CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2017

Contents

9:10 Classes

Functions of a Complex Variable, Week 1. (, Mark, Tuesday–Saturday)
Spectacular (and unexpected) things happen in calculus when you allow the variable (now to be called $z = x + iy$ instead of x) to take on complex values. For example, functions that are "differentiable" in a region of the complex plane now automatically have power series expansions. If you know what the values of such a function are everywhere along a closed curve, then you can deduce its value anywhere inside the curve! Not only is this quite beautiful math, it also has important applications, both inside and outside math. For example, functions of a complex variable were used by Dirichlet to prove his famous theorem about primes in arithmetic progressions, which states that if a and b are positive integers with $\gcd(a, b) = 1$, then the sequence $a, a + b, a + 2b, a + 3b, \ldots$ contains infinitely many primes. This was probably the first major result in analytic number theory, the branch of number theory that uses complex analysis as a fundamental tool and that includes such key questions as the Riemann Hypothesis. Meanwhile, in an entirely different direction, complex variables can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow. Dirichlet?s theorem is certainly beyond the scope of this class and heat conduction probably is too, but we should see a proof of the so-called "Fundamental Theorem of Algebra", which states that any nonconstant polynomial (with real or even complex coefficients) has a root in the complex numbers. We should also see how to compute some impossible-looking improper integrals by leaving the real axis that we're supposed to integrate over and venturing boldly forth into the complex plane! This class runs for two weeks, but it should be worth it. (If you can take only the first week, you'll still get to see a good bit of interesting material, including one or two of the things mentioned above.)

Homework: Recommended

Prerequisites: Multivariable calculus (including Green's Theorem; if the MV crash course doesn't get to Green's Theorem, it will be covered near the beginning of this class)

Related to (but not required for): Riemann Surfaces (W4)

Hyperplane Arrangements. (, Kevin, Tuesday–Saturday)
Suppose I want to cut a watermelon into pieces. With 0 cuts, I can make 1 piece: the entire watermelon. With 1 straight cut, I can make 2 pieces. With 2 cuts, I can make 4, and with 3 cuts, I can make 8.

I'm sure you see the pattern by now: with k cuts, we can make $\frac{1}{6}(k^3 + 5k + 6)$ pieces. These cuts are an example of a *hyperplane arrangement*, which is simply a collection of k hyperplanes in

n-dimensional space. We'll learn some powerful techniques for counting the number of regions that a hyperplane arrangement determines, and we'll maybe even see some connections to graph theory. Throughout, partially ordered sets will abound!

Homework: Recommended

Prerequisites: None.

Math and Brains. (🌶🌶, *Nora Youngs*, Tuesday–Saturday)
The brain is an incredibly complex organ, and even with impressive advances in neuroscience there are still many mysteries remaining around how it functions. In this class, we'll take a look at some of the mathematical models which have been used to understand how neurons work, and how they interact in both small and large networks. We'll also try to see how these models answer questions like: What is a memory? and: How does learning happen? Phrased another way, in this class we will think about what might be the best mathematical way to think about thinking.

Homework: Optional

Prerequisites: The willingness to suspend reality in order to construct a feasible model.

Planar Algebras. (🌶🌶🌶, *Noah Snyder*, Tuesday–Saturday)
We write mathematics, like we write words, on a line. This means you can multiply on the left or on the right, but not on top or bottom. In planar algebra, you can use the whole page to write your expressions multiplying in any direction you like. Planar algebras play important roles in operator algebras, knot theory, and quantum groups. One nice thing about planar algebra is it's a place where some very recent research is accessible without a lot of background, in particular most of the material in this class will be taken from papers published in the past 35 years, some as recent as last year!

Homework: Recommended

Prerequisites: Linear Algebra (vector spaces, bases, and dimension)

Set Theory. (🌶🌶🌶🌶, Steve, Tuesday–Saturday)
Sets appear everywhere in mathematics—it's very difficult to do math without sets. What about studying sets without math?

It turns out that sets are all we need to do math! We start with the emptyset, and build progressively more complicated sets with some basic operations (taking powersets, taking unions, . . .) and a couple more complicated operations. It turns out that from this modest beginning, we can build all of mathematics!

One particularly interesting question, which was asked in various ways during the early 20th century, is: what sets do we *actually need* in mathematics? One way to phrase this is to ask about *parts* of the whole universe V, which aren't the whole thing but still have "enough sets" that they satisfy the ZFC axioms (and maybe more!), and at the same time are easier to understand than V. This is called *inner model theory*, and is one of the main research areas in modern set theory. The goal of this class is ultimately to do some inner model theory, and see why it's super cool.

Still undecided? Inner model theory studies *iterable weasels.* Seriously. Google it. Weasels. Tell your friends!

Homework: Recommended

Prerequisites: Comfort with proofs.

Related to (but not required for): Cloudy with a Chance of the Continuum Hypothesis (W2); A Countable Hat Game and the Importance of Measurability (W4)

10:10 Classes

Analytic Number Theory. (🌶🌶🌶🌶, *Djordje Milicevic*, Tuesday–Saturday)
Did you know that a large majority of the numbers with a quadrillion digits have at least 30 but no more than 40 prime factors? (Really.) If you have a large prime, how far is the next one anyway? Is it more likely to end in digit 1 or 7? And how hard — or how important — can it be to locate all zeros of a single function (the Riemann zeta-function) that you can earn a million dollars for doing so, and what does this have to do with throwing a fair coin?

Hardly any collection of questions appears more disparate than these, but actually they all have two things in common: 1) they combine the beautiful and intricate multiplicative structure of the integers with the concepts and tools of calculus, the study of continuous change, and 2) we will talk about all of them this week! We will learn about arithmetic functions and their average orders, techniques of analytic number theory, characters, the Riemann zeta function, and the prime number theorem, and we will survey some landmark and contemporary developments in analytic number theory.

Homework: Recommended

Prerequisites: A little familiarity with number theory (divisibility, primes) and with single-variable calculus. You should know what derivatives and integrals are, but you do not have to have a lot of experience working with them.

Riemann and Series. (🌶🌶, Lara, Tuesday–Saturday)
The Riemann Hypothesis is one of the biggest open problems there are in mathematics. We'll begin this course by understanding what this hypothesis says. However, we'll have a far less lofty goal than proving it.

Instead the goal of this class is to find the values taken by the Riemann Zeta function (the function on which the hypothesis is based) at positive even integers. We'll develop the theory of Taylor series, see some of its applications and the problems it can be used to solve, finishing this section with understanding Euler's intuition about what $\zeta(2)$ should be. We'll then get acquainted with Fourier series and use them to prove that Euler was right and to come up with a recursive formula for $\zeta(2n)$.

Homework: Recommended

Prerequisites: A bit of familiarity playing with integrals; come talk to me if don't have this but are willing to work extra hard.

Systems of Differential Equations. (🌶🌶🌶, Mark, Tuesday–Saturday)
Many models have been devised to try to capture the essential features of phenomena in economics, ecology, and other fields using systems of differential equations. One classic example is given by the Volterra-Lotka equations from the 1920s:

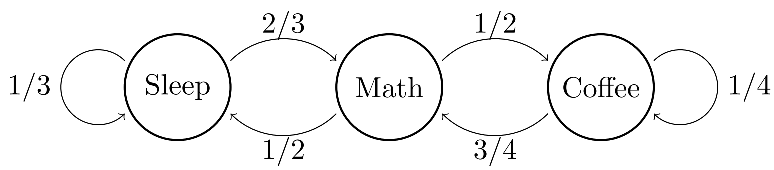
$$\frac{dx}{dt} = -k_1\,x + k_2\,xy\ ;\ \frac{dy}{dt} = k_3\,y - k_4\,xy\,,$$

in which x, y are the sizes of a predator and a prey population, respectively, at time t, and k_1 through k_4 are constants. There are two obvious problems with such models. Often the equations are too hard to solve (except, perhaps, numerically); more importantly, they are not actually correct (they can only hope to approximate what really goes on). On the other hand, if we're approximating anyway and we have a system $\frac{dx}{dt} = f(x,y)\ ;\ \frac{dy}{dt} = g(x,y)$, why not approximate it by a linear system such as $\frac{dx}{dt} = px + qy\ ;\ \frac{dy}{dt} = rx + sy$? Systems of that form can be solved using eigenvalues and eigenvectors, and usually (but not always) the general behavior of the solutions is a good indication of what actually happens for the original (nonlinear) system if you look near the right point(s). If this sounds interesting, come find out about concepts like trajectories, stationary points, nodes, saddle points, spiral points, and maybe Lyapunov functions. Expect plenty of pictures, and probably an opportunity for some computer exploration using *Mathematica* or equivalent. (If you don't want to get involved with computers, that's OK too; most homework will be doable by hand.)

Homework: Recommended

Prerequisites: Linear algebra (eigenvectors and eigenvalues), calculus, a little bit of multivariable calculus (equation of tangent plane)

Underhanded Tricks with Markov Chains. (, Misha, Tuesday–Saturday)
When I am not at Mathcamp, the activity I am doing is updated from hour to hour by the following rules (numbers on arrows denote probability):



This is an example of a Markov chain, and questions we might ask about it include the following:

- What fraction of the time am I doing work?
- How long will it take me to get home?
- What's the probability I'll go for a whole day without coffee?

All of these questions can be answered in a boring way: by solving systems of linear equations. In this class, we'll learn to solve them in more exciting ways: by defining a betting game about what I'm doing, granting me the power of time travel, or transforming the Markov chain into an electric network.

Homework: Recommended

Prerequisites: None; I'll make some offhanded references to linear algebra and graph theory, but you will not need either one to follow the class.

Vertex-Transitive Polytopes. (, Viv, Tuesday–Saturday)
We're going to spend this class studying highly symmetric shapes, for differing definitions of "highly symmetric" and "shapes." For starters, we'll define regularity and prove that the number of regular polytopes in n dimensions is given by one of my all-time favorite sequences:

$$1, \infty, 5, 6, 3, 3, 3, 3, 3, 3, 3, \ldots$$

Then we'll talk about relaxing regularity to vertex-transitivity. This gives us a lot more leeway to think about shapes that we'd really like to be able to call highly symmetric; for example, a soccer ball is vertex-transitive but not regular. We'll talk about what happens when we try to construct vertex-transitive shapes that have many vertices, and what happens with shapes in many dimensions. Along the way, we'll discuss discrete Gauss-Bonnet and Euler characteristic.

Homework: Recommended

Prerequisites: None.

11:10 Classes

Cloudy with a Chance of the Continuum Hypothesis. (, Angela, Wednesday–Thursday)
In the forecast Wednesday and Thursday: clouds!

Sometimes a cloud is defined to be a visible mass of condensed water vapour floating in the atmosphere, typically high above the ground. Not today. Today, a cloud C about a point p is a subset of the plane such that for every line through p, C intersects that line at only finitely many points. You might be wondering why you would care about sets like this. Well, it turns out that clouds are pretty weird.

Blurby seguey. The cardinality of the natural numbers is typically denoted $\aleph_0$, and the next biggest size of infinity (the smallest cardinality greater than $\aleph_0$) is denoted $\aleph_1$. The Continuum Hypothesis is a very famous statement that says that the cardinality of the real numbers is equal to $\aleph_1$; famous partially because it was the first statement that was proven to be independent of the ZFC axioms of set theory. Super famous.

Here's a simple statement about clouds: three clouds cover the plane. SURPRISE! It turns out this statement is actually equivalent to the Continuum Hypothesis. In this class we are going to prove the equivalence of these two statements. Get hyped. #gethypedforclouds

Homework: Recommended

Prerequisites: None.

Related to (but not required for): Set Theory (W2)

Coloring the Hyperbolic Plane. (, Ina, Friday–Saturday)
Here's a problem: color a plane such that no two points a unit distance apart are the same color. How many colors do you need? Turns out, nobody knows! Since when this problem was posed in the 1950s, we've only known bounds. A newer line of research examines the problem in the hyperbolic plane, where even finding bounds becomes more complicated—but also, potentially, more achievable. In this class we'll begin with our current knowledge about this problem on the flat Euclidean plane. Then we'll move into hyperbolic space, exploring current research—either for fun or to give you tools to dig into this research yourself!

Homework: Recommended

Prerequisites: None.

Related to (but not required for): What's It Like to Live in a Hyperbolic World? (W4)

Coloring Graphs on Surfaces. (, Marisa, Wednesday–Saturday)
The Four Color Theorem tells us that if we want to properly color a graph drawn on the plane, we need at most four colors to do it. There is a much cooler theorem which says that if you want to draw a graph on *literally any other surface* and then properly color the vertices, you will need at most this many colors:

$$chr(S) \le \left\lfloor \frac{7+\sqrt{49-24\chi(S)}}{2} \right\rfloor$$

...which looks mildly unattractive, but which you could absolutely prove this week. And in this class, you will indeed prove this, and check out lots of other interesting properties of graphs on surfaces along the way. Works for both orientable and non-orientable, to boot.

Homework: Optional

Prerequisites: Intro Graph Theory or equivalent. We'll be using concepts from topology (e.g. All Things Manifoldy), but I'll cover all the key points in class.

Related to (but not required for): All Things Manifoldy (W1); Euler Characteristic (W2); Shannon Capacity of Graphs (W2)

Cryptography, and How to Attack It, Week 1. (, Linus, Wednesday–Saturday)
In a *normal* cryptography course, you'd learn how codes like RSA work.

In *this* course, you'll learn how to use math to hack the people who took the other course. You won't just learn RSA, Diffie-Hellman, and (maybe) more: you'll learn how to break the Vigenere cipher; crack subtly incorrect implementations of RSA; and (theoretically) break the entire Cryptocat iPhone app circa 2013.

Homework: Recommended

Prerequisites: Have a good understanding of modular arithmetic. You should have internalized Euler's Theorem and the Chinese Remainder Theorem.

Related to (but not required for): Finite Fields (W2)

Finite Fields. (, Aaron, Wednesday–Saturday)
What do the rational numbers, complex numbers, and real numbers have in common, but not share with the integers? They are all fields; we can add, subtract, multiply, and divide elements in them. But which finite sets also have these properties? What possible sizes can such a finite set have? What are the possible subfields? These questions all have simple, beautiful answers which we will present in this course. Finite fields are crucially used throughout number theory, algebraic geometry, cryptography, and coding theory.

Homework: Required

Prerequisites: linear algebra, group theory

Related to (but not required for): Group Theory (W1); The Kakeya Conjecture (W1); Linear Algebra (W1); Prime Numbers (W1); Cryptography, and How to Attack It, Week 1 (W2); Ring Theory (W2); Quadratic Field Extensions (W3); Cryptography, and How to Attack It, Week 2 (W3); Finite Geometries (W4)

Required for: Classification of Subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$ (W3)

Mathematica Workshop. (, Beatriz, Wednesday–Saturday)
Mathematica software is a very useful tool that allows us to efficiently solve symbolic and algebraic problems. We'll learn how to perform several different computations, plot graphs, and create dynamic visualization using Mathematica.

Homework: Recommended

Prerequisites: None.

Required for: The Logistics of Zombies: Cobwebs and Chaos (W3)

1:10 Classes

Euler Characteristic. (, Apurva, Tuesday–Saturday)
Euler characteristic is a benign number that is computed by counting vertices and edges of graphs. This single number explains why a cyclone should have an eye, why you can eat a pizza without spilling all the toppings and why my hair looks messy no matter how much gel I apply. The entire branch of algebraic topology was invented to rationalize the existence of so powerful an invariant.

In this class we'll understand what it means for the Euler characteristic to be an invariant of surfaces and explore several geometric manifestations of it. We'll see proofs of the Sperner's lemma, Brouwer's fixed point theorem, hairy ball theorem and Gauss-Bonnet theorem using Euler characteristic.

Homework: Recommended

Prerequisites: None.

Related to (but not required for): The Fundamental Group (W1); Coloring Graphs on Surfaces (W2); Symmetries of Spaces (W3); How to Define the Square Root (W4); Riemann Surfaces (W4)

Evasiveness. (, Tim, Tuesday–Saturday)
We'll explore a conjecture in computer science that has been open for over 40 years, concerning the complexity of graph properties. One way to measure the complexity of a problem (like "Does this graph have a Hamiltonian cycle?") is by its time complexity — roughly, how long it takes a computer to solve it. Another important way to measure complexity is query complexity — roughly, how many questions you need to ask about the graph to sole the problem. The graph properties with maximum query complexity are called *evasive*, and the conjecture is that a huge class of graph properties — specifically, all those that are nontrivial and monotone — are evasive.

We'll trace the story of this conjecture through time from its conception in 1973, through a surprising appearance of topology in 1984, to the present day, including research from the past few years. Along the way, we'll see scorpion graphs, clever counting, collapsible simplicial complexes, transitive permutation groups, and hypergraph properties.

This class is directly related to my research, and in class we'll see a recent result of mine, along with its proof.

Homework: Required

Prerequisites: Group theory (normal subgroups, quotient groups). If you haven't seen graph theory, talk to me first.

Ring Theory. (, Susan, Tuesday–Saturday)
Ring theory is a beautiful field of mathematics. We cut ourselves loose from our usual number systems—the complexes, the reals, the rationals, the integers, and just work with . . . stuff. Stuff that you can add. And multiply. Rings are structures in which addition and multiplication exist and act as they "should." Polynomials, power series, matrices, real-valued functions on a set—wherever you have some way of defining an addition and a multiplication, you've got a ring.

Somehow, in throwing away the numbers that gave us our initial intuition about how addition and multiplication should work, we are left with a tool that is immensely powerful. Ring theory is the backbone of fields such as algebraic geometry, representation theory, homological algebra, and Galois theory.

This class will be a quick introduction to some of the basics of ring theory. We will cover the ring axioms, homomorphisms of rings, integral domains, and basic commutative localization theory.

Homework: Required

Prerequisites: None.

Related to (but not required for): Finite Fields (W2); Classification of Subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$ (W3)

Required for: Division Rings, Week 1 (W3)

Shannon Capacity of Graphs. (, Yuval, Tuesday–Saturday)
What do umbrellas have to do with text messages? As it turns out, quite a bit! In this class, we will use graphs to understand communication, and then use communication to understand graphs. In the end, we will use umbrellas to understand both communication and graphs.

Homework: Recommended

Prerequisites: Basic graph theory (Marisa's Week 1 class certainly suffices, but come talk to me if you haven't taken it; you might be prepared anyway.)

Related to (but not required for): Intro to Graph Theory (W1); Coloring Graphs on Surfaces (W2)

Turing and His Work. (→, Sam, Tuesday–Saturday)
It's hard to understate how remarkable of a person Alan Turing was. His contributions to mathematics are as broad as they are significant: he was instrumental in breaking encrypted Enigma messages, laid much of the groundwork for theoretical CS, wrote computer programs before anything even vaguely

resembling today's computers existed, and towards the end of his life, started working in mathematical biology. He is described as "shy and diffident" and "fairly clumsy," but also as a "a warm, friendly human being" who "was obviously a genius, but [an] approachable and friendly genius [who was] always willing to take time and trouble to explain his ideas" and who was "funny and witty." He was an avid fan of chess, both writing perhaps the first computer program to play chess and inventing his own form of chess-sprinting, and ran marathons at a near-Olympian level.

This class is a seminar in which we'll try to get to know who Turing was as a person through a variety of lenses. Day 1 will be primarily social history; we'll look at some of the major events in his life and get a better read on his personality. Then we'll transition to seminar, where we'll read and discuss some of his papers. Finally, we'll switch gears again, and try to learn about Turing Machines from one of Turing's papers!

Caveat Emptor: This is a seminar style course, so "homework required" means that some reading (15 minutes to an hour) will be required each night. The reading is vital to the class. On the plus side, you get to read actual mathematics papers, read papers written by Turing, and learn directly from Turing through them!

Homework: Required

Prerequisites: None!

Colloquia

Project Selection Fair. (The Staff, Tuesday)

A special colloquium-slot event: at the *Project Selection Fair*, held in the classrooms in Thompson as well as outside, you'll get a chance to ask staff about the projects they've proposed (project blurbs: available Saturday night). If you're interested in participating in a staff-supervised project at camp, whether it's already been proposed, or you have an idea of your own that you're excited about, make sure to come to the project selection fair, and fill out a preference form.

Farey Fractions and Plane Geometry. (*Noah Snyder*, Wednesday)

How do you find the best way to approximate an irrational number using rationals? How good are these approximations? These questions can be answered using certain properties of the Farey sequence. These patterns in the Farey sequence can in turn be proved using plane geometry. No background will be assumed (in particular, this talk should be understandable to people who haven't seen continued fractions while still being interesting to people who have.)

Algebra and the Internal GPS. (*Nora Youngs*, Thursday)

When you think of neuroscience experiments, one thing which may come to mind is a scientist in a lab coat running rats through a maze. But how exactly are those rats learning to navigate that maze? Part of the answer lies with a set of neurons called place cells, which are so named because they are specifically active for certain locations. We'll consider an algebraic way to extract useful geometric information from the neural data of place cells, and explore how that information can tell us something about the animal's environment.

Why are drums shaped by number theory sometimes louder than others? (*Djordje Milicevic*, Friday)

Simple harmonics, such as monochromatic light waves or heart rhythms or standing patterns of a vibrating string, are basic building blocks of analysis: a compound signal like sunlight or the sound of your favorite instrument is composed of (many) single-color bands or single-pitch tones.

On more general spaces, flat ones or those with some curvature, the role of simple harmonics is played by eigenfunctions, objects central in contexts ranging from spectral geometry, a field whose spirit was captured by Mark Kac's famous question "Can you hear the shape of a drum?", to quantum mechanics, where they represent "pure quantum states" and where their concentration of mass is closely related to geometry and dynamics.

After describing some of these fundamental modes and what they can tell us about the underlying spaces, we will discuss what eigenfunctions have to do with number theory (things like primes, or divisors, or Fermat's Last Theorem) and how additional symmetries of arithmetic or geometric nature can drive their exceptional behavior not generically observed or predicted by physical models.

Visitor Bios

Djordje Milicevic. Djordje is excited about questions that combine analysis (the study of continuous change) and number theory. That combination is actually pretty crazy, if you think about it. (Unless you think about it for a little longer, at which point it becomes too awesome.) Djordje took part in International Mathematical Olympiads and undergraduate competitions, has taught problem solving seminars for over 15 years, and co-wrote three Putnam exams. In his research on automorphic forms and analytic number theory, Djordje is particularly interested in objects known as Maass forms, basic building blocks of analysis on curved spaces with extra number-theoretic structure, as well as in central values of L-functions, which generalize the Riemann zeta function and guide phenomena ranging from distribution of primes and lattice points to quantum chaos. He enjoys talking to and learning from students, cooking, traveling, and good chocolate.

Noah Snyder. Noah Snyder is an associate professor at Indiana University studying 2-dimensional algebra. Instead of multiplying on the left or on the right, he'd rather be able to multiply on the top and bottom as well. He's also branched out into 3 dimensions as well. He also does research in knot theory, subfactors, quantum groups, and category theory. He was a mentor at Mathcamp from 2006-2008 and the AC in 2009, and a visitor in 2010-2016. He cowrote the 2008 and 2010 Mathcamp puzzle hunts, and the 2006 and 2011 MIT mystery hunts. He likes extinct animals, and baby tapirs.

Nora Youngs. Nora Youngs teaches at Colby College, and does research in applied algebra and mathematical neuroscience. She is fascinated by the brain, and delights in using unexpected mathematical tools to understand the way it works. Nora is an avid runner, loves to cook, and still gets super excited about train rides.