Dave

Description You've probably seen the "15-puzzle" before. That is the 4×4 grid with 15 movable tiles labelled 1–15 and a blank space to allow movement of the tiles. The object is to slide the blocks around until you end up in the winning configuration (usually just ascending order). Well, there's nothing special about a 4×4 grid. You can define a sliding block puzzle on any graph. So if you have a sliding block problem on some graph, the question is whether or not it is possible to get to the winning state from a given initial configuration. For instance, you've probably heard that if you forcefully switch the 14&15 while the puzzle is in the "solved" position, then the puzzle becomes impossible to solve. This question of "solvability" can be tackled using some group theory.

We're proposing two possible avenues of exploration:

We have two papers. The first goes through the solution of the 15-puzzle and its mathematics, and then discusses generalizations to other graphs. However, to get all the details for the generalization, there is another, more advanced paper to be read. Neither paper is that long, 12 and 11 pages, respectively. You could...

1. Simply read about the 15-puzzle (the 1st 8 pages of the 1st paper), or you could read the entire 1st paper to get a flavor for the generalizations (just an extra 4 pages of reading). This requires some comfort with the groups S_n and A_n and a little group theory, which could pick up along the way if you're motivated. This would be a $**$ project.
2. Read *both* papers so that you see the full argument for a sliding block problem on an arbitrary graph. This is a $****$ project, but it's perfectly accessible for someone who is comfortable with groups (in particular S_n and A_n), and some group actions. With that background, the 1st paper is a quick read. The second paper is quite a bit more challenging.
3. Reading and exploring—You could read as much of the 1st paper as you need to get the idea of how these puzzles work. Then you might want to try constructing your own puzzles and figuring out the mathematics of the solutions. This is probably a $***$–$****$ option.

Expected project output Perhaps a poster. This could definitely extend to a longer research project.

Difficulty level $***$–$****$

Mathematical prerequisites Good understanding of groups, in particular permutation groups. You can definitely pick up the graph theory along the way. See above descriptions for additional details.

Math you'll get to learn along the way A cool application of group and graph theory to some interesting puzzles.

Computer use None

Preferred number of project participants 1–3

Degree of supervision Roughly 2 meetings a week.

Is there a nontrivial homology on food?

Supervised by Dave

Description In this project, we will try to figure out whether there is a nontrivial homology on food. More importantly, we will answer the question of what the heck this first question means. Homology was originally developed in order to understand topological spaces, but mathematicians later learned that it could be applied in unexpected places. We will apply it in the most uexpected place of all.

Expected project output Hopefully, we will develop food homology, which may someday be as cool as foodtongue.

Difficulty level $****$

Mathematical prerequisites Linear algebra.

Math you'll get to learn along the way Homology.

Computer use None.

Preferred number of project participants As many as would like to do this.

Degree of supervision High at the beginning, low at the end.

Filters in Topology

Supervised by Anti and Alfonso

Description In point-set topology, we usually describe topological spaces using open sets. There are other ways to describe the same object, though; one of them is by giving a notion of convergence. However, you can't just talk about convergence of sequences; you have to talk about convergence of fancier things like "nets" or "filters". Our goal will be to discover how to define topological notions using convergence of filters or nets, rather than open sets. We (Anti and Alfonso) know some of the answers, but not all of them!

Expected project output Definitions of topology, compactness, continuity, and other things using filters or nets.

Difficulty level $***$

Mathematical prerequisites Point-set topology

Math you'll get to learn along the way Filters!

Computer use None.

Preferred number of project participants A few.

Degree of supervision Medium.

Symmetry Groups of Solids

Supervised by Leigh

Description In “Groups & Symmetry“, we talked about the groups of symmetries of the 5 regular solids (tetrahedron, cube, octahedron, dodecahedron, and icosahedron). But what about shapes like the cuboctahedron or the rhombic dodecahedron?

This project would be to take solids such as those mentioned above and use group theory and models (zome tool) to find their symmetries. An important starting point is to understand the following result:

Every finite rotational symmetry group is either cyclic, dihedral, or one of the symmetry groups of the five regular solids.

So the first idea to explore would be the rotational symmetries of these other solids – which ones are like cubes, and which ones are like dodecahedron?

Then we could try to play with the reflections. Here I don’t know any results (I’m sure there are some somewhere), so I don’t know how hard this idea is. We can play.

Expected project output A poster and/or some pretty models

Difficulty level $**$–$***$

Mathematical prerequisites Some group theory is required. Some knowledge of matrices or willingness to do some reading is also necessary. If you took “Groups & Symmetry“, you’re a perfect candidate.

Math you’ll get to learn along the way You’ll pick up some group actions along the way.

Computer use None required, but there’s some cool software out there for playing with models, if you want to investigate.

Preferred number of project participants 1–4

Degree of supervision Roughly 2 meetings a week.

Project Name

The Mathematics of Lights Out

Supervised by Alfonso and Leigh

Description Maybe you've seen an old electronic game called "Lights Out" or "Merlin's Magic Square". The game begins with a $n \times n$ grid of lights, each of which is "on" or "off". Pushing any button will toggle the lights adjacent to it, switching their state (on to off or off to on). The goal of the game is to get from the inital configuration to some set winning configuration, like "all lights off".

Questions abound here...

1. Given a particular intial condition, can the game be won?
2. What's an optimal strategy, and what is the maximum number of moves you would have to make to get to the solution?
3. What happens if you try to play this game on a torus?

Leigh will give a colloquium next week on a 3×3 version of the game, which will answer questions 1,2 for that puzzle. So the first step in this project is to go to that talk. We can also give you several papers published on this topic—one which explores a 3×3 version of the game that Leigh is presenting, and one which explores the 5×5 version of the game.

If you want to work on this project, you would be trying things like:

1. To what size grids can you generalize the strategies for the 3×3 game?
2. What happens to the answer of the questions if you change the effects of pushing certain buttons?
3. What if buttons have more than 2 states—red, green, and blue instead of on and off?
4. What happens to the game if you change the goal?
5. What *does* happen if you try to play on a torus?

The drawback is we do not have a copy of this game. We can probably find some sort of platform online, but it might be a slightly different variation.

Expected project output Perhaps a poster.

Difficulty level $**$–$***$

Mathematical prerequisites Linear algebra (or a willingness to pick it up)—just knowing the words "basis", "vector", "linear independence", "dimension" will work

Math you'll get to learn along the way A cool application of linear algebra.

Computer use Some optional time playing some sort of online version, to get a feel of the game.

Preferred number of project participants 1–4

Degree of supervision Roughly 2 meetings a week.

Period of the Fibonacci numbers modulo n

Supervised by Mark

Description If you look at the last digit (in base 10) of the Fibonacci numbers, you get the sequence

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0,$$
$$9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, 0,$$
$$1, 1, \dots,$$

which is periodic with (smallest) period 60. What happens if you replace base 10 by base n? Will the sequence always return to the starting values $1, 1$? If it does, what is the period, as a function of n? Investigating such questions, which can be started by experimentation, will lead you to some unexpected patterns, most of which you should have a shot at proving (although at least one leads to a somewhat notorious open problem).

NOTE: Some, although by no means all, of this material was discussed in the introductory number theory class last year. Therefore, in the spirit of discovery and the interest of fairness, this project is only open to new campers!

Expected project output Poster, probably.

Difficulty level $**$–$***$

Mathematical prerequisites Some number theory, in particular the Chinese Remainder Theorem and the Quadratic Reciprocity Law, would be helpful.

Math you'll get to learn along the way Any of the above that you don't know yet.

Computer use After a while you'll get tired of doing modular addition by hand, and you'll want to write an easy computer program (in whatever programming environment you're comfortable with) to help you out. At least one participant should be capable of doing this.

Preferred number of project participants 2 or 3

Degree of supervision Hard to predict—I'll be available at TAU and I'll try to be helpful (and give hints) if/when you get stuck. However, if you're not self-motivated, the project will likely fizzle, which would be too bad!

Unraveling the Human Knot Game

Supervised by Marisa and Yvonne

Description In the human knot game, people stand in a circle and reach out to each other. They join hands, with each person holding two different person's hands. If this is an ice-breaker, the people then try to unravel the configuration without letting go of each other's hands. If this is Mathcamp, we analyse the game.

The goal of this project is to come up with good mathematical models of this game that can answer some of the following questions:

- What knots are n people likely to make? What knots cannot be made? When can n people be unravelled?
- How many separate components can n people form?
- What's the probability that the people form a multiple component link?

Expected project output A model for the human knot game and potentially solutions to some of the above questions.

Difficulty level $**$ for students who know some knot theory, $***$ otherwise

Mathematical prerequisites If you have taken knot theory, you may be particularly interested in this project.

Math you'll get to learn along the way More fluency with knot theory, learning to how break down a large question into tractable, well-defined pieces.

Computer use Up to you.

Modelling hyperbolic geometry

Supervised by Yvonne

Description We will make various models of various hyperbolic things using media of your choice (yarn, cardstock, etc.). Using these models as a launching point, we will examine the notions of curvature, geodesics, and polygons on hyperbolic-y surfaces.

Expected project output Models of the hyperbolic plane and a poster or paper explaining them.

Difficulty level $*$

Mathematical prerequisites Euclidean geometry

Math you'll get to learn along the way Hyperbolic geometry, geometry of surfaces

Computer use None

Preferred number of project participants 1–3

Degree of supervision Moderate; I would like to meet with project participants at least briefly once every few days.

Classifying finite topological spaces

Supervised by Nina

Description This class is aimed at people who took the first week of Point Set Topology. Given three points, how many ways can put a topology on them? How does this question become more difficult when we only care about the different topologies up to homomorphism? With three points this can be done by hand. What about 4 points? What about k points?

Expected project output A computer program to classify finite topological spaces.

Difficulty $**$

Mathematical prerequisites None. If you don't yet know what a topological space is, you can learn.

Computer use Lots.

Preferred number of project participants 2–3

Degree of supervision Some.

SET in n dimensions

Supervised by Nina

Description This is project is part math and part game testing, and can vary in difficulty based on your math/computer experience. The game of set is played with four characteristics (color, shading, number, and shape), each of which can take on three different values. What is the game like if we change either the number of the characteristics or the different values they can take on? More specifically, in the $(4, 3)$ version of SET we all know and love, there exists a maximum number of cards we can look at and not find a set. There is no general answer to this problem for a (n, k) game of SET.

So where does the game testing come in? If we put three SET decks together and add another characteristic, i.e. play a $(5, 3)$ version of the game, how hard is it to play? How many cards do we want to put out to make the game fun? How do our brains adjust to the new characteristic?

This project is *very* open-ended. Students could write programs to tackle the problem, or come up with proofs of upper or lower bounds (for (n, k) or their choice, or general (n, k)).

Expected project output A new SET deck, a computer program, some upper or lower bound proofs, etc.

Difficulty level $*$ to $****$

Mathematical prerequisites Experience programming, some knowledge of $\mathbb{Z}/n\mathbb{Z}$, enthusiasm for SET would *help* but are not necessary.

Math you'll get to learn along the way Finite cyclic groups.

Computer use Optional.

Preferred number of project participants 2–8

Degree of supervision Flexible.

Counterexamples in Point-set Topology

Supervised by Nina

Description This is intended for students who have taken either week of Point Set Topology this summer at camp, or have taken it in the past. The number of definitions in point set topology can be mind boggling. Take the separation axioms for example (T_1, Hausdorff, regular, normal…to name a few). Which implies which? What counterexamples do we have to see that certain axioms are not equivalent? We don't need to restrict ourselves to the separation axioms to play around in this way; there are also various forms of compactness and connectedness. We'll work closely with the book *Counterexamples in Topology* as well as other books and resources on the internet to understand and/or concoct various examples and counterexamples.

Expected project output A document with students' descriptions and pictures of various examples and counterexamples in topology.

Difficulty level $**$–$***$

Mathematical prerequisites Point set topology.

Math you'll get to learn along the way More point set topology.

Preferred number of project participants 1–3.

Degree of supervision Flexible.

Two mathematical art projects

Supervised by Nina

Description Here are two ideas—pick one.

1. *Mathematical picture book.* There was an excellent illustrated "children's" book written about complex numbers: see http://mathforum.org/johnandbetty/. Pick your own favorite mathematical tidbit and write and illustrate a children's book on it. This gives you the opportunity to distill the essence of some kind of math, or some particular problem or concept you find fun or interesting. How can you teach a complicated concept in a fun and visual way? Possible topics: orientable and non-orientable surfaces, the real numbers, the Hairy Ball Theorem, surgery on manifolds, elementary number theory, graph theory. You get the idea—it can relate highly to a class your taking, or some area of math you've explored on your own.
2. *Escher-esque tessellations.* Using art supplies or Photoshop, create your own Escher-esque tessellations. This form of art relates closely to tilings of the plane. We'll first explore various ways to tile the plane with one or more shapes, and then study the tessellations of Escher to make our own.

Expected project output A picture book or tessellation(s).

Difficulty level $*$

Mathematical prerequisites None.

Math you'll get to learn along the way Tilings (for the second project).

Computer use Possible use of Photoshop, if I can get my hands on it, for either project.

Preferred number of project participants 1–5

Degree of supervision Some.

Setless sets in SET, in dimension 4 and beyond

Supervised by M@

Description It turns out that the card game SET is basically mathematically equivalent to playing tic-tac-toe in 4-dimensional space. There is a folklore theorem you may have heard that says every set of 21 cards contains a set. How do you prove this? We will work our way through a paper that proves it, and the approach is geometric. We will talk about affine subspaces over finite fields, but your intuition about points, lines, and planes is all you will really need to get started.

However what happens in higher dimensions, even approximately, is still very mysterious. In fact Fields Medalist Terence Tao recently listed it as one of his favorite unsolved problems.

Expected project output A nice understanding of the basic theorem for SET, and an intuition for how it might work in higher dimensions. Perhaps even a nice poster.

Difficulty level $**$

Mathematical prerequisites None, although linear algebra and basic number theory (really just mod 3 arithmetic) will be helpful.

Math you'll get to learn along the way Geometry over finite fields. Some combinatorics.

Computer use None.

Preferred number of project participants 1–5

Degree of supervision Weekly meetings.