Class descriptions—Week 4, Mathcamp 2005

Contents

Attention! Continuing classes in the 9 am and 10 am slots have switched places again for this week (back to where they were in Weeks 1 and 2).

New classes beginning in Week 4

Bayesian Statistics; or, Don't Listen to Anything They Teach You in School! (∗∗, Mira, 10 am, all week)

Statistics is the science of analyzing data in the presence of uncertainty or with incomplete information. Since there is little in the world that is certain, and information is always scarce, we humans can't go a day without doing some kind of statistics – in our routine cognitive functions, in science, in the political arena, etc.

When you start studying it in school, statistics at first looks a lot like math. Yet if you've ever taken a statistics class, you might have felt your inner mathematician getting increasingly disgruntled and annoyed – and with good reason! Many mathematicians shun statistics as not being a legitimate branch of mathematics. Among the general public too, statistics has a bad rep. Try to imagine the famous quote by Mark Twain ("There are three kinds of lies: lies, damned lies, and statistics") applied to mathematics. Impossible! So what's the difference?

The difference is that, unlike math, statistics (as it is usually done) appears to be just a scrap book of arbitrary tests and procedures, with no basic underlying principle. This attitude makes it easy to come up with a variety of "lies", through either negligence or malice. But there are serious problems with statistics even when it is done carefully and honestly. For instance, if you look at what some of the standard statistical tests are actually measuring, you will find that most of them are not asking the questions that they're supposed to be answering. Instead they're measuring something related, but different and much more convoluted.

Why is statistics so screwed up? There are interesting historical and philosophical reasons, and we'll discuss them. The good news is: there is an alternative. There is a way of doing statistics which is really math, which doesn't substitute artificial questions for the questions you actually want answered, and which makes perfect sense every step of the way. We only have a week, so we may not get to a lot of the technical stuff. However, I hope to give you a sense of how Bayesian statistics works and to convince you that it's the way to go. More and more people are starting to notice this approach, and a revolution in statistics may well be on its way. Get in on the action early!

Prerequisites: Basic probability theory. Calculus is helpful but *not* a prerequisite. There may be points in the course where I take 5 minutes to do something using calculus (especially if a lot of people have the background), but even without calculus, you should be able to follow all the important stuff without a problem.

Homework: optional.

Related to: Josh Tenenbaum's "Probability and the Mind" – but this course will be less mind and more probability.

Elliptic Functions (∗∗∗∗, Mark, 11 am, all week)

If you like periodic functions (such as trig functions) of one real variable, wouldn't be even better if your functions had two independent periods? This can happen for functions of a complex variable; just as the trig functions cos and sin can be used to parametrize a circle, "doubly periodic" functions can be used to parametrize cubic curves. We'll construct such functions and if time permits (and/or in fifth week) see how they lead to modular forms - the very things Holly has been talking about in her class. [However, although they're closely related, the two classes are independent, so you can take this one without having taken the other.] These are things I've been meaning to get to the bottom of for a long time, and the class may well feel at times as if we're all helping each other figure things out. As a result, there may be some rough spots, but that may also make it more exciting!

Prerequisites: The Cauchy integral formula from complex analysis (or willingness to accept that formula).

Homework: none.

Related to: Advanced Partitions and Modular Forms.

Fractional Graph Theory (∗∗∗, Ari, 10 am, Tue - Wed only)

There are many basic numbers associated to graphs, but they're all totally boring. Chromatic number, matching number, independence number... all plain old integers. How dull! What did we do to deserve this? In this class, we'll proceed rationally (get it?) to some surprisingly elegant results. Might the Four Color Theorem be easier if we used eight half-colors? (I spent a year trying to answer this question. The answer is, "Probably not.")

Prerequisites: basic linear algebra.

Homework: none.

Game Theory (∗∗, Pedro, 11 am, all week)

What does a mathematician understand a game to be? What sort of games are there for which there is a best strategy? How do you find a best strategy for a game? Why does game theory have applications in economics, political science, biology, psychology, mathematical logic, and even analysis? If your curiosity is peaked, this is your invitation to come and play. Since we have a short time, we might not answer all of these questions in an adequate way, but we'll do our best to answer some of them. We will cover two-person zero-sum games and a little bit of two-person general-sum games. We might also have a guest lecture on evolutionary game theory.

Prerequisites: A little bit of Linear Algebra would help.

Homework: optional.

Related to: Convexity, Linear Algebra.

The Geometry of Voting (∗∗, Miranda, 10 am, Thursday - Saturday only)

While some people are disappointed with the outcome of a vote, many people are also frustrated with the voting system itself. Is there a voting system which encapsulates the true wishes of the electorate? Arrows Impossibility Theorem states that given a small number of very reasonable criteria, the only "fair" voting system is a dictatorship. Does this mean that democracy is a failure? Of course not. While a perfect voting system may not exist, some are definitely better than others. We will develop mathematical tools that will allow us to visualize a large variety of voting systems geometrically. Using these, we will explore many electoral paradoxes. We will attempt to identify other criteria with which we can evaluate voting systems, so that we can find the "best of the worst."

Prerequisites: none. *Homework:* none.

String Logic (∗∗, Anti, 4 pm, all week)

The logic we usually use in mathematics is quite boring: things can only be true or false. There are a wide variety of more interesting logics, such as many-valued logics, fuzzy logics, and probabilistic logics, in which a statement can be "half true" or "true with probability 1/3." In addition, there are stranger logics like quantum logic, which (if the physicists are right) is actually the logic of the real world.

It turns out that while the usual way of writing mathematical formulas is adequate for the usual logic, it becomes inadequate when dealing with stranger logics. There is an alternate notation, however, called "string notation," which (like string) is very flexible and powerful. We'll define string notation and use it to study fuzzy and quantum logics, including a brief introduction to Quantum ElectroDynamics and maybe even Topological Quantum Field Theory.

Prerequisites: basic symbolic logic ($\exists$, $\forall$, etc.)

Homework: optional.

Women in Math (Ellen, Holly, Mira, Shilpa, probably with many others; 4 pm, Fri - Sat only)

The first day of the class will be about the past. We'll talk about the three most famous female mathematicians in history (not counting the last 50 years or so): Sophie Germaine, Sofia Kovalevskaya, and Emmy Noether. We'll say a bit about their mathematical results (yes, there will be some actual math in the class!), as well as the historical context in which these women lived and worked. There also will be some opportunity for discussion; bring stories of your favorite female mathematicians of the past to share with the class.

The second day of class will be about the present and the future. Obviously, women doing math today do not face the same kinds of obstacles that their predecessors did; why then are they underrepresented at the top math departments (and, for that matter, at Mathcamp)? We will talk a bit about hurdles like stereotype threat (an important phenomenon to know about, even outside the context of women in mathematics). Then we'll take the bull by the horns. Earlier this year, Larry Summers, the president of Harvard University, sparked a nationwide controversy by hypothesizing that in our time, the relative scarcity of women scientists and mathematicians is due primarily to innate biological factors rather than to external social forces. After making this statement, Summers got caught up in a whirlwind of politics and trouble, but his claim is at heart a scientific one, which is either true or false. We will read and discuss a debate by two of the world's leading cognitive scientists, Steven Pinker and Elizabeth Spelke, about the scientific merits of Summers' position. There is a lot to unravel here – statistics vs. determinism, the search for justice vs. the search for truth, averages vs. standard deviations. When feelings run strong, as they did in this case, it is easy to ridicule and demonize one's opponents. It is therefore important to see what an intelligent defense of Summers' claim does and does not involve – and similarly for an intelligent objection to Summers' claim. Whichever side of the debate you find more compelling, there is a lot to learn from both of the arguments. We look forward to an interesting discussion!

Homework: We will expect the participants in Saturday's class to have the read the Pinker-Spelke debate for homework.

Continuing classes that welcome new students

Abstract algebra ($***$, Mark, 10 am, weeks 3 - 4)

If you've seen some group theory, you can join us now as we start exploring rings and fields. Although the class has just one more week to run, I intend to propose at least one class for week 5 which builds on abstract algebra - whether that class will actually run, will be up to you!

Homework: optional

Combinatorics ($*$, Jackie, 9 am, weeks 1 - 4)

This week we will finish generating functions. Then (probably on Wednesday or Thursday) we will start the principle of inclusion and exclusion. For example, we'll look at questions like:

- Suppose n married couples attend a party where everyone is seated around a large circular table for dinner. How many ways are there to seat the $2n$ people if no one can sit next to

his or her spouse? What if we also require that the people must alternate man, woman, man, woman, ... around the table?
- Suppose that n people check their coats at the theater, but due to a mix up, the coats are handed back "randomly." What is the probability that no one gets his/her coat back?

If you'd like to join the class for the principle of inclusion/exclusion, talk to Jackie to find out what day we'll start.

Prerequisites: Basic counting principles (not generating functions); talk to Jackie for details

Homework: required

Combinatorics on grids and graphs: the highlights (∗∗, Ivan, 11 am, Tue only)

[Ivan didn't get to say much about grids and graphs in Week 3, due to illness. Now he gets just one class to make up for it! Here is the original class description: he'll choose something from among these topics.]

Combinatorics on grids. Mathematical games on finite and infinite chessboards, partition of chessboards into poliminoes, existence of combinatorial configurations on chessboards. Solving these problems doesn't require any theory, but as we will see, the problems can be arbitrarily difficult and interesting.

Combinatorics on graphs. We will prove Ore's theorem for the existence of Hamiltonian cycle, whose proof involves one very non-standard idea!

Combinatorial geometry (completely unrelated to Anti's Week 2 class). Coloring an orange and partitioning a figure into congruent parts are just the beginning! Have you ever been asked: "Why do you like mathematics?" If you want an answer to this question that is acceptable to everybody, this classes will give it to you.

Inequalities (∗∗∗, Ivan, 4 pm, weeks 3-4)

[Since Ivan was sick for much of week 3, the class has hardly started meeting, so if you haven't been going, you can probably catch up pretty easily. Here is the description again:]

We will start with some problems involving inequalities that requires only clever ideas, with no heavy machinery. After that, we will study the inequalities of Holder, Minkowski, Yaung, Jensen, Schur, Muirhead, Tchebyshev, and Karamata, with geometric interpretations for some of those inequalities. We will also study generalization of inequalities for the discrete (and maybe even the hard-core) Lebesgue integral. If time permits we will look at applications of inequalities in probability theory.

Prerequisites: none.

Homework: optional.

Point set topology (∗∗∗∗, Alfonso, 9 am, weeks 1 - 4)

While sequences are often the motivation to define a topology on a set, they are not in general enough to characterize a topological space. Here are some pathologies:

- Two different topologies on the same set may have exactly the same convergent sequences.
- A sequentially continuous map may not be continous.
- An accumulation point of a sequence may not be the limit of any subsequence.
- A compact set may have a sequence without convergent subsequences.
- A topological space may have no sequence with more than one limit, and yet not be Hausdorff

All of these bad behaviors are easily taken care of when we substitute sequences with nets, their uncountable generalization, which are the *correct* way of talking about convergence in general.

Prerequisites: If you have learned some point set topology in the past, but do not know how to deal with nets or filters (or how to prove Tychonoff's theorem in a very easy way), join us in Week 4.

Visitor classes

Bernoulli Numbers (∗∗∗, Dave, 11 am, all week)

Ever wonder why the Taylor series expansions for $\sin x$ and $\cos x$ are in every calculus textbook, but the Taylor series for $\tan x$ is nowhere to be found? The series for $\tan x$ involves a sequence of rational numbers called the Bernoulli numbers, which are startling in their ubiquity and importance in number theory. Perhaps the most striking example involves the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$: there is an exact formula for the values $\zeta(2k)$ of the zeta function at even integers, involving the Bernoulli numbers. The denominators of the Bernoulli numbers are well-understood; we'll figure out what they are. The numerators are much more mysterious: for example, they can be used to explain one reason why Fermat's Last Theorem is hard.

Prerequisites: Taylor series, basic number theory (generating functions are a plus)

Related to: Remember John Conway's talk on day 1?

Combinatorial Topology (∗∗, Katherine Crowley, 11 am, Wed - Sat only)

The first meeting of this class is really the Tuesday colloquium (see description). In the class itself, we will continue our combinatorial approach to investigating topological results by introducing several fundamental ideas in combinatorial topology, including vector fields, phase portraits, and winding numbers. This will enable us to study theorems such as the Poincare Index Theorem and the Lefschetz Fixed Point Theorem. We will conclude with two combinatorial analogs of continuously shrinking a topological space to a point, called collapsing and shelling. In particular, we will look at one question related to collapsing that was only recently solved, and a very similar conjecture related to shelling that is still open.

Prerequisites: none.

Homework: optional

Related to: Sunny Collapse, all the topology classes

Incredible polyhedral sculptures! (∗, George Hart, Tue - Thur only)

- **3D and 4D geometry with Zometool:** (Tuesday, 4 - 5:30 pm). Explore cool things you can build with Zometool and what it means to build projections of 4-dimensional objects. Fun small projects and a good background for the giant construction project on Wednesday (but not required).
- **Wooden sculpture assembly:** (Tuesday evening, see calendar board in Main Lounge for time and location) Assemble a beautiful, complex polyhedral structure made out of wood – a different one from last year.
- **The cantellated 600-cell – a 3D world premier!:** (Wednesday, 2 - ?? pm). The cantellated 600-cell is a 4-dimensional polytope that, as far as we know, has never been seen in the 3D world – until we build its projection out of Zome today. The structure is likely to be about as tall as a person, and will use 1380 bals and more than 5000 Zome struts! **The building will start during TAU and will continue into the 4 pm class slot and beyond, if necessary.** Come for as long or as short as you want: if lots of people participate, we should be able to finish it in about 4 hours.
- **Do try this at home:** (Thursday 4 - 5:30 pm) A smaller-scale polyhedral construction project using only paper, scissors, and tape.

Colloquia

Note: Colloquiua do not have star ratings: speakers are instructed to make colloquia interesting and accessible to all Mathcampers. One of this week's "colloquia" is a movie; another is a talk by one of our hosts – a professor from the Reed math department.

An Introduction to Combinatorial Topology (Katherine Crowley, Tuesday)

According to the Brouwer Fixed Point Theorem in topology, no matter how you throw a map of Portland onto the ground (in Portland), some point on the map will lie directly above the point on the ground that it represents. This is a standard result using sophisticated topological tools usually acquired in one's first year of graduate school. However, this and many other topological theorems are readily accessible via a combinatorial approach, using nothing more complicated than vectors and limits. I will introduce some key ideas of combinatorial topology and show how they can be applied to prove the Brouwer fixed point theorem.

The Fundamental Theorem of Algebra (Dave Savitt, Wednesday)

The Fundamental Theorem of Algebra—the theorem that every polynomial has a root in the complex numbers—was surprisingly slippery to prove. The earliest published proofs, by d'Alembert in 1746 and Gauss in 1799, were eventually seen not to be completely rigourous. (Ironically, Gauss was even attempting to rectify d'Alembert's lack of rigour!) In this fine tradition, we won't give a complete proof either, but we will give most of one. In particular we will develop some of Gauss's ideas about the geometry of polynomials to the point where we completely understand why every polynomial has a root.

Prerequisites: Complex numbers, including de Moivre's Theorem that $e^{i\theta} = \cos\theta + i\sin\theta$. We'll try to have a class on this during TAU on Tuesday.

The Math Life (1-hour movie, Thursday)

[Official blurb:] Why did a magician become a mathematician? How can a person see in four dimensions? What does a mathematical proof have in common with a Picasso portrait? "The Math Life" brings to life the human dimension of mathematics through lively interviews with Freeman Dyson, David Mumford, Ingrid Daubechies, Persi Diaconis, Michael Freedman, Fan Chung Graham, Kate Okikiolu, Jennifer Tour Chayes, Peter Sarnak, Steven Strogatz, and seven other mathematicians. These captivating luminaries vividly communicate the excitement and wonder that fuel their work as they explore the world through its patterns, shapes, motions, and probabilities. Computer animations and analogies drawn from the visual arts are incorporated, to maximize accessibility to the fascinating concepts discussed.

It really is a pretty interesting movie!

Geometry and Number Theory on Clovers (Jerry Shurman, Friday)

Gauss determined the values of n for which the circle can be divided into n arcs of equal length by ruler and compass. His argument, using algebraic structure to solve a geometric problem, is the first proto-example of *Galois theory.* Abel solved the division problem for the lemniscate, and his method contains the beginnings of *complex multiplication* and even *class field theory.* The solutions found by Gauss and Abel involve *Fermat primes*,

$$p = 2^a + 1.$$

Pierpont solved the circle problem with origami in place of ruler and compass, and his solution involves *Pierpont primes*,

$$p = 2^a 3^b + 1.$$

This talk places these results in the general setting of m*-clovers* for all positive integers m, including the cardioid when $m = 1$, the circle when $m = 2$, the lemniscate when $m = 4$, and a three-leaf clover when $m = 3$ that apparently has not been studied until now. Analyzing origami division of the three-leaf clover and the lemniscate seems to require results from Galois theory, complex multiplication, and class field theory, in contrast to the elementary arguments by Gauss and Abel that led into these areas.

Brief visitor bios

Katherine Crowley (St. Olaf College)

Katherine Crowley studies topological spaces like spheres, Mobius strips, Klein bottles, projective planes, tori, Bing's house with two rooms, and dunce caps. To answer interesting questions about such objects, she takes advantage of the relationship between topology and geometry, while employing a combinatorial tool called discrete Morse theory. The perspective that comes from working at the intersection of three exciting fields leads to interesting and beautiful pictures that she looks forward to sharing at Mathcamp.

George Hart (SUNY Stony Brook)

George Hart is both a professor of computer science and a mathematical sculptor; his work can be seen at www.georgehart.com. He is also one of the world's leading experts on Zometool.

David Savitt (University of Arizona)

Dave Savitt isn't really a visitor: he is Mathcamp's deputy director (aka "Ze Top Blueberry"), and, together with Mira, this year's academic coordinator (even if mostly in absentia). He is also responsible for introducing the Puzzle Hunt to Mathcamp and is the main organizer of this year's Hunt. Like Mira, he did admissions, so he knows all of your names: come and introduce yourselves!

Dave was Mathcamp's very first mentor; this is his 10th Mathcamp! His research (which is what kept him away from us this summer) is in number theory and arithmetic geometry. He's also a mean Scrabble player. Oh, and he's Canadian; he says "zed"; don't be alarmed. :)

Jerry Shurman (Reed College)

Jerry Shurman isn't really a visitor either; he is a professor at Reed, so it's more like we're the visitors and he's the host! He has been sitting in on several of our classes and colloquia for the past few weeks and has been extremely helpful in getting us settled in at Reed. His research is on modular forms.