WEEK 5 CLASS PROPOSALS, MATHCAMP 2018

Abstract. We have a lot of fun class proposals for Week 5!
Take a read through these proposals, and then let us know which ones you would like us to schedule for Week 5 by submitting your preferences on the appsys `appsys.mathcamp.org`. We'll keep voting open until sign-in on Wednesday evening. Please only vote for classes that you would attend if offered. Thanks for helping us create the schedule!

Contents

Aaron's Classes

Braid groups and the fundamental group of configuration space. (🌶🌶🌶, Aaron, 1 days) Define C_n to be the space of n-tuples of distinct points in the complex plane. In this class, we'll explain how to determine the fundamental group of C_n and what this has to do with polynomials.

Homework: None

[1] What is the number of subspaces of a vector space over $\mathbb{Z}/p$? How many invertible matrices are there over $\mathbb{Z}/p$? How many square-free polynomials are there over $\mathbb{Z}/p$? In this class, you'll work through exercises where you figure out how to answer questions like this.

Prerequisites: the fundamental group

Counting points over finite fields. (🌶🌶🌶, Aaron, 2–3 days)
Some might say this class is pointless. They would be wrong.[1]
Homework: Recommended
Prerequisites: Linear Algebra, familiarity with matrices, familiarity with addition and multiplication modulo primes.

Dimensional Analysis. (🌶–🌶🌶, Aaron, 1 days)
We'll give some examples of how dimensional analysis can let you find geometric formulas. For example, we'll be able to find the area of a triangle in terms of its side lengths, the area of an ellipse, and even use dimensional analysis to approximate the ratio of the density of the moon to the density of the sun.

Homework: None
Prerequisites: None.

Finite Fields. (🌶🌶🌶, Aaron, 3 days)
Define finite fields: fields having finite member number. Primal result: exists unique finite fields having member number equals primes raised toward powers.[2]
Homework: Required
Prerequisites: linear algebra, ring theory

Math is Applied Physics. (🌶, Aaron, 1 days)
You may have heard that physics is applied math, but math can also be applied physics! In this class, we will see how to use physics to prove the Cauchy-Schwartz inequality. Time permitting, we will also see how to use physics to prove the Pythagorean theorem.
Homework: None
Prerequisites: Willingness to trust your physical intuition. Some familiarity with physics (such as the notion of torque) would be useful.

Riemann-Roching out on the projective line. (🌶🌶–🌶🌶🌶, Aaron, 1–2 days)
Did you know Riemann was a rock star? Come to this class to hear some of his greatest hits![3]
Homework: Optional
Prerequisites: familiarity with complex numbers and polynomials

The Art of Live-TeXing. (🌶, Aaron, 1 days)
Have you wanted to type math notes in real time but were too intimidated by the prospect? If so,

[2] What do the rational numbers, complex numbers, and real numbers have in common, but not share with the integers? They are all fields; we can add, subtract, multiply, and divide elements in them. But which finite sets also have these properties? What possible sizes can such a finite set have? What are the possible subfields? These questions all have simple, beautiful answers which we will present in this course. Finite fields are crucially used throughout number theory, algebraic geometry, cryptography, and coding theory.

[3] How many rational functions are there which blow up at 3 and 7? You might guess there infinitely many. But if you work with the projective line instead of the complex numbers, there turn out to be a 3 dimensional space of such polynomials. In this class, we'll learn a simple formula to count how many rational functions there are with poles and zeros as specified points.

this is the class for you. I'll describe some tips and tricks to make live-TeXing a feasible, and even enjoyable, experience. Students are encouraged to bring their laptops to class.

Homework: None

Prerequisites: Minimal prior experience with latex would be helpful.

The dimension formula for modular forms via the moduli stack of elliptic curves. (🌶🌶🌶🌶, Aaron, 1 days)
Dimension usually measures how "big" something is (somehow). Modular forms are things from complex analysis that you might have met in Shiyue's class already. Stacks are probably things that you pile up and I'm pretty sure that elliptic curves show up in cryptography or something. Were any of those sentences true? What happens when you pile them all together? I have no idea at all but there's a formula![4]

Homework: None

Prerequisites: Shiyue's week 2 Modular Forms class, linear algebra

Aaron, Vivian's Classes

Galois Theory and Number Theory. (🌶🌶🌶🌶, Aaron, Vivian, 2–3 days)
Can you find an example of a polynomial which is irreducible over the integers but reducible mod every prime?

In this class, we'll see what this question has to do with Galois theory. We'll also investigate many other relations between Galois theory and number theory.

Possible further topics include:

(1) Galois groups of finite fields
(2) Cyclotomic extensions
(3) Kummer Extensions
(4) Extensions of local fields (and the infamous secret society of p-adics)
(5) Inseparable extensions
(6) Purely inseparable extensions
(7) Perfect fields

Homework: Recommended

Prerequisites: Galois Theory, some background in algebraic number theory may be helpful but it's not necessary that you took the week 1 class on algebraic number theory

Ania, Jessica's Classes

Cycles in permutations. (🌶🌶–🌶🌶🌶, Ania, Jessica, 2–3 days)
Let's take a random permutation. What is the average number of fixed points in it? What is the average number of cycles? What is the probability that is has exactly one cycle? What is the probability that elements 1 and 2 are in the same cycle? Come to the class to find out and solve riddles about prisoners and planes!

Homework: Optional

[4] The above blurb was written by a Ben for potential amusement value. Here is a more informative blurb:
Did you take Shiyue's week 2 class on modular forms, but find the proof of the dimension formula for modular forms a bit analytic for your liking? It turns out there is an algebraic proof by interpreting modular forms as sections of a certain line bundle on the moduli stack of elliptic curves. While there won't be enough time to precisely explain what is going on, we'll try to give some intuition for how the algebraic proof works. Although the words in the title may seem scary, the proof ultimately boils down to an elementary computation involving taking floors of multiples of 1/2 and 2/3.

Prerequisites: None.

Apurva's Classes

Galois Correspondence of Covering Spaces. (🌶🌶🌶, Apurva, 2 days)
Galois Correspondence shows up not only while studying algebra, but also in topology while studying covering spaces. In this class we'll understand this correspondence and also look at the construction of the universal covering space.
Homework: Recommended
Prerequisites: Should know the definition of Fundamental Group

MP3 $\neq$ JPEG. (🌶🌶, Apurva, 1–2 days)
Fourier analysis can be used to analyse and compress sound, but it totally sucks when it comes to images. Instead, for images one uses a refined version of Fourier analysis called Wavelet analysis. In this class we'll understand the problem with Fourier transforms and see how Wavelet transform solves it.
Homework: None
Prerequisites: Linear algebra, basics of Fourier analysis (eg Jeff's class)

The Quantum Spring. (🌶🌶–🌶🌶🌶, Apurva, 2 days)
How do groups and algebras show up in physics? What does the set of 2×2 trace 0 matrices have anything to do with the quantum mechanics? How do Linear Operators create and annihilate particles? Why is Linear Algebra the answer to all your prayers?
Homework: Optional
Prerequisites: Linear Algebra, specifically, Eigenvalues and Eigenvectors

There isn't enough Homological Algebra in the world. (🌶🌶🌶, Apurva, 2 days)
Linear algebra gives us the illusion that the world is an ideal place, homological algebra breaks that delusion. In this class, we'll understand the failure of exactness of tensor products and learn about the Tor and Ext functors.
Homework: Optional
Prerequisites: You should know how to compute homology of chain complexes.

Would I ever Lie Group to you? (🌶🌶, Apurva, 2 days)
Lie Groups are groups which are also manifolds. The easiest examples of Lie groups come from Linear Algebra as symmetries of vector spaces. We'll study these Matrix Groups and understand their connections with their geometry and physics.
Homework: Optional
Prerequisites: Linear algebra, Group Theory

Ben Dees's Classes

Bairely Sensible. (🌶🌶–🌶🌶🌶, Ben Dees, 1–2 days)
The Baire Category Theorem was not proved by bears and it doesn't involve category theory, but it is still an interesting and useful theorem. Essentially, what it says is that some spaces are too "big" to be the (small) union of "small" subsets of them. The statement (even when said fully formally) seems strange, but it turns out to be surprisingly useful.

In this course, we will begin by stating, explaining, and proving the Baire Category Theorem. After this, we shall begin to use this theorem as a hammer in search of nails. That is, we'll discuss a lot of results that can be proved from the Baire Category Theorem, which are often quite interesting. For example:

How do you prove that there are nowhere-differentiable functions approximating any continuous functions? The Baire Category Theorem.

There are functions that are discontinuous on the rationals and continuous on the irrationals. Are there examples of the reverse? The Baire Category Theorem provides the answer.

We'll investigate these topics and more, always looking for other interesting questions to which the answer is "The Baire Category Theorem."

Homework: Optional

Prerequisites: Some familiarity with metric spaces. Jeff's Metric Space Topology class provides more than enough background; Ben's Chaotic? Good! class also provides sufficient context about metric spaces. Ask if you're not sure and want some notes!

Finite Field Fourier Fun. (🌶🌶🌶–🌶🌶🌶🌶, Ben Dees, 1–2 days)

For functions on the real line, the Fourier transform (in some sense) decomposes them as sums of frequencies. In finite fields, this sense of frequency decomposition is perhaps more merely formal, but we can still sensibly define something that looks like the Fourier transform.

This class will look at a few theorems about the Fourier transform on finite fields. Facts about Gauss sums, Plancherel's inversion formula, and other fascinating results will be the focus of this course. Finding applications of these formulae to geometric combinatorics will be another possible topic of this course.

Homework: Optional

Prerequisites: Familiarity with finite fields

What is a "Sylow" anyways? (🌶🌶–🌶🌶🌶, Ben Dees, 2–4 days)

Recently, in 2004, the Classification of Finite Simple Groups was completed, finishing off decades of work spread out over hundreds of articles. We won't be able to prove this theorem in class (among other things, I don't know if anyone in the world knows all of the details), but we can look at one of the useful tools involved in the classification!

The Sylow Theorems are a few results arising from group actions, proven by the Norwegian mathematician Peter Sylow. These theorems provide a way to show that all groups of order 15 are cyclic, that there are no nonabelian simple groups of order less than 60, among other consequences.

What are these theorems? How can we use them? What are simple groups? What are a few other facts about groups that come in handy when looking for simple groups? What are some fun facts about Sylow, and how do you pronounce his name? These are some of the questions we'll investigate in this class.

Homework: Optional

Prerequisites: Group Theory

Yes, you can trisect angles. (🌶–🌶🌶, Ben Dees, 1 days)

Trisecting arbitrary angles with compass and straightedge alone has been known to be impossible since the development of Galois Theory.

On the other hand, since antiquity we have known how to trisect arbitrary angles, and divide them into 5 pieces (or six, seven, and so on) by use of a construction called a "quadratrix." This isn't a curve we can construct using compass and straightedge, but use of it in mathematical arguments dates back to at least 400 BCE.

In this class, we will learn about what this curve is and about its history, and how to use it to trisect angles. If we have time, we'll discuss other uses of the quadratrix and how we can approximate it with compass and straightedge.

Homework: None

Prerequisites: None

Brian Reinhart's Classes

A Slice of PIE. (🌶🌶, Brian Reinhart, 1 days)
If you've ever wondered about the sizes of things that aren't infinite, you might have run into the Principle of Inclusion-Exclusion:

$$\left|\bigcup_{A\in\mathcal{F}} A\right| = \sum_{S\subseteq\mathcal{F}} (-1)^{|S|-1} \left|\bigcap_{A\in S} A\right|$$

This is a formula which tells you the size of a union of a bunch of sets based on the sizes of their intersections. But... we use a lot of information about the sets, and end up with one very small piece of information. Shouldn't we be able to figure out more stuff if we know the sizes of ALL possible intersections?

In this class, we'll be taking a look at how we can modify inclusion-exclusion to be more general. First, we'll see how we can find the size of something called the "symmetric difference" of all of the sets. We'll also see how this is like cutting the union in half, and find out what happens when we cut into more than two pieces. Along the way, we'll see some generating functions and complex numbers, and get some insight into why Inclusion-Exclusion is so useful.

Homework: Optional

Prerequisites: You should be familiar enough with Inclusion-Exclusion to know how it applies to this problem: How many 3-digit numbers have an even first digit or a last digit divisible by 3 (or both)?

Dyusha, Michelle's Classes

Curse of Dimensionality. (🌶🌶, Dyusha, Michelle, 1 days)
As those of you who went to Pos lecture already know, oranges can be deceiving.

Since our very birth, all of us, and all of our progeny unto eternity, have lived and will forever live in a three dimensional world. For humans (even Po), an orange is just a healthy snack, but for n-dimensional beings (where n is large), life is not nearly as simple. Even essentials such as peeling an orange to reveal juicy flesh are no longer viable. For instance, training a neural network to discern good poetry from bad is simple in three or four dimensions, but optimizing thousands, if not millions, of factors, in a high-dimensional space is quite the challenge. Similarly, clever strategies to find the k nearest neighbors from a given point in 2 dimensions quickly fall apart when dimensions grow larger.

This funky behavior in large dimensions is known as the *Curse of Dimensionality*. In this class, well dive into several examples of things that work just fine in low dimensions, discuss how the curse arises, and propose some strategies to combat it.

Homework: None

Prerequisites: If you've heard about neural networks and vector spaces, that would be helpful but not at all required

J-Lo's Classes

Axiomatic Music Theory part 2: Rhythm. (🌶, J-Lo, 2–3 days)
Want more music theory? We talked a lot about pitch and intervals and triads and chord progressions in week 3, but very little about another crucial component of any musical piece: *rhythm.* Join a small group and explore how the Euclidean algorithm can be used to derive many traditional rhythmic patterns, find unexpected ways to transform rhythms in ways that preserve the set of time differences between pairs of beats (using a result known as the "hexachordal theorem"), and discover implications of the fact that pitch and rhythm are really the same thing.

Homework: Recommended

Prerequisites: None (in particular, Axiomatic Music Theory is not required)

Lattices. (🌶🌶, J-Lo, 3–4 days)
The Dread Pirate Riemann is back! But since we last saw him on the qualifying quiz, he's built a spaceship to explore 3-dimensional grids of planets — and just recently, he stole the secret to interdimensional travel and can now traverse 100-dimensional lattices.

How will he navigate these multidimensional seas? He can try to apply the tools of linear algebra, but while some tools (e.g. row reduction, change of basis) carry over to this discrete setting, others (eigenvectors, finding orthogonal vectors, etc) will be lost in translation.

Riemann's main goal, of course, is to find the treasure. He has a map that leads him to a point somewhere out in the 100-dimensional sea, and then declares "the treasure is on the nearest planet." It turns out that this simple-sounding instruction is so hard to solve that the treasure will likely never be found.

Homework: Optional

Prerequisites: Linear Algebra

Minus Choice, Still Paradoxes. (🌶🌶–🌶🌶🌶, J-Lo, 1–2 days)
The Banach-Tarski paradox says that a ball can be broken into finitely many pieces, and using only translations and rotations, can be rearranged into two balls, each the same volume as the original. The "Axiom of Choice" is a crucial part of this construction, and historically some people have used Banach-Tarski as an argument against this axiom.

But it turns out that paradoxical decompositions (breaking something into finitely many pieces and rearranging them into two copies of the original) exist even without the Axiom of Choice! This class will discuss some of these constructions, which are all based in group theory, and see what specific role Choice plays in the Banach-Tarski case.

Homework: None

Prerequisites: Group Theory

The Borsuk Problem. (🌶🌶, J-Lo, 1 days)
The *diameter* of an arbitrary shape is the largest possible distance between two points in the shape. In 1932, Karl Borsuk asked the following question:

Given a bounded subset of $\mathbb{R}^n$, can it be split into $n+1$ pieces, each of which has smaller diameter than the original set?

For example, if you try to split an equilateral triangle into two pieces, one of them will always contain two of the vertices, and so its diameter will not be any smaller than that of the original triangle . However, any figure in the plane can be decomposed into 3 pieces of smaller diameter.

Over the course of the 20th century, evidence began to grow that $n+1$ pieces would always be enough. That is, until 1993, when the first counterexample was found... in 1325 dimensions. Currently the smallest known counterexample is a 64-dimensional set which requires more than 65 pieces.

And in case you were wondering, the counterexamples come from graph theory. This class will explain what on earth graph theory has to say about this inherently geometric problem.

Homework: None

Prerequisites: Intro Graph Theory

The Music of Zeta. (🌶🌶, J-Lo, 1 days)
You can use the Riemann Zeta function to study music. One of the (many) things it measures is how many notes should be in a scale. (Spoiler: 12 is really good.)

You can use music to study the Riemann Zeta function. In this class, we'll hear what analytic continuation sounds like, listen to the primes as though an orchestra of Zeta zeros is playing a chord, and interpret the Riemann hypothesis as saying that no instrument in this orchestra is playing too loud.

Homework: None

Prerequisites: complex numbers up to Euler's formula

Yes, you can solve the Quintic. (🌶🌶🌶🌶, J-Lo, 4 days)
In this class we will describe a method for solving a general quintic equation.

"But that's impossible!" you may protest. Indeed, there is no solution *in radicals*. But if you're allowed to use other tools, such as modular forms, then everything changes.

Homework: Optional

Prerequisites: Modular Forms

Jeff's Classes

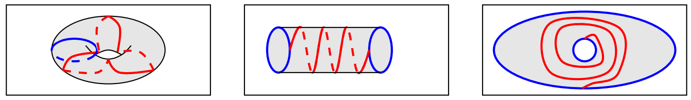
Combinatorial Topology. (🌶🌶, Jeff, 2–4 days)
So, you want to be a topologist because you love drawing pictures. But, youve never taken point-set topology[5] . How much can you prove about topology?
Turns out, quite a bit. In this class, well be developing simplicial complexes, which give combinatorial representations of topological spaces. Then, well look at discrete Morse theory, a combinatorial representation of a construction from differential topology. Along the way, well draw lots of pictures and diagrams, and get a feel for what topology should do, without messing around with all of those icky open sets.

Homework: Optional

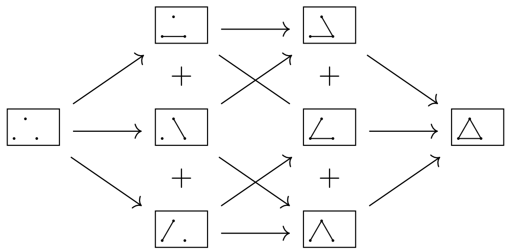
Prerequisites: None!

Jeff's Favorite Pictures. (🌶, Jeff, 2–4 days)
As you may have guessed from my other class proposals, I really really like drawing pictures. Here is a series of 4 one-offs which will explore some of my favorite mathematical diagrams from topology, geometry and combinatorics.
Drawing 3-Manifolds: How do mathematicians visualize higher dimensional manifolds? One way to do so is to cut and paste together pieces that we can visualize. In this class, we'll go into more detail on how to visualize 3 dimensional manifolds from 2 dimensional diagrams.

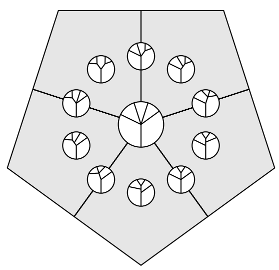
[5] Disclaimer: I have never taken point set topology



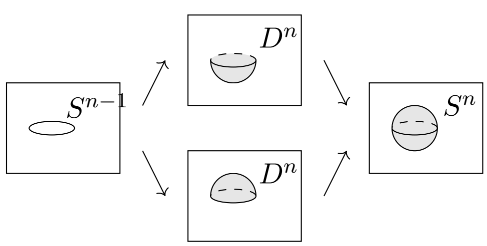
Graph Colorings: How can we efficiently count the number of ways to color a graph? We'll give a diagrammatic method to compute the number of k-color a graph, and prove some remarkable properties about this function. Time permitting, we'll also show how this method relates to many other graph invariants.



Moduli of Trees: How many different planar trees are there? One visual way to classify a group of mathematical objects is to fit them all into a topological space. We'll look at the moduli space of metric trees, a polyhedra which gives geometric structure to trees.



Visualizing Homology: Homology is a powerful mathematical tool that allows us to characterize mathematical spaces, generalize the concepts of sums and unions of sets, and understand what "equivalence relations between equivalence relations" mean. But what does it look like? The pictures from this course will provide some intuition on how mathematicians may have developed the cutting and pasting tools which eventually went on to become the field of homological algebra.



Homework: None
Prerequisites: None.

Knot theory: Khovanov Homology. (🌶🌶🌶🌶, Jeff, 2–4 days)
In our Knot Theory class, we developed a tool called the Jones Polynomial, which we could use to distinguish quite a few knots from each other. An open question in mathematics is whether this knot invariant can distinguish the unknot from every other knot. In this class, we'll develop the *Khovanov*

chain complex, a powerful combinatorially defined algebraic invariant of knots which generalizes the Jones polynomial. This invariant is known to detect the unknot, gives bounds on what kinds of surfaces your knot can bound, and gives an interesting example of a homological theory which truly does not arise as some kind of sheaf cohomology.

Homework: None

Prerequisites: You should be able to provide an example of a chain complex, and be comfortable with direct sums of vector spaces.

Quadrisecants?! (🌶🌶🌶, Jeff, 1 days)
Sometimes theorems lay bare the beautiful structures in mathematics. Hilbert's Nullstellensatz is an example of such a theorem. It ties together the worlds of algebra and geometry, creating new possibilities for exploration.
Some fields of mathematics — like Galois theory or measure theory — build new abstractions that change the way you look at the world.

And then, there are some theorems which are just kinda weird, and make you think, *wait, what?* Here is my favorite example.

Theorem: Let K be a non-trivial knot. There exists a straight line L which intersects K at 4 distinct points.

Weird, amirite??

Homework: None

Prerequisites: Fundamental Group

Systems and Signals VI: Return of the Rigor. (🌶🌶🌶🌶, Jeff, 2–4 days)
At the end of Systems and Signals, did you feel like something was missing? Do you wish you could do the whole course again, except with all the fun computer graphics replaced with ϵ's and δ's? If you found listening to *Africa* by Toto infuriating, and just wished that Jeff would actually define what the space of signals was at some point during the class, this might be the class for you.
We'll focus on developing a rigorous definition of Signals. In all likelihood, we'll only manage to prove that there are things called signals, and that we can take Fourier transforms of them. We'll talk about test spaces, tempered distributions, and actually define δ_0.

Homework: Recommended

Prerequisites: You should be really comfortable with integrals, sequences and series. In particular, you should be willing to accept the existence of the Lebesgue integral, and know some convergence properties of sequences of functions.

Important note: This class does *not* require Systems and Signals

Uranus has at least 2 storms. (🌶🌶🌶–🌶🌶🌶🌶, Jeff, 1–2 days)
True Life Story of Mathematics: At UC Berkeley, we're required to take a qualifying oral exam in our second year. The exam is moderated by 3 mathematicians, and an outside member not from the mathematics department whose job is to make sure the 3 faculty members on your committee are treating you fairly. They're also suppose to lob you some easy questions, like "what made you study this field," or "what is your favorite theorem?"
One of my subjects was differential topology, which is what I mostly study now. My outside committee member asked me the following wonderful question:

"Describe a result from your field in a way which anybody can understand."

I feel like differential geometry is full of these types of results, so I brought out my favorite result: that a generic smooth function on the sphere has 2 critical points. I tried to package this in a way which was simpler to state.

"On any planet, at any given moment, there are at least 2 storms."

I thought this was a clever way of stating the theorem. By replacing the construction of gradient vector field with wind, and by classifying critical points of a vector field as a storm, I figured I had discovered some elegant way of getting at the crux of the field. To which, the professor (who was a tenured member of the astronomy department) looked at me and stated.

"Uranus has no storm systems."

I ended up passing my qual, but not before the 3 mathematicians members of the committee spent 10 minutes trying to convince this astronomer that Uranus did indeed have 2 storms.

In this class, we'll prove that Uranus has at least 2 storms, and many other things about differential manifolds, by using the tools of *Morse Homology*, which provides incredibly strong inequalities for the number of critical points on a manifold.

Homework: Recommended

Prerequisites: Multivariable calculus, linear algebra

Jessica's Classes

The Poncelet-Steiner Theorem. (🌶🌶, Jessica, 2–3 days)
You can construct a lot of cool things with a straightedge and compass — like angle bisectors! And square roots! And equilateral triangles! You can also construct a lot of cool things with a straightedge and a given circle and its center — like angle bisectors! And square roots! And equilateral triangles! Wait, what?

As it turns out, anything you can construct with the straightedge and compass, you can also construct with just a straightedge and a given circle. Come to this class to make fun constructions, and prove the Poncelet-Steiner theorem along the way.

Homework: Recommended

Prerequisites: None!

Kevin's Classes

A Stupid Float Trick. (🌶, Kevin, 1 days)
Did you know that if x is a float, then "float"(1597463007 - "int"(x)/2) is quite close to 1/sqrt(x)? We will learn how floating point numbers work and understand this nonsense and where this number comes from.

Homework: None

Prerequisites: none

Dirichlet's Theorem on Primes a mod b. (🌶🌶🌶–🌶🌶🌶🌶, Kevin, 2–4 days)
Did you know there are infinitely many primes? Did you know there are infinitely many primes congruent to a mod b, as long as a and b are relatively prime? We will study the world of L-functions, which generalize the Riemann zeta function, and use them to prove this fact.

Homework: Optional

Prerequisites: Familiarity with complex numbers. Some knowledge of groups and basic complex analysis (such as the word "pole") is helpful.

Dirichlet's Unit Theorem. (🌶🌶🌶, Kevin, 2–4 days)
Want more algebraic number theory? In this class, we'll prove Dirichlet's unit theorem, which describes precisely how many units there are in a ring of integers. We'll see more lattices and maybe Minkowski will make an appearance!

Homework: Optional

Prerequisites: Familiarity with rings of integers. The Class Number is very helpful but not strictly required.

Hyperplane Arrangements. (🌶🌶–🌶🌶🌶, Kevin, 2–4 days)
Suppose I want to cut a watermelon into pieces. With 0 cuts, I can make 1 piece: the entire watermelon. With 1 straight cut, I can make 2 pieces. With 2 cuts, I can make 4, and with 3 cuts, I can make 8.

I'm sure you see the pattern by now: with k cuts, we can make $\frac{1}{6}(k^3 + 5k + 6)$ pieces. These cuts are an example of a *hyperplane arrangement*, which is simply a collection of k hyperplanes in n-dimensional space. We'll learn some powerful techniques for counting the number of regions that a hyperplane arrangement determines, and we'll maybe even see some connections to graph theory. Throughout, partially ordered sets will abound!

Homework: Optional

Prerequisites: none

Kevin, Viv's Classes

The Tennis Racket Theorem. (🌶🌶–🌶🌶🌶, Kevin, Viv, 1 days)
If you take a tennis racket (a generalized tennis racket, such as a book, is OK...try it!), and you fiddle with it, you may start spinning it. A generalized tennis racket can be spun in three ways: around the short axis, around the long axis, and around the middle axis. One of these spins is unlike the other, because the generalized tennis racket wobbles terribly from side to side.

We'll talk about why this happens.

Homework: Recommended

Prerequisites: Calculus, basic knowledge of or faith in linear algebra

Larsen's Classes

Brownian Motion. (🌶🌶🌶–🌶🌶🌶🌶, Larsen, 1–2 days)
If you stand at a point on a plane, and at every second you take a step in a random cardinal direction, you are doing a "random walk." If you instead take infinitesimally short steps every infinitesimal fraction of a second, you might be doing Brownian motion, which is the randomest of continuous motion. Does such a thing even exist and make sense? Yes, and we will use it to "randomly" solve some important differential equations.

Homework: Optional

Prerequisites: Comfort with calculus and probability. You should be familiar with: Partial derivatives, path integrals, differential equations, random variables, independence, and expected value.

Higher Homotopy Groups. (🌶🌶🌶, Larsen, 2–3 days)
In week 4 we will have a class on the fundamental group, also denoted π_1. But if there is a π_1, surely

there must be a π_2, and π_{17}! The "higher homotopy groups" are the natural generalization of the π_1, measuring the higher-dimensional complexity of a space. They are notoriously difficult to work with, but have some strange properties, for example they are all abelian.
Homework: Optional
Prerequisites: The Fundamental Group

Mapping Class Groups. (🌶🌶🌶, Larsen, 1–2 days)
If S is a surface, such as a torus, a symmetry of S is a continuous way of "rearranging" S on top of itself, and these symmetries form a group. In some cases, such as the sphere, all of these symmetries are just the identity in disguise. But in other cases, there are "twists" that put the surface back together in a fundamentally different way from how it started, and so our group is nontrivial. Come learn why the annulus is the best shape for knotting up other surfaces within themselves.
Homework: Optional
Prerequisites: Group Theory

One-Day Complex Analysis. (🌶🌶, Larsen, 1 days)
If you poke a hole in the complex numbers, they all stay in one piece—unlike the real line. Thanks to this and other facts, calculus with complex numbers works a lot better than regular calculus. Did you know that if a complex function is differentiable, it is infinitely differentiable? And if it is also bounded, it must be constant? After this one-day crash course, you will see why calculus is what makes complex numbers special.
Homework: Optional
Prerequisites: Calculus

Ping Pong: The Lemma. (🌶🌶, Larsen, 1–2 days)
In the field of Geometric Group Theory, the Ping Pong Lemma is a powerful tool for identifying free groups by analyzing their group actions—in this case, actions that resemble a game of ping pong. Among other things, we will use the Ping Pong Lemma to prove the Banach-Tarski paradox, which states that a ping pong ball can be taken apart and put back together as two ping pong balls.
Homework: Optional
Prerequisites: Group Theory

LAURA'S CLASSES

Cayley's Tree Formula. (🌶🌶, Laura, 1 days)
We'll learn about Cayley's formula for counting the number of labeled trees on n vertices. This theorem has a lot of nice combinatorial proofs, and we'll hopefully look at three of them that rely on clever bijections and double-counting arguments!
Homework: Recommended
Prerequisites: None

Combinatorial Game Theory. (🌶🌶, Laura, 2–4 days)
We'll look at some examples of two player impartial games and the winning strategies, and learn about how to figure out the winning positions for a game and assign nimbers to positions to figure out how to win!
Homework: Recommended
Prerequisites: None

The Matrix-Tree Theorem. (🌶🌶🌶, Laura, 1 days)
The matrix-tree theorem gives a way to compute the number of spanning trees of a graph as the determinant of a certain matrix connected to the graph. The proof uses a nice bijection of terms in the determinant with directed graphs to show how all terms cancel except for precisely the ones corresponding to trees!
Homework: Recommended
Prerequisites: Familiarity with determinants

Linus's Classes

HyperRogue. (🌶, Linus, 1 days)
The game HyperRogue takes place in the hyperbolic plane. Play HyperRogue in the computer lab! Learn about the hyperbolic plane.
Homework: None
Prerequisites: None

The Maths of Peppa Pig. (🌶🌶🌶🌶, Linus, 1 days)
Let's examine some of the advanced mathematical topics featured in the short film series Peppa Pig. This class will be a disjointed, ill-structured compilation of random mathematical thoughts I have while watching Peppa Pig next week.
Homework: None
Prerequisites: Get ready for algorithms and maybe topology.

Linus, at least's Classes

Problem-Solving: The Just Do It method. (🌶🌶🌶, Linus, at least, 1 days)
You are navigating a 50-by-50 maze on your computer when suddenly your monitor shuts off. You don't remember where the maze's walls are. You only remember that you are in the top-left corner, the exit is in the bottom-right corner, and there is some path from you to the exit.

Is there some sequence of arrow key presses you can do that guarantees that you exit the maze at some point?

Learn the "Just Do It" method, a problem-solving technique in combinatorics that solves problems like this one by revealing that they are actually trivial.
Homework: None
Prerequisites: None

Marisa's Classes

Almost Planar. (🌶🌶, Marisa, 3–4 days)
People think of planarity as a property of graphs that's binary: either you're planar or you aren't. But if the latter, can we tell how *far* a graph is from planar? In this class, we'll look at several different ways of measuring closeness to planarity, from the structural to the space-bending. The format will be inquiry-based, so you'll be discovering and proving results yourself.
Homework: Optional
Prerequisites: Intro Graph Theory

King Chicken Theorems. (🌶, Marisa, 1–2 days)
Chickens are incredibly cruel creatures. Whenever you put a bunch of them together, they will form

a pecking order. Perhaps "order" is an exaggeration: the chickens will go around pecking whichever chickens they deem to be weaker than themselves. Imagine you're a farmer, and you're mapping out the behavior of your chickens. You would like to assign blame to the meanest chicken. Is it always possible to identify the meanest chicken? Can there be two equally mean chickens? Are there pecking orders in which all the chickens are equally mean?
Homework: None
Prerequisites: None

Latin Squares and Finite Geometries. (–, Marisa, 1–3 days)
In 1782, Euler conjectured an answer to the following yes/no question: is it possible to arrange six *regiments* consisting of six officers each of different *ranks* in a 6×6 square so that no rank or regiment will be repeated in any row or column? We know the answer now, but surprisingly, the question remained open until 1901. In this class, we'll be exploring combinatorial design questions like this one through the lenses of Latin Squares (like Sudoku puzzles) and Finite Geometries (like the Fano plane).

Homework: Optional
Prerequisites: None

Mismatched. (, Marisa, 1 days)
If we have ten Mathcampers and their ten sarongs, out of all of the 10! ways of handing each camper a sarong, what is the probability of matching them all wrong (so that nobody gets their own sarong back)? Which is more likely - getting them all wrong, or getting at least one right?

This short class on the Hat Check Problem will use my favorite approach: counting the number of 1-factors in the graph $K_{n,n}$ minus a 1-factor by solving a recurrence relation.
Homework: None
Prerequisites: Basic graph theory, plus either familiarity with the Taylor Series for e^x or willingness to take 5 minutes of calculus on faith.

MARISA, SHIYUE'S CLASSES

The Stable Marriage Problem. (, Marisa, Shiyue, 1 days)

N single men and N single women want to pair up and get married. These are their names and preferences:

- Jeff: Susan > Shiyue > Jessica > Viv > Ania
- Ben: Ania > Susan > Viv > Shiyue > Jessica
- Agustin: Ania > Viv > Shiyue > Susan > Jessica
- Pesto: Jessica > Shiyue > Viv > Ania > Susan
- Tim! : Jessica > Ania > Susan > Viv > Shiyue
- Ania: Jeff > Pesto > Tim! > Agustin > Ben
- Jessica: Agustin > Ben > Jeff > Tim! > Pesto
- Susan: Agustin > Pesto > Tim! > Ben > Jeff
- Shiyue: Agustin > Tim! > Jeff > Ben > Pesto
- Viv: Agustin > Pesto > Ben > Tim! > Jeff

Is it possible to make everybody happy? Obviously not since almost everybody wants to marry Agustin. But is it possible to at least create a stable situation? For instance, it is a bad idea for Agustin to marry Shiyue and for Viv to marry Pesto, because then Agustin and Viv would prefer each other rather than staying with their partners, so they will run away together. How can we at least avoid having a run-away couple? Is there more than one way to do it? What is the best way to do it? And what if Shiyue and Viv decide that marrying each other is better than marrying Agustin?

Homework: None

Prerequisites: None

Mark's Classes

Cyclotomic Polynomials and Migotti's Theorem. (🌶🌶, Mark, 1 days)

The cyclotomic polynomials form an interesting family of polynomials with integer coefficients, whose roots are complex roots of unity. Looking at the first few of these polynomials leads to a natural conjecture about their coefficients. However, after the first hundred or so cases keep confirming the conjectured pattern, eventually it breaks down. In this class we'll prove a theorem due to Migotti, which sheds some light on what is going on, and in particular on why the conjecture finally fails just when it does.

Homework: Recommended

Prerequisites: Some experience with complex numbers, preferably including complex roots of unity; some experience with polynomials

Elliptic functions. (🌶🌶🌶–🌶🌶🌶🌶, Mark, 4 days)

Complex analysis, meet elliptic curves! Actually, you don't need to know anything about elliptic curves to take this class, but they will show up along the way. Meanwhile, if you like periodic functions, such as *cos* and *sin*, then you should like elliptic functions even better: They have two independent (complex) periods, as well as a variety of nice properties that are relatively easy to prove using some complex analysis. Despite the name, which is a kind of historical accident (it all started with arc length along an ellipse, which comes up in the study of planetary motion; this led to so-called elliptic integrals, and elliptic functions were first encountered as inverse functions of those integrals), elliptic functions don't have much to do with ellipses. Instead, they are closely related to cubic curves, and also to modular forms. If time permits, we'll use some of this material to prove the remarkable fact that

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{k=1}^{n-1}\sigma_3(k)\sigma_3(nk),$$

where $\sigma_i(k)$ is the sum of the i-th powers of the divisors of k. (For example, for $n = 5$ this comes down to

$$1 + 5^7 = 1 + 5^3 + 120[1(1^3 + 2^3 + 4^3) + (1^3 + 2^3)(1^3 + 3^3) + (1^3 + 3^3)(1^3 + 2^3) + (1^3 + 2^3 + 4^3)1],$$

which you are welcome to check if you run out of things to do.)

Homework: Recommended

Prerequisites: Functions of a complex variable, in particular Liouvilles Theorem.

From Counting to a Theorem of Fermat. (🌶–🌶🌶, Mark, 1 days)

A standard theorem stated by Fermat (it's actually uncertain whether he had a proof) states that every prime p congruent to 1 modulo 4 is the sum of two squares. (On the other hand, if p is 3 modulo 4, it has no hope of being the sum of two squares.) There are many proofs of this theorem, but perhaps the weirdest one, due to Heath-Brown and simplified by Zagier, uses just counting no "number theory" at all! In this class we'll see at least that proof, and maybe some others and/or related proofs of other things.

Homework: Recommended

Prerequisites: None

Primitive roots. (🌶🌶, Mark, 1 days)
Suppose you are working modulo n, and you start with some integer a and multiply it by itself repeatedly. For instance, if $n = 17$ and $a = 2$ you get$2, 4, 8, 16, 15, 13, 9, 1$ and then you're back where you started. Note that on the way we haven't seen all the nonzero integers mod 17; however, if we had used $a = 3$ instead we would have gotten $3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1$ and cycled through all the nonzero integers mod 17. In general we can ask when (that is, for what values of n) you can find an a such that every integer mod n that's relatively prime to n shows up as a power of a (such an a is called a *primitive root* mod n). We may not get much beyond the case that n is prime, but even in that case the analysis is interesting. In particular, we'll be able to show that a exists in that case without having any idea of how to find a, other than the flat-footed method of trying $2, 3, \dots$ in turn until you find a primitive root.
Homework: None
Prerequisites: None.

Quadratic reciprocity. (🌶🌶–🌶🌶🌶, Mark, 2 days)
Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?
(1) "Is q a square modulo p?"
(2) "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. You'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101is a square modulo 9973, you'll be able to answer a lot more quickly, whether or not you use technology!
Homework: Optional
Prerequisites: Some basic number theory (if you know Fermats Little Theorem, you should be OK)

Simplicity itself: A_n and the "other" A_n. (🌶🌶–🌶🌶🌶, Mark, 2 days)
The monster group (of order roughly $8 \cdot 10^{5}3$) gets a lot of "press, but it's not the largest finite simple group; it's the largest exceptional finite simple group. (Reminder: A simple group is one which has no normal subgroups other than the two "trivial ones; by using homomorphisms, all finite groups can be "built up from finite simple groups. The complete classification of finite simple groups was a monumental effort that was completed successfully not far into our new millennium.) What about the unexceptional finite simple groups? They come in infinite families, and in this class we'll look in some detail at two of those families: the alternating groups A_n and one class of groups of "Lie type", related to matrices over finite fields. (If you haven't seen finite fields, think "integers mod p" for a prime p.) By the way, the simplicity of the alternating groups plays a crucial role in the proof that in general, polynomial equations of degree 5 and up cannot be solved by radicals (there is no "quintic formula"). We'll prove that A_n is indeed simple for $n \geq 5$, and we should be able to prove simplicity for the other class of groups also, at least for 2×2 matrices.
Homework: None
Prerequisites: Basic group theory and linear algebra; familiarity with finite fields would be helpful, but not really necessary.

The Cayley-Hamilton Theorem. (🌶🌶🌶, Mark, 1 days)
Take any square matrix A and look at its characteristic polynomial $f(X) = \det(AXI)$ (the roots of this polynomial are the eigenvalues of A). Now substitute A into the polynomial; for example, if A is a 4×4 matrix such that $f(X) = X^4 6X^3 X^2 + 17X8$, then compute $f(A) = A^4 6A^3 A^2 + 17A8I$. The

answer will always be the zero matrix! In this class we'll use the idea of the "classical adjoint" of a matrix to prove this fundamental fact, which can be used to help analyze linear transformations that can't be diagonalized.
Homework: None
Prerequisites: None.

The Magic of Determinants. (🌶🌶, Mark, 3 days)
Determinants, which are numbers associated to square matrices, have many useful properties; for example, they are needed to give a general formula for the inverse of a matrix. Unfortunately, determinants are often defined in a very *ad hoc* way (using Laplace expansion) which may obscure what is really happening. This class will give a "better" theoretical framework as well as some geometric intuition, and I'll try to at least give an outline of the proofs of all the main computational properties of determinants, such as the Laplace expansion.
Homework: Optional
Prerequisites: Matrix multiplication, and the idea of a linear transformation.

Wedderburn's Theorem. (🌶🌶🌶, Mark, 1 days)
Have you seen the quaternions? They form an example of a division ring that isn't a field. (A division ring is a set like a field, but in which multiplication isn't necessarily commutative.) Specifically, the quaternions form a four-dimensional vector space over $\mathbb{R}$, with basis $, i, j, k$ and multiplication rules

$$i^2 = j^2 = k^2 = 1, ij = k, ji = k, jk = i, kj = i, ki = j, ik = j.$$

Have you seen any examples of finite division rings that aren't fields? No, you haven't, and you never will, because Wedderburn proved that any finite division ring is commutative (and thus a field). In this class we'll see a beautiful proof of this theorem, due to Witt, using cyclotomic polynomials (polynomials whose roots are complex roots of unity).
Homework: Recommended
Prerequisites: Some group theory and some ring theory; familiarity with complex roots of unity would help.

Mira's Classes

Bayesian Statistics; or, Dont Listen to Anything They Teach You in School! (🌶🌶–🌶🌶🌶, Mira, 3–4 days)
Statistics is the science of analyzing data in the presence of uncertainty or with incomplete information. Since there is little in the world that is certain, and information is always scarce, we humans cant go a day without doing some kind of statistics in our routine cognitive functions, in science, in politics, etc.

But the traditional way of doing statistics – the stuff you learn in AP Stats – is deeply flawed.[6] For example, many branches of science that have been relying on standard statistical techniques for decades now find themselves in the midst of a replication crisis: it turns out that many scientific results previously thought to be "statistically significant" cannot be replicated and are likely false. As for mathematicians, they have long shunned statistics as "not real math", but just an arbitrary bag of tricks and tests.

Why is statistics so screwed up? There are interesting historical and philosophical reasons, and well discuss them. The good news is: there is an alternative. There is a way of doing statistics that is really math, that doesnt substitute artificial questions for the questions you actually want answered,

[6]Don't take AP Stats if you can help it! Talk to Mira if you want to know more.

and that makes perfect sense every step of the way. We only have a few days, so we may not get to a lot of the technical stuff. But I hope to give you a sense of how Bayesian statistics works and to convince you that its the way to go.

Homework: Recommended

Prerequisites: The first day or two has no prerequisites. After that, you'll need calculus.

Hark! The bells are ringing a Hamiltonian cycle on a truncated octahedron! (, Mira, 1 days)

For over 400 years, people who ring bells in English churches have been looking for Hamiltonian cycles on Cayley graphs of S_n (although they would have been very confused if you told them that this was what they were doing). Come find out about this unexpected application of group theory.

Homework: None

Prerequisites: know the definition of a group and a Cayley graph.

Information Theory and the Redundancy of English. (, Mira, 4 days)

NWSFLSH: NGLSH S RDNDNT!! (BT DN'T TLL YR NGLSH TCHR SD THT)

The redundancy of English (or any other language) is what allows you to decipher the above sentence. It's also what allows you to decipher bad handwriting or to have a conversation in a crowded room. The redundancy is a kind of error-correcting code: even if you miss part of what was said, you can recover the rest.

How redundant is English? There are two ways to interpret this question:

- How much information is conveyed by a single letter of English text, relative to how much could theoretically be conveyed? (But what is information? How do you measure it?)
- How much can we compress English text? If we encode it using a really clever encoding scheme, can we reduce the length of the message by a factor of 2? 10? 100? (But how will we ever know if our encoding is the cleverest possible one?)

Fortunately, the two interpretations are related. In this class, we will first derive a mathematical definition of information, based on our intuitive notions of what this word should mean. Then we'll prove the Noiseless Coding Theorem: the degree to which a piece of text (or any other data stream) can be compressed is governed by the actual amount of information that it contains. We'll also talk about Huffman codes: the optimal way of compressing data if you know enough about its source. (That's a big "if", but it's still a very cool method.)

Finally, we'll answer our original question – how redundant is English? – in the way that Claude Shannon, the father of information theory, originally answered it: by playing a game I call Shannon's Hangman and using it as a way of communicating with our imaginary identical clones!

The class is 4 days long, but you can skip some of the days and still come to the others. Here's how it works:

Day 1:: Introduction and definition of information. (Required for the rest of the class.)

Days 2, 3:: Noiseless coding and Huffman codes. (The mathematical heart of the class, where we'll prove the Noiseless Coding Theorem.)

Day 4:: Shannon's Hangman and the redundancy of English. (You can come to this class even if you don't come on Days 2 and 3 – you just need the material from Day 1.)

Homework: Recommended

Prerequisites: None

Mathematics of Polygamy (and Bankruptcy). (🌶, Mira, 1 days)
Here is a passage from the Mishnah, the 2nd century codex of Jewish law:

A man has three wives. According to their prenuptial agreements, when he dies one of them should get 100 [silver pieces], one should get 200, and one should get 300.

If his total estate is 100, they split it equally.

If the estate is 200, then the first wife gets 50 and the other two get 75 each.

If the estate is 300, then the first wife gets 50, the second one gets 100, and the third one gets 150.

Similarly, any joint investment with three unequal initial contributions should be divided up in the same way.

For 1800 years, this passage had baffled scholars: what could possibly be the logic behind the Mishnah's totally different ways of distributing the estate in the three cases? Then, in 1985, a pair of mathematical economists produced a beautifully simple explanation based on ideas from game theory. They showed that for any number of creditors and for any estate size, there is a unique distribution that satisfies certain criteria, and it turns out to be exactly the distribution proposed in the Mishnah. The proof is very cool, based on an analogy with a simple physical system! See if you can figure out this ancient puzzle for yourself, or come to class and find out.

Homework: Recommended

Prerequisites: None

No, you can't just vote your conscience. (🌶🌶, Mira, 1 days)

> *No one pretends that democracy is perfect or all-wise. Indeed it has been said that democracy is the worst form of Government except for all those other forms that have been tried from time to time...* – Winston Churchill

You may have heard of Arrow's Theorem: it says that if you want your voting system to satisfy certain reasonable-sounding conditions, then your only option is a dictatorship. But this class is *not* about Arrow's Theorem, because Arrow's Theorem is not depressing enough: its definition of a voting system is so restrictive that it barely ever applies in practice.

The Gibbard-Satterthwaite theorem is less famous, but I think it's much more depressing. It says that if a voting system satisfies two very simple criteria,

(a) if candidate A is preferred by all the voters then A wins;
(b) the system is not a dictatorship,

then this system is vulnerable to strategic voting whenever there are more than two candidates. In other words, there is at least one voter who can obtain better results by voting dishonestly than by voting honestly. Democracy can *always* be gamed. In this class, you won't necessarily learn how, but you'll learn why.

Homework: None

Prerequisites: None

The politics of rounding fractions. (🌶🌶, Mira, 1 days)

> *God help the state of Maine when Mathematicks reach for her and undertake to strike her down!* – Representative Littlefield (R, ME), 1901

The US Constitution mandates that "representatives ... shall be apportioned among the several states ... according to their respective numbers". This is usually taken to mean that the number of representatives in each state should be proportional to its population. But exact proportionality is not possible: for example, California cannot have 54.37 representatives. The same issue arises in countries where seats in parliament are apportioned to parties based on the percentage of votes each party received. Once again, what do you do with fractions of representatives?

This is the *problem of apportionment*, and it's a lot trickier and more interesting than might appear at first glance. Over the course of US history, Congress went through five different apportionment methods, always accompanied by fierce political debates. The method that we currently use was proposed in 1921 by a Harvard mathematician (!), and its was adopted by Congress on the recommendation of the US Academy of Sciences (!!). As far as I know, the US is the only country in the world that uses a method of apportionment that was derived by a mathematician from first principles!

Homework: Optional

Prerequisites: None

Misha's Classes

Information Theory and Counting. (🌶🌶🌶, Misha, 2 days)
In Mira's Week 5 class, information entropy is used for reasonable things, such as measuring the information communicated by learning the answer to a question.

This class is about off-label uses for information entropy: we will use it to estimate binomial coefficients, solve coin-weighing problems, and bound the permanent of a matrix.

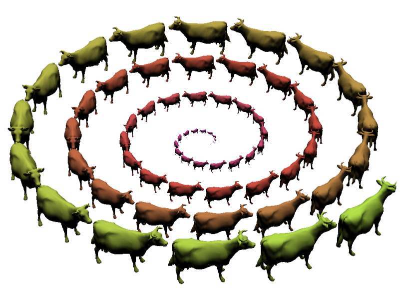
Homework: Optional

Prerequisites: The definition of entropy (at least the first day of Mira's class)

Mathematica Workshop. (🌶🌶, Misha, 2 days)
All campers get a free 1-year license for Mathematica from Wolfram.[7] But do you know how to use it?

In this class, you will learn how to do things in Mathematica, including:

- Drawing pictures such as the one below:



- Solving crossword puzzles.
- Various tricks I've learned for dealing with hard combinatorial problems.
- Using the incredibly extensive help function.

There will be time for you to work on doing your own cool things in Mathematica, which I will try to help you with.

Homework: Optional

Prerequisites: None.

[7] Also, if you're going off to college, many universities provide free Mathematica licenses for their students.

Problem Solving: Tetrahedra. (🌶🌶🌶🌶, Misha, 1 days)
In the nine years from 1964 to 1972, every IMO competition contained a question with a tetrahedron in it. Since then, no such question has showed up again. In this class, we go back to the halcyon days of yore and solve as many of these problems as we can.

(This is a repeat of the class I taught in 2016.)

Homework: None
Prerequisites: None

Prüfer Codes and Counting. (🌶🌶, Misha, 1–2 days)
The trees on vertices $v_1, v_2, \ldots, v_n$ correspond exactly to Prüfer codes: $(n-2)$-tuples of integers from $\{1, 2, \ldots, n\}$.

This gives us a nice way to count all such trees, but in this class we will go beyond that, and use Prüfer codes to solve more specific counting problems.

Homework: Optional
Prerequisites: Graph theory

The Hales–Jewett Theorem. (🌶🌶🌶🌶, Misha, 4 days)
Ramsey theory is a branch of combinatorics about proving that sufficiently large structures have ordered substructures.

Why is it called "Ramsey theory" and not "Ramsey collection of similarly flavored results"? Because of theorems such as the Hales–Jewett theorem that connect them, letting us prove many results from just one and giving a unified way to improve our bounds on "sufficiently large".

Imprecisely speaking, the Hales–Jewett theorem is about tic-tac-toe. If you color a $3 \times 3 \times \cdots \times 3$ grid with 1000 colors, then in sufficiently many dimensions you are guaranteed to find 3 collinear points all of one color, and this is true for all values of 3 and 1000.

We will give one or more proofs of this theorem and possibly compare how good they are. We will also see some of the ways that other results in Ramsey theory fall out of the Hales–Jewett theorem.

Homework: Optional
Prerequisites: None

Yes, you can square the circle. (🌶🌶, Misha, 1 days)
Since ancient times it has been an open problem to use compass and straightedge to construct a square and a circle with equal area.

That is, it has been an open problem until it was proved to be impossible in 1882.

But we'll do it anyway: in the hyperbolic plane.

Homework: None
Prerequisites: None

Nic Ford's Classes

Special Relativity. (🌶🌶, Nic Ford, 4 days)
Around the beginning of the twentieth century, physics was undergoing some drastic changes. The brand-new theory of electromagnetism made very accurate predictions, but it forced physicists to come to grips with a strange new truth: there is no such thing as absolute space, and there is no such thing as absolute time. Depending on their relative velocities, different observers can disagree about the length of a meterstick, or how long it takes for a clock to tick off one second.

In this class, well talk about the observations that forced physicists to change their ideas about space and time, and how the groundwork of physics has to be rebuilt to accommodate these observations.

We will see how, as Minkowski said, "Space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind union of the two will preserve an independent reality." At the end, we'll also briefly look at how to revise the classical definitions of momentum and energy and see why we should believe that $E = mc^2$.

Homework: Optional

Prerequisites: none

Pesto's Classes

Closed Quasigeodesics. (🌶🌶, Pesto, 1 days)
A *closed quasigeodesic* on the surface of a polyhedron is a loop which has at most π worth of angle on both sides of it, everywhere. (So, on a face, it's a straight line; across an edge there's only one way it can go, and at a vertex there may be many or no ways for it to go.) We'll talk about why these always exist and how to find one.

Homework: None

Prerequisites: None

Graph Minors. (🌶🌶🌶, Pesto, 2–4 days)
The graphs K_5 and $K_{3,3}$ are nonplanar. In fact, *Wagner's Theorem* says that a graph is nonplanar if and only if it contains one of those two graphs as a "minor". We'll define minors, prove that theorem, and talk about why K_5 is hardly necessary in that statement. Also, we'll state the most famous unsolved problem in graph theory, a generalization of the Four-Color Theorem, and prove it for some special cases.

Homework: Recommended

Prerequisites: Graph theory: Understand the first sentence of the blurb.

How to Get Away from Dead Guys. (🌶🌶, Pesto, 1 days)
To get his PhD at MIT, Adam gave a talk on how get away from dead guys if you are in some area A and can run at rate 1, and they have to stay out of A but can run at a rate $z > 1$; you get away if you get to the edge of A with no dead guy at the same spot, but the dead guys try to stop you and be at the same spot on the edge of A. I'll tell you how to get away for some z, how the dead guys can stop you for some z, and why it's hard to say who wins for some more z.

Also, like a dead guy who can't talk well, I'll stay in the game of four.

Homework: None

Prerequisites: If you don't know what the statement "The problem of finding a set of k vertices in a graph such that every edge contains at least one is NP hard" means, ask before class.

Line Graphs. (🌶🌶, Pesto, 1–2 days)
Each pilot for a certain airline regularly flies back and forth between two cities. Pilots can talk to each other if they're ever in the same city at the same time (and flights are delayed often enough that any two pilots whose flights share a city will be able to talk to each other).

If G is the graph of cities connected by flights, then the graph of pilots connected by being able to talk to each other is called the *line graph* $L(G)$ of G.

What graphs are line graphs of some graph?

Homework: Optional

Prerequisites: None.

Multi-Coefficient Solving of Problems. (🌶🌶🌶🌶, Pesto, 3–4 days)
Polynomials are a frequent topic of Olympiad-style competitions, since there are many and interesting problems using them. For instance, if $a_1, \dots, a_n$ are distinct real numbers, find a closed-form expression for

$$\sum_{1\leq in} \prod_{1\leq j\leq n, j\neq i} \frac{a_i + a_j}{a_i - a_j}.$$

Dont see the polynomials? Come to class and find them.

Homework: Required

Prerequisites: Linear algebra: be able to write at least two bases for the vector space of polynomials of degree at most 3 in x.

Nonzero-sum games. (🌶, Pesto, 1 days)
We'll play and talk about combinations of (Iterated) (Community) (Hidden) Prisoner's Dilemmas (with punishment). Ask me at TAU about their relationship to theories of ethics.

Homework: Optional

Prerequisites: None

Shiyue's Classes

Matroids. (🌶🌶–🌶🌶🌶, Shiyue, 2 days)
What is a "matroid?" It might sound unfamiliar, but all of you have actually already met matroids many times in your math life. We will introduce first a family of very intuitive and widely-used computer algorithms called greedy algorithms - a "greedoid, and see typical examples of "graphic matroids. Both are examples of "matroids." Then we will look at some cool properties of matroids – matroids being a generalization of vector spaces, realizability of matroids and their fantastic invariants – Tutte polynomials. Behind the simple definition of matroid, we will be pleasantly surprised that they have recently given rise to many very cool geometric, topological and combinatorial problems and solutions!

Homework: Recommended

Prerequisites: None.

Riemann Surfaces and Algebraic Curves. (🌶🌶🌶–🌶🌶🌶🌶, Shiyue, 3–4 days)
We will review some complex analysis ande start with manifolds, compact surfaces, euler characteristic and orientability. Then we will talk about definitions, properties, examples of Riemann surfaces, algebraic curves, complex projective curves. Our goal is to cover maps from $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$, maps of elliptic curves, Riemann-Hurwitz formula and its application.

Homework: Recommended

Prerequisites: Complex analysis; some familiarity with projective space.

Tropical Plane Curves. (🌶🌶–🌶🌶🌶, Shiyue, 2–4 days)
Happy Halloween! We saw some creature named tropical geometry going around TAU. But what is that? Tropical Geometry is an emerging subfield of algebraic geometry. Technically speaking, it provides a modern degeneration technique to replace algebraic varieties with combinatorial objects. Classical algebraic geometers study the interplay between polynomials and their zeros. But zeros could be hard to study. In 1990s, people discovered that transforming all our polynomials into tropical semiring, which is $\mathbb{R} \cup \{\infty\}$ with the usual addition and multiplication replaced with taking the min/max and addition, will turn polynomials into piecewise linear functions and their zeros into

polyhedra. Now we can do combinatorics to tackle these algebraic geometry problems! This course will cover basic arithmetic in tropical semiring, tropical plane curves, Bezouts Theorem for tropical plane curves, etc.
Homework: Recommended
Prerequisites: Some ring theory will be useful

SUSAN'S CLASSES

Continued Fraction Expansions and e. (, Susan, 3 days)
The continued fraction expansion of e is

$$1+\cfrac{1}{0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{\ddots}}}}}}}}}}}}}}$$

Okay, but seriously, though, why?!?! Turns out we can find a simple, beautiful answer if we're willing to do a little integration. Or maybe a bit more than a little? No previous experience with continued fractions necessary. Come ready to get your hands dirty—it's gonna be a good time!
Homework: Recommended
Prerequisites: None.

Diamond, Forcing, and CH. (, Susan, 1–2 days)
What is the diamond axiom, what does it have to do with the Continuum Hypothesis, and how can we prove that it is consistent with ZFC? In this class, we will answer all three of these questions! We will (re)define the diamond axiom, prove that it implies that the Continuum Hypothesis is true, and present the forcing proof of its consistency.
Homework: Optional
Prerequisites: The Continuum Hypothesis

Irrationality Proofs. (–, Susan, 1 days)
I'm sure you can prove that $\sqrt{2}$ is irrational. Let's do some weirder things! In this class, we'll prove the irrationality of $\sqrt{2}+\sqrt{3}$, $\log_5(7)$, e, and π!
Homework: None
Prerequisites: None

Ordinal Arithmetic. (, Susan, 1–2 days)
Maybe you've heard of Hilbert's Hotel, a mathematical thought experiment about a hotel with infinitely many rooms, all in a row. This turns out to be an excellent way to think about the ordinal numbers!

We know what the operations $+$, $\times$, and ˆ mean on the natural numbers. In this class, we'll talk about what happens when we extend them to infinite sets. We'll give a simple definition of the ordinal numbers, and talk about how to visualize their addition, multiplication, and exponentiation.
Homework: None
Prerequisites: None.

Why Susan Is Wrong. (, Susan, 2–3 days)
I owe you all an apology. In Week 1, I promised you that my Stupid Games On Uncountable Sets would use only the axioms of Zermelo-Fraenkel set theory. Unfortunately, it turns out that we needed choice! You see, it turns out that assuming the Axiom of Determinacy, we can prove that the club filter on ω_1 is an ultrafilter. How embarrassing! Come see the proof.

Homework: Optional

Prerequisites: Stupid Games on Uncountable Sets

Tim!'s Classes

Calculus without Calculus. (, Tim!, 1–4 days)
If you've taken a calculus class in school, you've surely had to do tons and tons of homework problems. Sometimes, calculus knocks out those problems in no time flat. But other times, the calculus solution looks messy, inelegant, or overpowered. Maybe the answer is nice and clean, but you wouldn't know it from the calculation. Many of these problems can be solved by another approach that doesn't use any calculus, is less messy, and gives more insight into what is going on. In this class, you'll see some of these methods, and solve some problems yourself. Some example problems that we'll solve without calculus:

- Jessica is 5 cubits tall and Larsen is 3.9 cubits tall, and they are standing 3 cubits apart. You want to run a string from the top of Jessica's head to the top of Larsen's head that touches the ground in the middle. What is the shortest length of string you can use?
- Ania rides a bike around an ellipical track, with axes of length 100 meters and 150 meters. The front and back wheels (which are 1 meter apart) each trace out a path. What's the area between the two paths?
- A dog is standing along an inexplicably straight shoreline. The dog's person stands 20 meters way along the shoreline throws a stick 8 meters out into the water. The dog can run along the shoreline at 6.40 meters per second, and can swim at 0.910 meters per second. What is the fastest route that the dog can take to get to the stick?
- Where in a movie theater should you sit so that the screen takes up the largest angle of your vision?
- What's the area between the curves $f(x) = x^3/9$ and $g(x) = x^2 - 2x$?

Amaze your friends! Startle your enemies! Annoy your calculus teacher!

Homework: Recommended

Prerequisites: Some calculus will be useful for context, but we won't actually use calculus (that's the point).

Error-Correcting Codes. (, Tim!, 3–4 days)
Ben and I are secretly planning to take over the camp. Shhh, don't tell anyone; it's a secret! Of course, we have a secret code. Ben sends me messages by knocking or tapping four times on my door. For instance, knock-tap-knock-tap means "I've hacked the class schedule and replaced every class with Category Theory", tap-tap-tap-knock means "Tonight is the night to steal Susan's idol of power", and so on.

One night, Ben knock-knock-knock-knocks on my door, but I mishear it as knock-tap-knock-knock. So, instead of the message "Let me in", I respond to the message "May Day! Burn down the dorms!".

This is a setback.

The problem is that if I mishear even one of the knocks, I get the wrong message-phrase. But there is a solution! There are codes that are error-detecting — if I mishear one of the knocks/taps, I'll know just from what I heard that something has gone wrong. Even more amazingly, there are codes that are error-correcting — if I mishear one of the knocks/taps, then the knocks/taps I do hear will tell me exactly what I misheard and what the correct message was supposed be. It seems too good to be true, but the simplest error-correcting codes are easy to construct, and the best ones are used in real-life computers and computer systems all over the world (and all over the solar system — the Cassini probe that crashed into Saturn last September really wanted to make sure its photos and data get transmitted back to earth correctly before its mission ended).

We'll see the power and magic of these codes!

Homework: Recommended

Prerequisites: Linear Algebra

Infinitesimaller Calculus: Sequences, Integrals, and Democracy. (🌶🌶🌶, Tim!, 2–3 days)
In Infinitesimal Calculus, we constructed the hyperreal numbers, and started to use them to construct pretty proofs about continuous functions and derivatives, ending with the Extreme Value Theorem. But there is so much more calculus to prettify!

We'll handle sequences and their limits with ease, we'll define integrals the way you've *always* wanted to, and we'll bask in the Fundamental Theorem of Calculus.

And hey, since we've been talking about hyperreal numbers in terms of voting, we're actually pretty well prepped to do some voting theory! We'll use ultrafilters to prove Arrow's Impossibility Theorem, which says that the only fair voting system is a dictatorship. But instead of hyperreal numbers, we'll use hyper-something else!

Homework: Recommended

Prerequisites: Infinitesimal Calculus

Viv's Classes

Dysfunctional Analysis. (🌶🌶🌶, Viv, 2–4 days)
There's a lot of facts that we'd generally like to be true about normed vector spaces. For example, we'd like to think that the unit ball is compact. Or that the dual space of the dual space is our original vector space (why else would we call it a dual?). It turns out we only want these things to happen because we only like to think of vector spaces that are very small. In this class, we'll think about some examples of big vector spaces, and watch the world crumble around us.

Homework: Recommended

Prerequisites: Linear Algebra

How to Juggle. (🌶–🌶🌶, Viv, 1–2 days)
In this class, you will learn to juggle...in theory.

We'll discuss juggling sequences, a mathematical model for juggling that revolutionized the juggling world!

Homework: None

Prerequisites: None!

Wreath Products. (🌶🌶–🌶🌶🌶, Viv, 2 days)
You may have heard of a couple of ways to take products of groups: for example, directly, or semi-directly.

Wreath products are another way! They're also my favorite way. We'll define them and discuss examples, like the lamplighter group or the Rubik's Cube group.

Homework: Recommended

Prerequisites: Group Theory