CLASS DESCRIPTIONS—WEEK 1, MATHCAMP 2005

Basic Number Theory ($**$, Mark, 10am, week 1 of 2)

For thousands of years, number theory has fascinated professional and amateur mathematicians alike. (Some of the amateurs, such as the 17th century lawyer Fermat and the modern-day theoretical physicist Dyson, are not to be underestimated!) Our old friends, the integers, have many beautiful and subtle properties. Although quite a few mysteries about them still go unsolved (and some spectacular open questions will likely be mentioned), we'll mostly look at some striking and important properties of the integers that *are* well understood. Several of these, long regarded as "pure" mathematics without much practical use, now have important applications in such fields as computer science and cryptography.

Prerequisites: none.

Homework: optional.

Required for: Sums of Squares (Week 3), and much of the rest of mathematics.

Related to: almost any course in algebra or number theory.

Basic Problem Solving ($*$–$**$, Dorin Andrica, 9am, week 1 of 5)

During the next five weeks we intend to discuss some useful problem- solving techniques: induction, contradiction, the pigeonhole principle, the extremal principle, invariants, basic identities and applications, simple ideas in proving inequalities, the computation of some important sums and products, some basic problems in number theory, and methods of solving some non-standard equations.

Prerequisites: none.

Homework: required.

Calculus and Numerics ($**$, Shilpa, 4 pm, week 1 of 2)

These days, when you hear the word "calculus", you probably think of derivatives and integrals – the mathematical theories of change and accumulation. But "calculus" actually comes from the Latin word for "pebble": historically, people used pebbles or stones to do calculations. In this class, true to the etymology of the word, we will not only study the theory of calculus, but also apply this theory on the computer to get actual numerical results. This isn't just a matter of plugging numbers into formulas. All of calculus hinges on the concept of a limit: what happens when you get "infinitely close" to the desired answer. That's all very well for mathematicians, but how do you tell a computer to get infinitely close? Perhaps you can replace "infinitely close" by "very close" or "close enough" – but how close, in practice, is close enough, and how do you tell?

This class will involve extensive use of the computer lab, with students writing programs based on concepts in calculus.

Prerequisites: basic computer programming (offered Week 1 during TAU)

Homework: required.

"Antirequisite": If you already know some calculus, you probably shouldn't come to this class: even though the approach is unusual, there won't be enough new material to keep you interested.

Required for: Conservation Laws (Weeks 3 and 4). Also helpful for Inequalities (Weeks 3 and 4), Dimensional Analysis (Week 3), Bayesean Statistics (Week 4).

Calculus Without Calculus ($**$, Brenda, 11am, Tue-Wed)

If you've ever taken a calculus class, you've almost certainly seen certain types of problems. Without a doubt, you've learned how to find the equations of tangents to curves. In all likelihood, you've learned how to maximize an area with a given shape and perimeter, and minimize the perimeter of a region of given shape and area. You've probably also seen the ol' "swim-and-run" problem of finding the route that minimizes the amount of time it takes to swim to the shore, and then run to a certain place on land. As it turns out, all of these problems – and more – can be solved without evaluating a single limit or derivative. In this brief course, we'll exploit the geometric properties of diagrams, and we'll explore some powerful inequalities that let us solve optimization problems swiftly and elegantly. Come to Calculus Without Calculus, and learn the math your calculus teacher doesn't want you to know. Previous knowledge of calculus isn't necessary, and in fact, those who have taken a calculus class before may find themselves distressed to learn just how much calculus they've used unnecessarily in the past.

Prerequisites: none.
Homework: optional.
Related courses: hmmm... calculus?

Combinatorics (by guided discovery)($*$, Jackie, 4pm, week 1 of 4)

Enumerative combinatorics is all about the art of counting. We will be thinking about questions such as:

(1) How many ways are there to rearrange the letters in the word MATHCAMP?
(2) In how many ways may we distribute k identical books onto the shelves of a bookcase with n shelves? (There are no limits on the number of books per shelf.)
(3) The Fibonacci sequence begins 1,1,2,3,5,8,13,... with each consecutive entry being the sum of the previous two. Find a formula for the n-th entry.
(4) A group of n married couples come to a party and sit at a round table. In how many ways can they sit so that no person sits next to his/her spouse?

Working in a group with other students, you will get to discover for yourself the theorems and big ideas of elementary combinatorics. Come learn to count!

Prerequisites: none.
Homework: required.
Related courses: Enumeration Celebration (Weeks 3 - 4) covers more advanced combinatorial techniques. Two weeks of this course will ensure that you have the necessary background.

Circles and Inversion ($**$- $***$, Ivan, 4 pm, week 1 of 2)

We start, simply, with circles – a much more powerful tool in geometric problems than many people realize. We will see some very complicated problems that can be solved by applying a simple theorem about circles - if only you can find where the circle in the problem is hiding.

We then investigate a transformation in geometry called *inversion*, which is analogous to symmetry with respect to a line. Inversion is a symmetry with respect to a circle. To understand the analogy, we will need to develop the notion of conjugated points and learn some basic projective geometry. We will get to some interesting constructions requiring only a ruler (no compass!), such as constructing the tangent line to a circle. The final goal of the course is to convince you of the following statements (one formal, one not):

- A line is really just a circle of infinite radius whose center is infinitely far from our eyesight.
- All constructions in geometry that can be done with a ruler and compass can be done with just a compass.

Prerequisites: high-school geometry, the basics of vectors (we can go over these as necessary)
Homework: required.
Related course: A Taste of Projective Geometry

Convexity ($**$–$***$, Ellen, 9am, week 1 of 1)

A convex set is one which doesn't contain any "dents": if two points are contained in a convex set, then any point on the interval between them must also be in the set. Can you think of 3 convex sets in $\mathbb{R}^2$ such that the intersection of any two of them is nonempty but the intersection of all of them is empty? How about 4 convex sets in $\mathbb{R}^2$ such that the intersection of any 3 of them is nonempty but the intersection of all of them is empty?

The magic of convexity makes the first possible and the second impossible. In this one-week course, we will discuss convexity and the surprising powers this simple concept holds. After this week, we will move on to polytopes, which are the higher dimensional analogue of polygons. (For a sneak peak at polytopes, see Sam and Matt's course this week!)

Prerequisites: If don't know linear algebra, it is highly recommended that you take it concurrently. I will state all the results from linear that you need to know, but they will make a lot more sense to you if you have a little background in the subject.

Homework: Required.

Related courses: This course can serve as a good introduction to the Polytopes course starting in Week 2, but is not required that course. It is also related to Sam and Matt's Polytopes course (Weeks 1 and 2).

If it is Fair, it is Nim($**$, Alfonso, 9am, week 1 of 1)

We have some baskets of apricots in front of us. We take turns eating the apricots. On your turn, you can eat as many as you want, as long as they are all from the same basket. Then it is my turn to do the same. The winner will be the one who eats the last apricot. Apart from getting a stomachache, will you be able to beat me?

This course will study combinatorial game theory, more specifically impartial games. An example is Nim, the apricot game described above. We are not interested in playing games (although apricots are delicious), but rather on analyzing games, obtaining winning strategies and understanding why they work. We will prove one very surprising result: any impartial game is Nim is disguise. Among other things, in this class, you will learn everything you need to know in order to solve problem 10 on the Qualifying Quiz.

Prerequisites: none.

"Antirequisite": If you already know the mex rule and that every impartial game is a nimber, please do not come to this class. If you just know how to win at nim, you are still welcome to join the class.

Homework: optional.

Introduction to Partitions($**$–$***$, Holly and Jackie, 9am, week 1 of 2)

A *partition* of a positive integer n is a nonincreasing sequence of numbers that add up to n. For example,

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1,$$

so there are five different partitions of 4. Many seemingly simple problems in partition theory are difficult to solve in simple ways. For example, the number of ways to partition n into odd parts is equal to the number of ways to partition n into distinct parts. This is easy to check for small n (try it!) – but why is it true? It turns out that there are some pretty sophisticated ways to handle this and other problems about partition identities, using generating functions. Some even simpler-sounding questions about the number of partitions of n (e.g. how fast does the number of partitions grow as n gets large?) turn out to be still more difficult and will have to wait until Advanced Partitions and Modular Forms (Weeks 3 and 4).

Prerequisites: none.

Homework: required.

Related courses: Advanced Partitions and Modular Forms (Weeks 3 and 4)

Linear Algebra ($***$, Mira and Sam, 10 am, week 1 of 2)

Linear algebra is one of the most useful methods in mathematics, rich in applications both to the real world and to pure math. One could argue that most of what mathematicians (and physicists, and engineers, and economists) do with their time is try to reduce hopelessly complicated non-linear problems to linear ones that can actually be solved. Thus for many applied fields, the most important math to know is not calculus, but linear algebra.

In this class, we will try to give you some feel for the subject: a bit of the basics, a glimpse of some more advanced topics (eigenvectors and eigenvalues), with hopefully time for a few fun applications – anything from ecology to Google!

Prerequisities: none.

Homework: required.

Required for: Sums of Squares (Week 3). Also very helpful background for the two courses on polytopes.

The Method of Archimedes ($*$, Sam, 1pm Fri)

This is still the most beautiful piece of mathematics I have ever seen. Using physical reasoning, involving balancing solids on a lever, Archimedes guessed the volume of a sphere, the center of gravity of a hemisphere, the volume of a paraboloid of revolution, the volume of the intersection of two cylinders inscribed in a cube, and much, much more...

In the language of modern calculus, you could say that Archimedes's method was to replace a three-dimensional integral that no one could do with a four-dimensional integral that looked even harder, which he then solved through a miraculous leap of cleverness. And he did this nearly two thousand years before calculus was invented!

We will retrace Archimedes's steps, starting from the area of a triangle and the area of a circle. No calculus necessary.

Prerequisites: none.

Homework: none.

Model Theory ($****$, Pedro, 4pm, week 1 of 4)

This course is an introduction to mathematical logic via model theory. Let's start with a quote from Bruno Poizat:

> "There could be no mathematics without numbers, and a mathematician who knew how to count to a hundred, and even beyond, without error, would never be considered a specialist in [number] theory. We can likewise say that basic model theory is to logic today what simple arithmetic is to mathematics [...] Model theory is the least logical branch of mathematical logic, yet it is the branch whose foundations no logician can ignore."

A word of caution with regard to the word "model": scientists in general tend to use the word model in a very different way than logicians. For example, a model of the stock market could be a computer program that attempts to simulate what goes on in a real-world stock market: the stock market is the real thing, and the model is the representation of the real thing. When a logician talks of models, though, she is refering to the real thing. The representation of the real thing is what is called a "theory", or a set of sentences in an appropriate "language". It is sometimes the case that we can pin down a nice, elementary language in which we can write sentences (usually called axioms) describing a certain kind of structure. Say, in the case of groups, a suitable language includes a name for the neutral element of the group, and a symbol for the group binary operation. We can write down elementary "logical" sentences in this language that are true in all groups; we call this the "theory" of groups, and each group is now called a model of this theory.

The goal of model theory is to come to as complete an understanding as possible of the family of mathematical structures satisfying some set of axioms, and of the relations between structures and the sentences true in them. One neat application of model theory is the construction of structures that are richer than the one we started with, but that satisfy the same elementary properties. For example, we can build the field of the hyperreals, which contains the real numbers, and in which there are infinitesimals and infinitely large numbers. The important feature of this field is that it satisfies something called the *transfer principle*, which allows us to prove statements about the real numbers using the bigger structure which were not so easy to prove with out it.

Prerequisites: set theory (can be taken concurrently), previous experience with abstract mathematics.

Homework: required.

Multivariable Calculus Crash Course ($***$, Mark, 11am, week 1 of 1)

In real life, most quantities depend on several variables (such as the coordinates of a location, the time, the temperature, etc.). As a result, ordinary (single-variable) calculus isnt enough to solve most problems. In this class, youll see some of the basics of calculus of several variables, and as time permits some cool applications in- and outside math, such as:

- if youre in the desert, what direction should you head in to cool off as quickly as possible?
- what is the total area under a bell curve?
- what force fields are consistent with energy conservation?

I also hope to cover Greens Theorem, which will be needed (although you can of course take my word for it) in next weeks class on complex variables.

Prerequisites: single-variable calculus.

Homework: optional.

Required for: Complex Analysis (Week 2)

Related courses: Advanced Partitions and Modular Forms (Weeks 3 and 4), Conservation Laws (Weeks 3 and 4)

Olympiad Problem Solving ($****$, Dorin Andrica, 10 am, week 1 of 5)

This is a continuing problem-solving class intended for experienced problem solvers. Topics we will cover using problems given in various contests and olympiads include: inequalities, geometric inequalities, polynomials, complex numbers, Diophantine equations, basic invariants of a triangle, barycentric co-ordinates and applications, complex numbers and geometry, advanced problems in trigonometry. Some of the above topics are used in MOSP and in the training of the Romanian IMO team.

Prerequisite: problem-solving experience.

Homework: required.

Orbispace ($**$, Anti, 9am, week 1 of 1)

We all know that we live on the surface of a sphere: the Earth. That implies, for example, that if you were to set out in any direction and walk in a straight line (disregarding minor problems like oceans getting in the way), you would eventually get back to your starting point. But what if, instead, we lived on the surface of a donut? Or a cylinder, or a cone? If we were two-dimensional creatures and lived *in* the surface, what would we see? How could we determine whether we lived on a sphere, or a donut, or a more complicated shape? What does the point of a cone look like from *inside* the surface of the cone? We'll also explore some even stranger spaces we could live in, like Klein bottles and pool tables.

Prerequisites: none.

Homework: required.

Related courses: point-set topology, combinatorial topology (Week 4)

Proof Techniques ($*$, Dan, 11am, week 1 of 1)

First of all: you should absolutely take this class if you haven't seen any of: proof by induction, the pigeon hole principle, or proof by contradiction. They're essential at camp. You should also take this class if you're not comfortable with writing proofs.

On to the description: Proof techniques is going to be a romp through some of the most amazingly elegant proofs of elementary mathematics. We're going to warm up with some old stand-bys you might be amazed to find out that you can rigorously prove (if you haven't seen it before): that there are infinitely many primes, and that the square root of two is irrational. ¿From there, we'll explore some really beautiful arguments in number theory, geometry, combinatorics, and a bit of graph theory.

This class will teach you how to form a good mathematical argument, and (through daily homework assignments) it will also teach you how to communicate that mathematics with others. You might take this class because you need it, you might take it because it'll be fun, or you might take it because you'd like to improve your writing skills: in all cases, you are welcome to it!

Prerequisites: none.
Homework: required.
Required for: everything!

Point-set topology (∗∗∗∗, Alfonso, 9am, week 1 of 4)

You probably know that the sequence $1, 1/2, 1/3, 1/4, \ldots$ converges to 0. This means that the elements of the sequence are closer and closer to 0. But what does the sequence *Alex, Aaron, Justin, Michelle, Lisa, Alan,* ... converge to? To answer this question, I need to tell you what *closer and closer* means in the space of campers; that is, I need to define a topology on the space of campers.

A topology is one of the most general structures we can put on a space. Topology studies "rubber spaces", because we are allowed to bend and twist, as long as we do not cut and glue. Actually, a topologist is defined as somebody who cannot distinguish between a cup and a daugnought.

Point set topology is a field where everything derives logically from axioms. Starting from the basics, we will quickly learn how to deal with the most abstract spaces, and how to not to take anything for granted. Nice well-behaved spaces are boring. During weeks 1, 2 and 3 we will learn the basics, so that in week 4 we can attack pathological topological spaces, where nothing is like what you are used to, and where we can get past sequences to the land of nets and filters.

Prerequisites: If you are not fluent with cardinalities, take Set Theory in week 1.
Homework: required.
Related courses: anything topological!

Set Theory (∗∗, Anti and Pedro, 11 am, week 1 of 1)

At a very young age, we learn to count finite sets, and decide that a set of three blocks and a set of three people share a property of "three-ness." But how do we count infinite sets? Perhaps you've seen a proof that you can't count the real numbers with the natural numbers, but how many different kinds of infinity are there? How do we add and multiply infinite numbers? Can we use proofs by induction for them? And how big do they really get? At the heart of set theory is the study of infinity, and the idea that no matter how high you count, you can always count still higher.

Prerequisites: none.
Homework: required.
Required for: point-set topology, topos theory, model theory.

A Taste of Projective Geometry (∗∗, Brenda, 11am Th-Sat)

Welcome to the real projective plane - where points and lines are interchangeable, and where the surprising power of a simple straightedge - without compass - is revealed. The projective plane, which has its roots in 17th century art, is an extension of the Euclidean plane, and contains a point of intersection for every pair of parallel lines. We'll show how this new space, while behaving quite differently from any Euclidean one, provides us with some elegant and powerful methods for investigating the familiar Euclidean plane.

Prerequisites: none.
Homework: optional.
Related courses: Circles and Inversion.

Theoretical Computer Science($***$, Dan, 10am, week 1 of 3)

In a lot of computer science courses, one studies *algorithms:* finite processes that output an answer to a given (inputted) problem. We're going to think bigger. We're going to ask "what can a computer do, period?" Then we're going to ask "what can a computer do quickly," trying to determine what problems can be solved in the real world.

Moreover, this will all be mathematically rigorous: we'll define a version of a computer (a Turing machine) using only mathematics, and use it to study the fundamental question sof computation.

Within a couple of weeks, I expect to cover the boundaries of computation (decidable and recognizable languages) as well as the great open question of computer science: is it true that $P \neq NP$? This problem is one of Clay Math's "Millenium Prize Problems," and worth a million dollars to whoever solves it. After that, we'll start exploring some more advanced topics, altering our model of computation (probabilistic computation and interactive proofs) and also considering what happens when we limit different resources of computation, such as space, but also alternation.

Prerequisites: none.

Homework: mostly optional.

Zoology of Polytopes: You think you know, but you have no idea. ($***$, Matt and Sam, 4pm, week 1 of 2)

The theory of polytopes, which include polygons, cubes, octahedra, dodecahedra, etc., and their higher dimensional cousins, is one of the oldest subjects in mathematics, and has been one of the most exciting and active fields of research in recent years, with deep connections to algebraic geometry, topology, combinatorics, and "mirror symmetry" in physics. This class will be a "zoology" of higher-dimensional polytopes. We will be constructing interesting examples with special attention to their combinatorial properties and numbers of faces. Some of the beasts in our zoo may surprise you!

Prerequisites: Some experience with linear algebra might be helpful, but is not required.

Homework: There will not be homework for this class, but come prepared to work on problems in class.

Related courses: This class will be symbiotic with Ellen's sequence of classes on convexity and polytopes. Take both, and feel the synergy!

What's next: a preview of selected classes from Weeks 2 - 4

Abstract Algebra ($**$– $***$, Mark K, Weeks 3 and 4)

Why can't fifth-degree polynomial equations, like $x^5 - 3x + 17 = 0$, be solved using something like the quadratic formula, although fourth-degree equations can? How can you analyze the symmetries of geometric figures, like star polyhedra, or the workings of Rubik's cube? How do physicists predict the existence of certain elementary particles before they test their predictions by expensive experiments? Why can't you trisect an arbitrary angle using straightedge and compasses (after all, bisection is easy)? To answer any of these questions you need abstract algebra: the study of groups, rings, and fields. This class offers an introduction to that beautiful area (alas, we may not get beyond group theory), which to most mathematicians is to high-school algebra as chocolate is to cardboard. Tastes differ, of course, but chances are you'll like abstract algebra!

Prerequisites: none.

Related courses: Groups via Generators/Relations (Week 2)

Advanced Partitions and Modular Forms ($****$, Holly, weeks 3 and 4)

Combinatorics meets complex analysis! Welcome to the fast and furious world of modular forms, which have applications in many areas of number theory, geometry, and analysis. Modular forms are analytic functions on the upper half of the complex plane that have sneaky and interesting transformation properties. In particular, we will see that modular forms are an incredibly powerful tool when applied to the study of partitions. For example, the number of partitions of 7,000,005 is huge and no one has ever calculated it, but we know that it must be a multiple of 7. Similarly, the number of partitions of 4,999,999,004 is a multiple of 5. These are special cases of the amazing Ramanujan Congruences which will be our culminating result in this class.

Prerequisites: Complex Analysis (Week 2) or good understanding of Taylor series

Related courses: Partitions (Weeks 1 and 2)

Complex Analysis ($***$, Mark, Week 2)

This class will explore some of the spectacular changes that occur when you have been doing calculus (differentiating and integrating) and you allow your variable x (now to be called $z = x+iy$) to take on complex values. As time permits, we'll look at some of the nice applications, such as the fundamental theorem "of algebra" and a method to compute certain improper integrals. If you like multivariable calculus, you should love this class!

Prerequisites: Multivariable Calculus (Week 1)

Related course: Advanced Partitions and Modula Forms (Weeks 3 and 4)

Conservation Laws and Traffc Jams ($***$- $****$, Shilpa, weeks 3–4)

What do waves and the flow of traffic on a busy road have in common? They are both modeled using partial differential equations derived from conservation laws – whether conservation of water or conservation of cars. The equations in the two cases are different, but related; the two problems are, in a sense, inverses of each other. By analyzing these equations, we can give a mathematical interpretation for many real-world phenomena. For instance, we will see that traffic jams are analogous to floods, and will model the way a traffic jams can arise seemingly out of nowhere.

Prerequisites: calculus. Multivariable calculus may be somewhat helpful, but is not at all required.

Polytopes ($***$, Ellen, weeks 2–4)

In 2D, they're called polygons; in 3D, polyhedra; in 4D, polychora; in general, polytopes!

If you like combinatorics, theorems where pictures provide the intuition, and the idea of studying geometry in n dimensions, this is your chance. Meant as an extension to the convexity course in Week 1, for the next three weeks we will be discussing the combinatorial properties of polytopes, and the crazy things that can happen when a convex hull is not a polytope (a face of a face which is not a face!)

In particular, we will look at the f-vector of a polytope P, which is a vector listing the number of faces that P has of each dimension. We will be working towards the beautiful duality theo rem, which states that for any polytope with f-vector $(x_0, x_1, \ldots, x_{d-1})$, there exi sts a polytope (the dual) whose f-vector is $(x_{d-1}, \ldots, x_1, x_0)$. I hope to end with Steinitz's theorem, a surprising result stating that any 3-connected planar graph is the 1-skeleton of a polytope.

Prerequisites: Linear Algebra (Weeks 1 and 2) and Convexity (Week 1) are highly recommended.

Related to: Zoology of Polytopes (Weeks 1 and 2)

Topos Theory ($****$, Anti and Ari, marathon, week 3)

Long ago, people believed nothing existed outside the earth we live on, making life here unique. But eventually, other planets and other solar systems were discovered. This raised the possibility that life could exist on other planets, although it might be very different from life on this planet and appear quite strange at first, and also the possibility that we could travel between planets and benefit life on both worlds by the interaction. Recently, similar possibilities have been raised with respect to entire different universes.

The word "topos" is Greek for "place" or "space" and is the analog in mathematics of a planet, or a universe. Just as nearly all life we know lives on earth, nearly all mathematics we know lives in a single topos: the topos of sets. For a long time, it didn't even occur to most mathematicians that there could be other toposes, other "universes" in which to do mathematics, and not until the development of formal logic and category theory in the mid-20th century did the language even exist in which to ask the question. It turns out that the answer is Yes! Not only are there many other toposes, we can travel between them, and the mathematics in both is benefited thereby. However, the mathematics in these "alternate worlds" can sometimes be quite strange and barely recognizable.

In this class, we'll explore the basics of topos theory, starting with introductions to the necessary logic and category theory and then developing the theory of toposes and similar worlds. Our main example, other than the familiar topos of sets, will be a topos called "the nonstandard topos." The nonstandard topos is one of the easiest to understand, as toposes go, but it is quite strange enough. Its main difference from the topos of sets is that there exist real numbers which are "infinitely small" and "infinitely large." One benefit of this is that by traveling back and forth between the topos of sets and the nonstandard topos, we can do calculus without limits, by simply changing our variables by an infinitely small increment.

This will be a "marathon" class, meaning it will take up almost every class period during week 3. It will also be largely Moore Method, meaning that for much of that time, rather than listening to us lecture, you will be either working (individually or in groups) on the proofs of exercises and theorems we hand out, or presenting your proofs to the rest of the class for comments and critiques. This will require a lot of work, energy, and dedication, on our part as well as on yours, but it is a spectacular way to dive deeply into a subject and come to a deep understanding of what is going on. Due both to the subject matter and the format, expect this to be a *very* hard class—but, we hope, just as rewarding!

Prerequisites: experience with mathematical abstraction and lots of stamina!