Class descriptions—Week 2, Mathcamp 2006

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Classrooms

Most of our classrooms are new this week. We will continue to have Jones 213. In addition, we will be teaching in the dinning halls of five fraternities:

- on the basement of ΣN (named B on your campus map).
- on the first floor of $\Phi\Delta\Theta$ (named D on your campus map).
- on the basement of $KA\Theta$ (named E on your campus map).
- on the basement of ΣX (named F on your campus map).
- on the basement of $B\Theta\Pi$ (named G on your campus map).

The colloquia will happen at McIntyre 103, except on Saturday, when we use McIntyre 003. George Hart's "Xtreme geometry – Group construction" will happen in the main lounge of Anderson/Langdon (1st floor), as will do the Project Fair on Friday.

Classes beginning in Week 2

Algorithms ($**$, fMatt, 11am, week 1 of 1)

When we study algorithms, we want to know not only how to solve computational problems, but how to solve them efficiently; sometimes the only difference between an intractable problem and one that can be solved in polynomial time is just a little bit of insight! We'll be taking a look at some basic (but incredibly useful) concepts in algorithms such as divide and conquer, dynamic programming, parallel processing, and if time permits, a few really delicious algorithms on graphs. We'll be looking at these concepts both from a more theoretical, mathematical point of view to see where they come from, as a well as a practical point of view to see how they can make our programs run harder, better, faster, stronger.

Examples will be given in pseudo-code, so previous programming experience is not necessary (although it certainly won't hurt). There will be a few short, required homework problems each day, plus some longer, optional problems to chew on at your leisure.

Prerequisites: the first week of "Theoretical computer science".

Homework: required

Related to: "Theoretical computer science" (Weeks 1–3).

Archimedes's school of rock ($*$, Sam, 11am, week 1 of 1)

Archimedes sought to find harmony with the way of the universe and to stick it to the man through the pursuit of beautiful mathematics and physics. For this week, we will seek to follow in his footsteps. The mathematics will include various combinations of geometry and combinatorics – we will compute the volume of a sphere as Archimedes did, using physical reasoning about balanced levers (instead of ugly integral calculus computations, since calculus was not invented for another two thousand years),

and we will also consider the number of ways of putting together the pieces of a children's puzzle to form a square, and the number of grains of sand that it would take to fill the universe. We may or may not set the sails of invading ships on fire using polished bronze parabolic shields.

Homework: none.

Prerequisites: none.

Related to: to "Combinatorics" (Week 1), "Combinatorial geometry in problem solving" (Week 2).

Classical logic ($**$, Kenny, 9am, week 1 of 2 (Weeks 2 and 3))

One of the most central things mathematicians do is to prove theorems. (It's certainly not the only thing they do, but that's a separate topic.) The general way this works is by starting with some axioms (approximately, things that are obviously true, or that we stipulate to define just what it is we're talking about) and then seeing what "*follows from*" them. Logic is the study of just what this *follows from* means.

Aristotle had a notion that explained some of this, but it was not until the 19th century, in the work of Gottlob Frege, that all of the patterns of proof in mathematical reasoning were finally discussed rigorously. In this class, I will teach you a modern version of Frege's system. If I have time, this will culminate in a proof of Gödel's celebrated Completeness Theorem, which basically shows why this system can encompass all of mathematical proof. (Unfortunately, the proof of Gödel's more famous Incompleteness Theorem, which shows some of the limits of what can be proved, won't fit.)

Prerequisites: none.

Related to: math.

Complex Analysis ($***$, Mark, 9 am, week 1 of 1)

We'll start with a quick review of complex numbers and possibly a bit of unfinished business from the multivariable calculus crash course (Green's theorem, if we didn't get to it; this theorem allows you to find the area of an irregularly shaped region just by walking around it with a suitable instrument and recording some information!). Then it's on to the spectacular changes that occur when you are doing calculus and you allow your variable x (now to be called $z = x + iy$) to take on complex values. Not only is this material very interesting, but there are surprising applications both in- and outside mathematics – often in areas you wouldn't expect, such as number theory. For example, complex analysis is used in proving Dirichlet's famous theorem on primes in an arithmetic progression, which says that if a and b are positive integers with $gcd(a, b) = 1$, then the sequence $a, a+b, a+2b, a+3b, \ldots$ contains infinitely many primes.

Prerequisites: "Multivariable calculus" (may not be needed until the second day).

Related to: "Pathway into analytic number theory" (Week 3).

Homework: recommended.

Fermat's Last Theorem for polynomials ($***$, Noah, 10am, Tu–Fr)

If n is a positive integer which is 3 or greater, then there are no non-constant polynomials $f(x)$, $g(x)$, and $h(x)$ such that $f(x)^n + g(x)^n = h(x)^n$. I have a truly marvelous proof of this proposition which this blurb is too narrow to contain. However, it will fit quite nicely into a four day class. In order to prove this result, we will discuss why polynomials are just like numbers (but easier!) and we will prove *the abc theorem.*

Prerequisites: basic number theory.

Related to: "Rational numbers... in space!" (Week 1), "Prime number theorem for polynomials" (Week 4), "Pathway to analytic number theory" (Week 3).

Homework: recommended.

The Fifteen theorem ($****$, David R, 11am, week 1 of 1)

The Fifteen theorem provides a beautiful and unexpected answer to a problem in number theory. A quadratic form is a function like $x_1^2 + x_1x_2 + x_2x_3 + x_3^2$: a sum of monomials, each of degree 2. We will be interested in quadratic forms with integer coefficients, and particularly those that

only take on nonnegative values. We say that such a quadratic form represents n if we can plug in integers for $x_1, \ldots, x_r$ and get n. For example, the quadratic form above represents 34 because $(-1)^2 + (-1)(2) + (2)(5) + (5)^2 = 34$. We may ask when a quadratic form represents all positive integers? Is there a simple way to tell?

The Fifteen theorem states that if a quadratic form represents all integers between 1 and 15, then it represents all positive integers. We will prove this theorem, then move on to generalizations. For example, when will a quadratic form represent all primes?

Prerequisites: linear algebra.

Homework: recommended.

Related to: number theory.

Fractal dimensions ($**$, Mike, 11am, Tu–We)

If there is justice in the universe a line is 1-dimensional. So is a circle. And a sphere and the surface of a donut are 2-dimensional. And we live in 3 dimensions. But what if we live on Sierpinski's gasket? Or a Koch snowflake? How do we do justice to our world? What dimension would we live in? Why would we care?

By "fractal", we mean an object that is "rough" in ways that are hard to control. Complicated things can happen on fractals. They arise naturally as attractors of chaotic dynamical systems. As mathematicians we want ways to describe these objects. In this class we will explore what it might mean to be 1/2-dimensional, or $(\log 3/\log 2)$-dimensional.

Homework: none.

Prerequisites: none.

Introduction to graph theory ($**$, Marisa, 9am, week 1 of 1)

Take a map of the United States and a little box of crayolas. The Four Color Theorem says it's possible to "properly" color the states in such a way that adjacent states (sharing a border, not just a corner) get different colors and use only four colors. It's one of the simplest theorems to state, but extremely difficult to prove: it was conjectured in 1850 but not proved until 1976, when Appel and Haken gave a computer-based proof. There's still no proof that's checkable by humans! Mathematicians are divided on whether or not it's okay for proofs to rely so heavily on computing power, but we all agree that the Four Color Theorem is true.

We're not going to prove the Four Color Theorem in the four days of class this week, but we *are* going to prove the Five Color Theorem (the regions of a any map can be properly colored in five colors), and lots of other neat tidbits from Graph Theory. We'll start with basic definitions and theorems (many of which you'll prove by discovery) and then touch on trees and connectivity, Eulerian cycles and Hamiltonian circuits (bringing along the Knight's Tour and the Travelling Salesman Problem), labelling, and lastly the Five Color Theorem.

Prerequisites: "Combinatorics" would be helpful, but not necessary.

Homework: recommended.

Required for: "Topological graph theory" (Weeks 3–4).

Related to: "A random walk from the marriage theorem" (Week 1).

Linear programing ($**$, Ellen, 10am, week 1 of 1)

Contrary to the vibes its name may evoke, linear programming has nothing to do with computer programming. Rather, linear programming is all about polyhedra and optimization. Many interesting questions (such as: in what order should this computer complete these tasks in order to minimize computation time?) can be answered using a linear program. The world was somewhat surprised and certainly pleased when it was proved no more than 30 years ago that linear programs can be solved in polynomial time. In this class, we will discuss what a linear program is, as well as the key ideas that go into the simplex method of solving linear programs.

Homework: optional.

Prerequisites: First week of "Linear algebra" (Week 1).

Related to: "Theoretical computer science" (Weeks 1–3), "Semidefinite programming" (Week 4).

Paradoxes in Probability and Decision Theory (Kenny, 10am, Th–Sa)
This course consists of three independent lectures. Each one of them begins with a famous paradox.

- *The Monty Hall Paradox* (∗, 10am, Th)
  You're on a game show, and there are three doors, one of which has a prize behind it. You pick one of the doors. To make things exciting, rather than just giving you what's behind that door, the host (Monty Hall) chooses a different door (which doesn't have the prize) and shows what's behind that. Now, he gives you a choice - do you stick with what's behind your door, or do you switch to the third door?
  Many people make the wrong decision here, perhaps because they're making a slight slip in interpreting how the game is set up. The moral of the story is that a slight change in the setup of your story can have a big effect on your correct choices, and that sometimes the *way* you learn something is as important as *what* you learn.
- *Bertrand's Paradox* (∗∗, 10am, Fr)
  Say you've got a circle with an equilateral triangle inscribed in it. What's the probability that a random chord in the circle is longer than the sides of that triangle?
  Say there's a factory that makes cubical boxes whose sides have lengths between 0 and 1 chosen at random. What's the probability that one of these boxes has volume less than 1/8?
  It turns out that the answers to these two problems are the same. The moral of the story is that when you say "random", you've said almost nothing.
- *Simpson's Paradox* (∗, 10am, Sa)
  In 1910, the following remarkable statistics were discovered, when comparing death rates from tuberculosis in New York and Richmond:
  (1) The death rate for African Americans was lower in Richmond than in New York.
  (2) The death rate for Caucasians was lower in Richmond than in New York.
  (3) The death rate for the total combined population of African Americans and Caucasians was higher in Richmond than in New York.
  Situations like this seem paradoxical at first, but can in fact be explained by some very simple facts about fractions, and arise surprisingly often.
  The moral of the story is that you have to be very careful when trying to read probabilities (or frequencies) as implying something about causation.

Prerequisites: none.
Homework: none.
Related to: "Probabilistic combinatorics", "Classical logic", "Information theory" (Weeks 1–2).

Probabilistic combinatorics (∗, M@, 1pm, week 1 of 1)
The probabilistic method is one of the most beautiful techniques available in discrete mathematics. It allows one to prove the existence of an object with desired properties, without having to actually construct it. The way we do this is to take a random object, then show that it has the desired properties with positive probability, and conclude that there must be at least one! In some cases this gives a much easier existence proof than constructive methods, and in others, there are no constructions known. In this one week course we will briefly survey this area, with particular attention to graph theory, Ramsey theory, and geometry.
Homework: optional.
Prerequisites: none.
Related to: "Combinatorics" (Week 1), "Introduction to graph theory" (Week 2).

Projective geometry (∗∗, D.A., 1pm, week 1 of 2 (Weeks 2 and 3))
We live in a three-dimensional world, but we like to draw things we see around us on flat, two-dimensional paper. How can you do this systematically and accurately, and how do your drawings

change as you change your point of view? Projective geometry has its roots in these questions, so it's no coincidence that many of its fundamental theorems are named after French architects.

Projective planes also turn out to be great places to do mathematics. Unlike Euclidean or hyperbolic planes, there are **no** parallel lines in a projective plane – any two lines always intersect. Theorems always come in pairs, often unifying several theorems from Euclidean geometry, and they can all be deduced from a few simple axioms. Notions like distance and "betweenness" don't matter much, but questions of incidence – whether three lines intersect in a common point, or three points lie on a common line – become central.

In the first week of the course, we'll examine some of the prettiest incidence theorems, sometimes using a little bit of linear algebra to prove them. We'll also learn to love the principle of duality, which lets you interchange lines and points.

In the second week, we'll look at strange variants of the projective plane. You can define numbers using projective geometry, and these numbers can be quite exotic – they might not be commutative, or even associative! We'll also try to understand some projective planes topologically, and get a glimpse of higher-dimensional projective spaces.

Talk to me if you're interested, but unsure about your linear algebra background – we can probably cover what's needed in a short crash course.

Homework: required.

Prerequisites: a little linear algebra.

Related to: "Hyperbolic geometry" (Week 1), "Linear algebra" (Weeks 1,3), "Vectors and matrices, spinors and tensors (Weeks 1–4)", "The fifteen theorem" (Week 2), "Complex analysis" (Week 2).

The stable marriage problem ($**$, Rob, 10am, Sa)

The emperor of Graphland issues a decree: everyone must get married right now. You, the court mathematician, are presented with the task of matching up every citizen with a suitable spouse, given everyone's preferences, and you want to make sure that no pair of unmarried people will want to cheat and run off with each other.

Knowing the Stable Marriage Algorithm, you know how to do this. But does this lead to true happiness for everyone involved? And what happens when Graphland gets more progressive and same–sex marriage becomes allowed?

Aside from this far–fetched story, you'll see how this algorithm is useful (and used) to solve real-world problems.

Prerequisites: none

Homework: none

Understanding infinity ($*$, Dan, 9am, week 1 of 1)

You can count, but can you count infinitely high? We'll begin by figuring out how to compare infinite sets. Are there more rational numbers than integers? Are there more real numbers than rationals? (Note: the answer to at least one of these is **not** "yes!") What does "more" actually mean when you're talking about infinite sets?

After we answer questions like those, we'll explore ordinals and cardinals, which in some ways act and look like numbers — but they count infinite sets instead of just finite sets, like the boring old numbers you're used to.

In addition to teaching some amazingly cool math, this class is going to teach you about how mathematicians use notation. If you've been in your Mathcamp classes and gotten confused at all the writing on the board, this class will help to clarify just how you translate from math-speak to intuition you can really get your hands on. We'll end the class with a look at different definitions in mathematics and how to understand what they're trying to tell you.

Homework: recommended, but not tons of it.

Prerequisites: none.

Related to: math.

Continuing classes with an entry point

Fermat's dream (∗∗∗∗, Miljan, 1pm, week 2 of 1)
If you want to join us in Week 2 in our discovery trip of the proof of Fermat's last theorem, please talk to Miljan.

Moore Method Topology (∗∗∗∗, M@, 10am, week 2 of 4)
This is a continuing course, and Week 2 is another opportunity to enter. We covered the basics of metric spaces in Week 1, and now we will leave the metrics far behind and start in on topological spaces and properties such as compactness and connectedness. We will inverstigate a few "pathological" spaces, such as the line with two origins and the Hawaiian Earring. Please talk to M@ in person if you'd like to join.

Vectors and matrices, spinors and tensors (∗∗∗, Anti, 1pm, week 2 of 4)
There is an entry point for this class at the beginning of the second week. If you already have something of a geometric feel for how vectors behave, and you want to see the abstract theory of vector spaces and linear transformations, feel free to join us this week. (You should be prepared for a few surprises, however, since we're treating some familiar vector operations a bit differently. Talk to me if you want to find out more.)
Prerequisites: some experience with vector algebra and matrices.

Visitor classes

Combinatorial geometry for problem solving (∗∗–∗∗∗, Ivan Matic, 1pm, Fr–Sa)
The task in these problems is proving the existence or non-existence of some crazy geometrical configurations in the plane or space. The beauty is that many of them are solved in the least likely among all the unlikely ways. This topic is closely related to some geometric inequalities, and I make no warranty whatsoever that we won't solve some other geometrical problems.

John Conway's Class (?, John Conway, 11am, We–Sa)
NTBA.

Xtreme geometry – Group construction (∗, George Hart, 11am, Tu)
We will be assembling a foam sculpture–puzzle. This assembly requires 15 people coordinated moving together, and it has never been tried before!

The Seven Millenium Problem Series

On May 2000, in order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts selected the Seven Millenium Prize Problems. They focused on important questions that had resisted solution over the years, and allocated an award of $1M for the solution of each one of them. In this series of colloquia we will present these problems to you, in the hope that you will some day solve them and endow Mathcamp with the award!

The Navier–Stokes equation (Ivan Matic, 4pm, Fr)
Traffic flow, the flow of a river or the air, and the behaviour of bees can all be described using partial differential equations. Unfortunately, these partial differential equations are most often unsolvable for mathematicians. Although nature always finds the way to solve these equations, in some cases it is very difficult for us even to prove the existence of a solution! The most famous such equation is the Navier–Stokes equation, which describes the flow of incompressible fluids.
During the class I would like to tell you something about the people who solved the problem, their plans for spending the money, and the main ideas used in the solution.

However, at the moment that I am writing this blurb, no one has found the solution yet. In case nobody solves the problem before the colloquium, I'll have to figure out an alternative way to use the time. I will tell you what is the form of this famous differential equation, what you are required to prove, and why modern mathematics can't make any serious attempt to solve this problem.

Staff Research Colloquium Series

In this series of colloquia, our staff members will tell you about their very own research!

Efficient approximation of convex bodies (Ellen, 4pm, Th)

A variety of questions in computation and optimization can be stated as questions about convex bodies. For example, questions like "in what order should this computer complete a set of computations to minimize the total running time?" can be answered by finding a vertex of a polytope. For the convex bodies associated with many interesting and important problems, including the one just mentioned, deciding whether a particular point belongs to the body is computationally hard. My research concerns finding an approximate solution which is easier to compute. This can be thought of as a convex geometry version of the $\mathrm{P}\stackrel{?}{=}\mathrm{NP}$ question. One can think about it in the following way: the membership question ("is a point x in the convex set X?") is in general NP-complete. Showing that any convex set can be efficiently well-approximated would point towards P=NP. Showing that any good approximation of a general convex set must be computationally hard would suggest P≠NP.

Brief visitor bios

John Conway (Princeton University – math)

One of the most creative thinkers of our time, John Conway is known for his ground–breaking contributions to such diverse fields as knot theory, geometry of high dimensions, group theory, transfinite arithmetic, and the theory of mathematical games. Outside the mathematical community, he is perhaps best known as the inventor of the "Game of Life".

Ivan Matić (UC Berkeley – math)

Ivan Matić is a graduate student at the department of mathematics at UC Berkeley, and a former mentor at Mathcamp. He became well known at Mathcamp 2005 for drawing 101 circles in a single lecture. His areas of interest include partial differential equations, probability, and analysis.

He also likes problem solving and, with several friends, he maintains the website `http://www.imo.org.yu` . He is a coauthor of the book *The IMO Compendium*, which contains problems and solutions from all previous IMOs and IMO shortlists. Look for it among the selection of Springer books that you may receive at the end of camp.

New staff bio

Kenny Easwaran (mentor, UC Berkeley, philosophy)

Kenny, whose hair has been every colour of the rainbow, is a fourth year PhD student in the Group in Logic and the Methodology of Science at Berkeley, a program jointly run by the math and philosophy departments. He started out primarily working on set theory and mathematical logic, but now his main research is on the philosophy of probability. In addition to set theory, his interests are primarily philosophical, about the role of proof and explanation in mathematics, as well as interests in philosophy of language and the theory of knowledge.