CLASS BLURBS—WEEK 5, MATHCAMP 2008

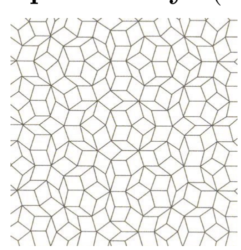
Contents

Classes at 9:30TW and 11:30 HF

Stirling's Formula. (, Mike, Tue)
$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$. Yes, that one.

Prerequisites: Calculus

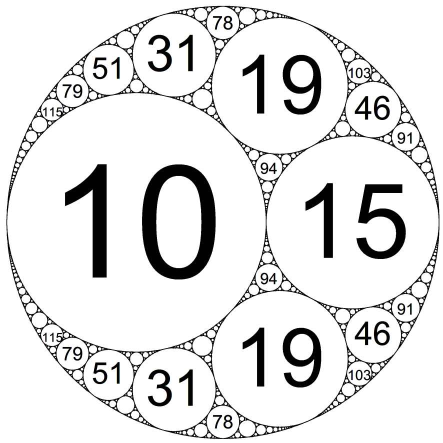
Penrose Tilings and Quasiperiodicity. (, Dan, Wed)



For a long time it was not known whether there existed a finite set of tiles that could tile the plane aperiodically but not periodically. In fact such sets of tiles do exist. The most famous examples of this phenomenon are Penrose tilings. In addition to looking cool, they have many interesting properties. We see that Penrose tilings have a kind of self-similarity, that any finite configuration of Penrose tiles appears infinitely many times in every Penrose tiling, and that Penrose tilings are two-dimensional slices of a four-dimensional periodic pattern.

Prerequisites: None.

Apollonian Circle Packings. (, Laura Zehender, Fri)



This is an Apollonian Circle Packing: start with four mutually tangent circles, and inscribe circles in each of the "curvy triangles" made by three other circles. Then inscribe circles in the new curvy triangles that were made in the last step, and so on. Besides making a really cool picture, Apollonian Circle Packings have the interesting property that if the four circles you start with have integer curvatures, all of the circles in the packing will have integer curvatures (the curvature of a circle is the inverse of the radius; the smaller the circles, the larger the curvature. For example, the numbers in the circles in the above picture are curvatures.) In this class, I will show first how Apollonean Circle Packings are generated with inversion, then why all of the curvatures are integers.

Prerequisites: None

Quadratic Forms. (- , David, Tue-Fri)

A quadratic form is a polynomial like $x^2 + 3xy - yz + 2z^2$, where every monomial has degree exactly 2. You may be aware that every positive integer can be written in the form $x^2 + y^2 + z^2 + w^2$. But did you know that every integer can be expressed in the form $x^2 + 2y^2 + 4z^2 + 14w^2$?

The first two days of this class will be , and will introduce quadratic forms, lattices with inner products, and the problem of representability. The third and fourth days will be : we'll sketch the proof of the fifteen theorem, which gives a necessary and sufficient condition for a quadratic form to represent all positive integers, and I'll explain the Hasse principle, which allows one to detect whether a quadratic form represents zero non-triviallly.

Prerequisites: Elementary Number Theory, Linear Algebra. We'll be using inner products, but will go over them briefly in class.

Logic is Complete. (, *Kenny Easwaran*, Tue-Wed)

How do we know when one mathematical claim follows from another? Most of the time this is unproblematic, since we know it when we see it, but sometimes it's not immediately obvious whether there might be some sort of special case that you left out. In the late 19th century logicians came up with a formal language for representing logical statements, and a series of rules for manipulating these formal sentences that correspond to certain intuitively valid arguments. But it wasn't until 1930 that Kurt Gödel showed that these rules are in fact complete - that is, if there is a "proof" that you can't complete by using these rules, then there must in fact be some sort of "special case" where the premises are true but the conclusion is false. Thus, these logical rules are all you need for doing logically correct proofs.

Prerequisites: None, though some familiarity with set theory and/or logic may be helpful.

Homework: Recommended

Logic is Incomplete. (, *Kenny Easwaran*, Thu-Fri)

In the early 20th century it seemed to many mathematicians that there was so much progress in recently developed rigorous mathematical methods that we would be able to find a small set of axioms for set theory, and that all mathematical truths could be proven from those. However, in 1931, after proving his Completeness Theorem, Gödel showed that this was false even for basic arithmetic. For any reasonable set of axioms (that is, one where we can recognize whether or not a given sentence is an axiom), there is a sentence of the form "All natural numbers have property X" that can neither be proved nor disproved. The way that Gödel proved this is by showing that all of the logical operations for manipulating sentences and proving them can actually be represented in numerical terms, and showing that the notion of truth can't be represented in these terms. We will go through as much of this proof as we have time for, though many of the technical details will be left for homework.

Prerequisites: None, though some familiarity with number theory will be helpful.

Homework: Recommended

Mathematically Maximize Your Odds (All's Fair in Love and War). (, Shoe, Tue-Wed)

If you love superhits from the 1980's you've heard the famous words: Love is a Battlefield. For all of us (even those who don't love 1980's superhits) this statement resonates with wisdom and truth. And so for those of us hoping to win the battle of love, it seems reasonable that we should use every means at our disposal. Luckily for mathcampers, you have a natural advantage - I am of course referring to math!

This class will tap into the awesome power of combinatorics to analyze different situations from the battlefield, and help you (yes YOU!) maximize your odds in love. For example, if you have a set of boys and a set of girls, and each boy has an order of preferences of girls he likes, and each girl has an order of preferences of boys she likes, how can you match them up so that nobody will ever have the urge to be unfaithful? And how can you garuantee that the matching will be optimal for the boys (or for the girls)? Or, if a finite sequence of suitors is walking by you one by one (some more desirable than others) how do you decide when to stop them and choose one? As you can see this might be the most applicable math class you ever take. (*Disclaimer: the mentor makes no guarantees that this class will actually help you succeed in finding love, and takes no responsibility for any embarrassing situations or "dating disasters" which result from said class*)

Prerequisites: None

Homework: Listening to 1980's superhits

Fermi Problems. (, Noah, Thu)

How many professional truckers are there in the United States? What's the volume of the Empire State Building? In this class we'll learn to solve problems like these by cleverly guessing the answer rather than using *Google*. We'll take challenge questions from other staff and campers and solve them as a group.

Prerequisites: None

Homework: None

Irrationality Measure. (, JR, Tue-Wed)

We know that some real numbers are rational, while other are irrational. Is there any meaningful way to say that certain irrational numbers are 'more irrational' than others? One way measure the irrationality of a number is to see how closely that number can be approximated by rational numbers whose denominators are not too large. The irrationality measure of an irrational α is one way to quantify how well α can be approximated by rationals. In this class, we will define irrationality measure, and prove a theorem about the irrationality measure of roots of polynomial equations. This theorem will enable us to construct a real number α which cannot be the root of any polynomial with rational coefficients; that is, we will provide an explicit construction of a *transcendental* number. Time permitting, we will discuss the Littlewood conjecture, an unsolved problem closely related to irrationality measure.

Prerequisites: Knowledge of the mean value theorem

Homework: Optional

Modeling Populations with Differential Equations. (, Shoe & JR, Thu-Fri)

If you've ever been chased by bears, you've probably wondered "Does the growth rate of the bear population in this region depend on the number of campers living here?" In this class you will have the opportunity to answer this question.

Differential equations are equations such as $f'(x) = c + kf(x)$ in which the values of the function are related to the derivatives of a function. It turns out that equations of this sort are often useful in modeling when the rate of change of a quantity depends on the amount already present. As an example, the above formula could represent a population of rabbits, where the more rabbits there are, the faster they breed. Furthermore, we can use systems of these types of equations to model the impacts of different species on each other.

In this class we will give you a brief introduction to differential equations, and show you how to create systems of equations to model many types of species interactions (predator-prey, parasite-host, 2 species competing for a single resource, vampires vs. werewolves, etc. . .). After a brief lecture you will create your own population models (whatever you want!) and use *Excel* to approximate solutions and graph what happens.

Prerequisites: Calculus

Classes at 10:30

Quantum Groups. (, Noah, Tue-Thu)

Quantum groups are like groups, but not as commutative. Now I know you're thinking that groups can already be non-commutative, but quantum groups are even more non-commutative. In this class I'll attempt to describe the simplest quantum group $U_q(\mathfrak{su}_2)$ and its relationship to knot theory.

Prerequisites: Group representation theory is required after the first day, Ring Theory I will be helpful

Homework: Optional

Class Field Theory. (, David, Fri)

Class field theory was developed in the early twentieth century and provides an explicit description of abelian extensions of $\mathbb{Q}$, as well as a number of results on abelian extensions of arbitrary number fields and p-adic fields. It's impossible to do justice to this topic in one day; I'll try my best.

Prerequisites: Algebraic Number Theory, p-adics and Galois Representations. Or a desire to be completely lost.

Mathematical Logic through Puzzles. (, Alison & Waffle, Tue-Fri)

Suppose you are a perfect logician and you have been sent to investigate a strange organization known as the Mentor-Camper Society of Prevaricators. All members of this organization are one of two types: the Mentors, who always tell the truth, and the Campers, who always lie. One day, you meet a member who says to you: "You will never believe that I am a Mentor." As you think over the implications of her statement, you become more and more puzzled, and begin to worry that maybe you are not a perfect logician after all...

This class will be a by discovery class in which we explore puzzles and paradoxes of the type above. We'll investigate different concepts of what means to be a "perfect logician" and reason about reasoners who reason about themselves and who reason about how they reason about themselves. The conclusions we reach will have important ramifications for mathematical logic: we'll ultimately apply what we know to prove Godel's Incompleteness Theorems, which in essence say that any sufficiently complex formal system, if it is consistent, must be unable to prove its own consistency. In particular, no consistent formal system can prove every true statement.

Prerequisites: None

Homework: In-class

Measure and Martin's Axiom. (, Susan, Tue-Thu)

Are you an analyst who's dying to find out what happens between countability and continuum? Are you a logician who would love to explore the mysteries of measure theory? Then this is the class for you! On day one, we will discuss filters on posets, and how we can use Martin's Axiom to build objects that don't exist in standard set theory. On day two, we will discuss what it means for a subset of the real numbers to have measure zero, and we will show that countable unions of measure-zero subsets have measure zero. Then, in the spectacular conclusion, on day three we'll bring the two concepts together to show that assuming Martin's Axiom, the union of *any* collection of fewer than $2^{\aleph_0}$ measure-zero sets will have measure zero. A beautiful proof, and an absolute must-see for the analyst dabbling in logic, or for the logician dabbling in analysis!

Prerequisites: None, though depending on background, you could skip day one or day two.

How to Force $\diamond$. (, Susan, Fri)

I've told you it's consistent. Waffle's told you it's consistent. Isn't it about time you saw the proof?

Prerequisites: Continuum Hypothesis

Related to: Infinite Trees (weeks 2 and 3)

Voting Theory. (, Alfonso, Tue-Thu)

When a large group of people have to make a decision together, bad things can happen. For example, suppose that a group of 10 campers is trying to decide which game to play tonight. Suppose further that 3 of them want to play Mao, and the remaining 7 would prefer to play *any* game they can possibly think of other than Mao. If the remaining 7 are divided between 5 or 6 different games, a strict plurality election system will force them to play Mao, even though a majority of the 10 campers would prefer any other candidate to the winner.

It seems, then, that the plurality election system is unfair. What could we do to make it fair? Which election system is the most fair? What does "fair" mean, anyway? Come to this class and find out. *Warning:* Your faith in democracy may vanish after this class.

Note: This class blurb was shamelessly stolen from former Mathcamp mentor Dave Jensen.

Prerequisites: None

Math that Makes Ugly Noises. (, Jonathan Love, Fri)

People have very different opinions about what qualifies as "good music," but at least in the majority of Western culture, there seems to be a very strong consensus that some groups of notes sound better when put together than other groups of notes. This class will be used to introduce and analyze one of the fundamental properties of music: harmony (putting together groups of notes), and the related concepts of consonance ("nice-sounding") and dissonance ("ugly-sounding"). In the class, we will discuss how the musical scale as we know it today was developed, why certain pitches were chosen above others to create the 12-note scale, and we'll be looking at the waveforms produced when two sound waves are added together to come up with a good explanation for why some pairs of notes sound better when played together than others. Come with ear plugs, tough eardrums, or a willingness to cringe; there *will* be ugly noises.

Prerequisites: Knowledge of basic properties of sound waves. No musical experience required, but it would be helpful to know at least what a note is.

Classes at 11:30 TW and 9:30 HF

Introduction to Analytic Number Theory. (, JR, Tue-Wed)

In his entire career as a mathematician, Riemann wrote just one paper on number theory. This paper, only 8 pages long, would revolutionize the study of number theory to this day. In his paper, Riemann discusses a function $\zeta(s)$, which today is called the Riemann zeta function. Riemann discusses the connections between this function and the distribution of the prime numbers. He also comments that "it is very probable that all roots" of $\zeta(s)$ have real part equal to $\frac{1}{2}$. This statement, known as the Riemann

Hypothesis, has become one of the most important conjectures in analytic number theory, and indeed in all of mathematics.

What sort of problems are analytic number theorists trying to solve? What does analysis have to do with number theory? How is this related to the Riemann Hypothesis, and why do we care? Come to this class to find out.

Prerequisites: Basic calculus. A little complex analysis would be very helpful.

Homework: Optional

Polya Counting Method. (- , Dan, Tue-Wed)

Were you stumped by the relay problem about the probability that two permutations commute? This class can help! We will learn about the Redfield-Polya Theorem, a very useful theorem for counting things with symmetries. In addition to solving the relay problem, we will also learn how to count the number of ways to color polyhedra and the number of nonisomorphic graphs with a given number of vertices.

Prerequisites: You should know what a group is.

Homework: Optional

Related to: Generating Functions (week 1), Combinatorics of Permutations (week 3), Problem Solving in Combinatorics (week 3)

Unique Factorization and Ideals. (, Alison & Dan, Thu-Fri)

You're familiar with the integers, and perhaps the Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. But what about more exotic things like

$$\mathbb{Z}[\sqrt{-5}] \text{ or } \mathbb{Z}[\sqrt{-14}]?$$

Can we, for example, find a unique factorization of any number in these rings into a product of prime numbers? We will see that the answer is no. However, all is not lost! We'll learn the right way to generalize the concepts of "number" and "prime number" to "ideal" and "prime ideal" and show that we can find a unique factorization of any ideal into prime ideals.

Prerequisites: Number theory; familiarity with rings and linear algebra will be helpful

Homework: Recommended

Problems To Which I Do Not Know The Answer. (, *James Tanton*, Tue-Wed)

In the short series of sessions, James Tanton will invite you to explore — through both tactile and intellectual play — curious mathematical questions about the world that are easy to ask but seem mighty tricky to answer. See if you can make a splash in the world by proving something brand new! Partial success might be in reach. Total success would be a dream come true.

James will begin each session by briefly presenting a series of ideas. Whatever grabs the imagination and delight of the participants at the time will then be the theme for the session - and we can change our minds half way through! These sessions will be very interactive and based on shared discussion and discovery. Lots of play to be had!

Here are some sample ideas:

- **Bicycle Tracks**

  Ride a bicycle through a puddle and notice the two tracks (why two?) that it leaves behind. These curves have some wonderful mathematical properties: just from their shape one can determine the direction of travel of the bicycle and the length of the bicycle. But James has a question that no one can seem to answer except in the trivial case: Can two bicycles of different lengths produce identical tracks?
- **Base One-And-A-Half**

  In base two, the number ten appears as 1010, in base three as 101, in base four as 22, and in base ten as 10. Most surprising, in base one-and-a-half it appears as 2101. Let's explore base one-and-a-half and examine its connection to the famous Collatz conjecture.
- **Counting Permutations**

  Is it possible to place the numbers 1 through 10 in the boxes below and respect the inequalities shown?

  $$\square < \square > \square > \square < \square > \square < \square > \square < \square < \square$$

  Is there more than one way to accomplish this task? How many solutions are there? Can problems like these always be solved?

  Now take a strip of paper, fold it in half multiple times and then straighten it out. This produces a sequence of creases along the strip. If we regard each valley crease as "greater than" and each mountain crease as "less than," then the strip reads as a permutation problem akin to the one above.
- **Candy Games**

  There are multiple non-standard ways to share candy among people. Let's play some candy games and encounter unsolved mathematical problems along the way!

- **Flat Braids**
  People with very long hair sometimes divide their hair into three strands, braid them, and then tie the three loose ends together with a rubber band. Is it possible to push the tied ends back through the braid and untangle the hair-do completely? Mathematically the answer is (almost) YES if each strand is considered a thin string and twisting along the length of the strand does not count. But if twisting is not allowed, then, well, there are open questions to be explored!
- **Other Topics Include** (but are certainly not limited to):
  Layered Tilings
  Pile Splitting
  The Locker Problem Run Amok
  Dots and Dashes and Beyond
  Pile Splitting
  Langton's Ant on Finite Graphs

Prerequisites: None

Counting with Bijections. (, Josh Alman, Thu)
Counting can be really hard. For instance, say we wanted to find the number of non-decreasing positive integer sequences of length n, whose ith term is at most i. Counting these with "normal" methods would require solving annoying recurrence relations or breaking the problem into a huge number of cases for big enough n. However, if would could find a bijective function from the set of these sequences to a set that is easier to count, it would make our task much simpler. This method almost always involves cool insights and very elegant ideas, and although combinatorial in nature, it can extend to other areas of math in surprising ways. In this class, we'll count the sizes of some sets and prove some identities, including relations involving Fibonacci numbers. We will also use exclusively bijective means to prove Fermat's Little Theorem.

Prerequisites: None. If you are already very familiar with Catalan numbers and proving Fibonacci identities bijectively, you might not learn very much new material.

The Math of Projective SET. (, Mathieu, Fri)
Most of you have probably played the card game SET, and some of you have also played the game Projective SET (or its variant, Linear Dependence). If you're wondering what these games have in common, and why one of them is called projective, come to this class! Among other things,

we'll see why SET is really about finding lines in affine 4-dimensional space over the field with 3 elements, why Projective SET is really about finding lines in projective 5-dimensional space over the field with 2 elements, and why there is always a set in Linear Dependence.

Prerequisites: Linear Algebra

Homework: Playing Games

Related to: Geometry and Transformations (weeks 2-4)

A Proof from "The Book"! (Partitions and Franklin's Proof). (, Mark, Tue-Wed)

A *partition* of a positive integer n is a way to write n as the sum of positive integers, where the order doesn't matter, and traditionally the terms are written in nonincreasing order. For example, the partitions of 7 are 7, $6+1$, $5+2$, $5+1+1$, $4+3$, $4+2+1$, $4+1+1+1$, $3+3+1$, $3+2+2$, $3+2+1+1$, $3+1+1+1+1$, $2+2+2+1$, $2+2+1+1+1$, $2+1+1+1+1+1$, $1+1+1+1+1+1+1$. There are 15 in all, and we write $p(7) = 15$. There is a spectacular formula for $p(n)$ due to Hardy, Ramanujan, and Rademacher, but it's quite hard to prove and even to state. There is also an elegant recursive formula for $p(n)$, due to Euler:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15)...$$

We'll prove that formula using a beautiful combinatorial argument by Franklin - apparently, this was the first-ever substantial piece of mathematical research by an American-born mathematician.

Note: If you took Alison's generating functions class in week 1, you'll already have seen most or all of what I would do in this class.

Prerequisites: No fear of summations

Homework: Optional, perhaps

Rescuing Mathematical Nonsense. (- , Mark, Thu-Fri)

The topics below are not really related, but they have a common theme: Sometimes you shouldn't dismiss a piece of mathematics too quickly, even when it is obviously flawed; there may be a way to rescue it, and better yet, to make it useful.

- **Rescuing Divergent Series** ( ) Tue Consider the infinite series $1 - 1 + 1 - 1 + ...$ What is its sum? Maybe $(1-1) + (1-1) + ... = 0$, maybe $1 + (-1+1) + (-1+1) + ... = 1$. Medieval mathematicians were quite perplexed by this, and one even thought the issue had theological significance. Now surely it's nonsense to think that the "real" answer is $\frac{1}{2}$, just because the answers 0 and 1 seem equally good, right? After all, how could the sum of a series of integers be anything other than an integer? Prerequisites: A bit of experience with the idea of convergence (perhaps from a calculus class covering infinite series, or from real analysis)

- **Rescuing Euler's Sketchy Result** (–), Wed

  Among Euler's prodigous output is a paper that tries, among other things, to justify the equation $1 + 2 + 4 + 8 + ... = -1$. He acknowledges that you wouldn't expect a sum of positive numbers to come out negative, and proceeds to give another "example" of a similar phenomenon. The general opinion about this particular work is that anyone who writes dozens of groundbreaking papers should be forgiven the occasional lemon. Meanwhile, is there some way that Euler's equation for $1+2+4+8+...$ can be rescued? We'll see that there is - in the right context (which we'll introduce), that of the 2-adic numbers. Prerequisites: An understanding of convergence, though a bit of elementary number theory might be useful too.

Discrete Calculus. (– , Mathieu, Tue-Wed)

Single variable calculus has some pretty useful techniques to offer, but it has an annoying drawback: because of the need to take limits all the time, it only works for functions on the real numbers (or in other "continuous" contexts). Fortunately, much of the theory and many of the techniques can still be made to work in the context of functions on the integers (that is, sequences of numbers) without too much trouble, and this leads to the theory of discrete calculus.

In this class, we will find discrete analogues of many of the mainstays of calculus like the product rule, the power rule and the exponential function (but not the chain rule, sadly), prove the Fundamental Theorem of Discrete Calculus, and use all of this to compute some sums such as

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

relatively painlessly. In the process, the Stirling numbers of both kinds will make an appearance, and we'll see how they are related.

Prerequisites: Knowing techniques from calculus would be useful

Homework: Optional

Related to: Generating Functions (week 1)

Hyperreal Analysis. (- , Don & Waffle, Thu-Fri)

When Newton and Leibniz first invented calculus, they talked about everything in terms of infinitesimal (infinitely small) quantities. For example, the derivative of a function f was defined as the quotient $f(x + \epsilon)/\epsilon$ for an infinitesimal ϵ. This remained the standard approach for over a century; Euler is particularly well-known for his intricate manipulations of infinitesimals. In the nineteenth century, mathematicians began to question the rigor of infinitesimals, developing the modern "ϵ-δ" definitions which are free of infinitesimals. Infinitesimals eventually became pariahs in the mathematical world, used only by physicists who had no regard for rigor.

Then, in the early 1960's, a mathematician named Abraham Robinson used sophisticated methods of model theory to construct an extension of the real numbers called the *hyperreals*. The hyperreals are the real numbers together with infinitesimal and infinite numbers. However, in a certain sense the hyperreals “look” just like the real numbers, and anything you can do with real numbers can be done with hyperreals. This means that hyperreals can be used to do calculus using infinitesimals. In this class, we'll construct the hyperreals[1] and then develop basic calculus with them.

Prerequisites: Basic calculus

Related to: Real Analysis (week 1)

Classes at 1:30

Olympiad Problem Solving in Computer Science. (, AJ Jayakumar, Mon)

If you enjoy problem solving and/or want to train for Olympiads such as the USACO (USA Computing Olympiad), then you should check this class out.

We'll be taking a look at some interesting problems from Computer Science Olympiads around the world. We'll also be discussing the ‘common’ algorithms encountered on Olympiads and some new approaches to ‘spice up’ some well-known algorithms (such as Dynamic Programming). If you aren't too familiar with certain algorithms and just want to learn more about solving problems/puzzles using a computer, you are free to come to this class too.

Prerequisites: Some problem solving experience. Knowing how to use (to some extent) a programming language is highly recommended.

Related to: Olympiad Problem Solving (weeks 1, 2, and 4). Sort of.

Secrets in Inequalities. (, Tuan Le, Wed)

This is the class for anyone who is curious and fascinated about the beauty of solving and proposing inqualities. You may find that solving an inequality is a very difficult task, especially “amazing” Olympiad inequalities which seem to be out of reach. It turns out that you can solve those problems easily once you know the new methods, or the tricks to tackle them (yes, there are many unbelievable tricks in solving this “game”). Moreover, you can learn how to use AM-GM, Cauchy-Schwarz, Holder or Vornicu-Schur inequalities efficiently. For example, how can you tackle the following problems using every tool that you know (until now!):

[1] “Construct” should not be taken too literally; the construction uses the Axiom of Choice!

Given $a, b, c > 0$, prove that:

$$\frac{a^2+b^2+c^2}{ab+bc+ac} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

Can you find an elementary solution to this appealing inequality? If you can, do you see where the constant 2 comes from on the RHS? If you can't answer either one of these questions, you should come to this class. If you can, then feel free to skip this class. Hope to see many of you next week!

Prerequisites: An understanding of basic algebra, the AM-GM inequality, and Cauchy-Schwarz's inequality

Intro to Lambda Calculus. (, Jimmy Koppel, Thu)

Here's an interesting operation for you to consider: You start with a string consisting of variables and parentheses (maybe preceded by something of the form $\lambda v.$) like $x(yz)zyx$. Then, you take the next of a sequence of such strings, say $\lambda w.wy(wy)$, and ask you to substitute it into the first string for each occurrence of x, yielding $(\lambda w.wy(wy))(yz)y(\lambda w.wy(wy))$.

By the way, using this operation, you can compute the answer to any question the most powerful computer possible can answer.

Prerequisites: Don't be confused by the name; this class has nothing to do with differential calculus. It does, however, have a lot to do with computer science, though programming experience should be unnecessary.

Introduction to Riemann Surfaces. (, David Corwin, Fri)

Have you ever been fascinated by the fact that $e^{2\pi i} = 1$? Have you wondered how we can define the natural logarithm function on all the complex numbers? Isn't the logarithm of 1 equal to 0, but also $2\pi i$, $-2\pi i$, and $4\pi i$? Have you wondered how we can define the square root function when most numbers actually have two square roots? Have you always wanted to divide by 0? It turns out that these questions can be answered using Riemann surfaces, objects which look very much like the complex plane on a local scale but whose overall structures may be very different. Riemann surfaces both explain the basic questions mentioned above and apply to many areas of modern mathematics, from proving that there's no quintic formula, to differential and algebraic geometry, and even to number theory. Sam Payne even mentioned them in his colloquium on sandpiles and chip firing. I won't quite get into all of this, but I will introduce some interesting new ideas and hopefully de-mystify the concept of a Riemann surface.

Prerequisites: An intuitive idea of topology. Complex analysis is not required but does provide a larger context for the material.

Homework: Optional problem set, including lots of forays into deeper topics

Related to: Complex Analysis (weeks 2 and 3), Modular Forms (week 3), Geometry and Transformations (week 3)

Two Games and a Code. (, Mira, Tue-Wed)

Game 1: 20 questions with a liar. I'm thinking of a number from 1 to 16. You are trying to guess my number by asking me Yes/No questions about it. The catch is that, for exactly *one* of your questions, I am allowed (but not required) to lie. How many questions do you need to ask if you want to be sure of guessing my number? (It's pretty easy to see that you can always do it with 9 questions – but can you do better?)

Game 2: The coolest hat game of them all. This game is played by a team of 7 players. Each person is randomly assigned a red or blue hat. Everyone can see everyone else's hats, but not their own. However, the hat assignments are independent, so seeing everyone else's colors gives a player no information about her own.

At a signal from the host, all the players simultaneously either guess their own hat color or say "pass." The team wins if there is *at least* one correct guess and *no* incorrect guesses (passes are OK). What guessing strategy should the players adopt to maximize their probability of winning? (It's pretty easy to see how they can play to win with probability 1/2 – but can they do better? Try it for 3 players first.)

What do these problems have to do with each other? It turns out that the perfect strategies for both games involve a beautiful mathematical structure called the *Hamming Code*, invented by Richard Hamming in 1947 to "correct" computer errors caused by (literal) bugs. The Hamming Code seems to pop up everywhere, from digital communication to projective geometry. In this class, we'll solve both of the above games (and play them!). We'll also talk a little more generally about error-correcting codes. Nim — the universal game from combinatorial game theory — may also make a brief guest appearance.

Prerequisites: None. If you are familiar with the Hamming code, you should figure out how to play these games on your own (a fun challenge!) and skip the class.

Homework: None, but it might be fun to demonstrate the hat game in the talent show – it looks quite impressive! If we can recruit enough people, we can also do it with 15 players, which looks even more impressive. However, if we do this, we'll need to rehearse for about 40 minutes some time after the class; that would be the only "homework."

Primes in Arithmetic Progressions. (, JR, Thu-Fri)

The study of analytic number theory is said to have begun with Dirichlet's 1837 proof that, if a and q are relatively prime, then there are infinitely-many primes p which are congruent to a mod q. His proof was based on the study of a particular family of functions, known today as Dirichlet L-functions.

In this class, we will introduce Dirichlet L-functions, and work through the proof of Dirichlet's theorem.

Prerequisites: Basic group theory (familiarity with finite abelian groups, and in particular the multiplicative group of integers mod n), a little complex analysis

Homework: Recommended

Elliptic Functions. (- , Mark, Tue-Thu)

So what do those doubly periodic functions look like, that can't be analytic everywhere and that were mentioned in the complex analysis class? And why are they called "elliptic"? And what do they have to do with elliptic curves? Besides answering questions like these, with any luck this class will get to a proof of one of my all-time favorite formulas:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k)\sigma_3(n-k),$$

where $\sigma_m(n)$ denotes the sum of the mth powers of all the divisors of n (including of n itself). Given that this amazing result is true, there "ought" to be a combinatorial proof, but as far as I know no one has any idea how to find one, so we have to make do with the one using elliptic functions!

Prerequisites: Complex analysis

Homework: Optional, perhaps

Related to: Multiplicative Functions (week 2)

Meet [Dead] Hilbert. (, Mark, Fri)

Some people claim that Hilbert was the last great mathematician to really understand all the major developments in mathematics through his time - after him, mathematics branched out so much that no single person could master the entire field. (Others feel that, years later, von Neumann did essentially manage.) So what sort of things did Hilbert do, and how did he react when the foundations of mathematics were thrown into an unexpected crisis by Russell's paradox?

Prerequisites: None

Scandalous Curves. (, Mike, Tue-Wed)

If someone asks you to draw a random continuous function on the real line, you might easily draw them a sine curve or a cubic polynomial. If you're feeling sneaky you might put in a few jagged corners, or a cusp, or a piece of the Devil's staircase to mess with their calculus. If you really could choose a "random" continuous funtion (which you can't), you'd almost certainly choose one that wasn't differentiable at any point at all!

These functions were once so despised that a mathematician did not speak of them in public. They were considered more of a gotcha example, contrary to the order of more decent and respectable boring functions like sine. In more modern times, they occur quite commonly, for example in Brownian motion and Markov partitions of dynamical systems.

In this class we'll exhibit a few such functions explicitly, and if time permits talk about notions in which such functions are in fact generic amongst continuous functions.

Prerequisites: You should know the rigorous definition of derivatives in terms of limits.

Related to: Real Analysis (week 1), Measure and Integration (week 2 and 3)

How Would You Like Your Fractal Served? (, Mike, Thu)

If you buy an expensive fractal sculpture and hang it over your bed but you're just too lazy to roll over and look at it (and no one can ever seem to get your lemonade mixed right), what shape of shadow would you like it to cast on the ground? Your favorite haiku? The Mona Lisa? It turns out you can have basically whatever you want. In theory one could build a fractal sundial that would give you a digital time readout when positioned correctly in the sun, though for practical reasons this isn't possible.

In this class we'll cover a pretty construction that lets us find a subset of Euclidean space whose shadow in (essentially) any direction is (essentially) what we want it to be. This construction is quite hard in n dimensions, but we will talk a little about two dimensions, and what's needed to go further.

Prerequisites: None

Related to: Measure Theory

The Weierstrass Approximation Theorem. (, Waffle, Fri)

In calculus, you learn how to write some functions as Taylor series. What this really means is that you're writing a function as a limit of polynomials. However, only very special kinds of functions have Taylor series—there are infinitely differentiable functions whose Taylor series do not converge to the function. On the other hand, the *Weierstrass Approximation Theorem* says that *any* continuous function can be approximated by polynomials. More precisely, any continuous real-valued function on a closed bounded interval is a uniform limit of a sequence of polynomials. We'll prove this, and see how the proof generalizes to let us approximate arbitrary continuous functions by other kinds of functions as well.

Prerequisites: Real Analysis

Related to: Real Analysis, Fourier Analysis, other analysis classes

Colloquia

Truth and Objectivity in Mathematics. (*Kenny Easwaran*, Tuesday)

What are we talking about when we do mathematics? If numbers and sets and other mathematical entities really exist, then it seems that they're not the sort of thing we could ever see or feel or smell, and thus it's hard to see how we can have any sort of reliable knowledge about them. If they don't really exist, then it looks like we have to treat mathematics as a kind of useful fiction, and say that mathematical statements like "There are infinitely many prime numbers" are literally false, though they may be "true according to the fiction." But then it's hard to see in what sense there could be an objective fact of the matter about anything that goes beyond the axioms - or for that matter, what could even make the axioms we use objectively correct, rather than an arbitrary personal decision. I will consider the famous completeness and incompleteness theorems of Kurt Gödel (which may be discussed in my classes) and argue that regardless of the metaphysical status of mathematical objects, the relation between numbers and the logical nature of proof means that there must be some notion of objective truth that goes beyond the axioms.

Life on a Sphere. (*James Bernhard*, Thursday)
In this talk, we will introduce the subject of differential geometry by taking a careful look at the intrinsic geometry of a sphere, not unlike Gauss did with his geodetic survey work about 200 years ago. We'll start by figuring out what should be considered analogous to a straight line in a curved space such as the sphere and proceed from there to investigate pairs of lines and then triangles on spheres. This will lead us to discuss some of the fundamental theorems in differential geometry.

Visitor Bios

Kenny Easwaran. (University of Southern California, Australian National University)
Kenny was a camper at Mathcamp in 1998, and returned as staff in 1999, 2002, 2003, 2004, and 2006. He started out primarily interested in mathematical logic and set theory, but over the course of his PhD in Berkeley's program in Logic and the Methodology of Science, he realized he was much more interested in philosophical questions than purely mathematical ones. His research focuses on how probability can help explain the ordinary notion of knowledge (with particular interest in the question of how to deal with probability 0) and also on how mathematicians justify the axioms used in mathematics and why they don't like probabilistic reasoning. He started his first job last year as a professor in the philosophy department at USC, but is currently on leave as a post-doc at ANU.

James Tanton. (, St. Mark's Institute of Mathematics)
James likes taking elementary ideas in mathematics and pushing them in strange and unusual directions. He works with students of all ages and their teachers to look for really cool ways to prove advanced theorems with very basic and elementary tools (light bulbs, triangular palaces, bits of felt, dots and boxes) thereby proving to the world that mathematics - real, creative and truly exciting mathematics - is absolutely accessible to all. He has written a book or two, has worked at all levels of mathematics education, founded an outreach Institute, and has conducted research in algebraic topology and a splash of number theory. But his major goal and pleasure is to advance true joy in thinking about mathematics, much like Mathcamp!

James Bernhard. (, University of Puget Sound)
James Bernhard's original mathematical training was principally in the areas of differential geometry and algebraic topology. From there, his interests broadened, and his current research interests lie primarily in the areas of bioinformatics and biostatistics. He enjoys learning about and discussing problems in many other mathematical fields as well. He is a professor in the Department of Mathematics and Computer Science at the University of Puget Sound.

Mr. (C. S., Pancake)

Mr. Pancake (who is fine with you calling him CS or Pancake) is a highly regarded researcher in an untold number of fields. He holds a Bachelor's Degree from Tacoma Community College, and honorary PhD's from many prestigious universities, such as Cambridge University (in England), the Indian Institute of Technology, Tel Aviv University, Hong Kong University, and more. His research has touched on such disparate fields as immunology, nanotechnology, and cartography, and his recently been focusing strongly on his study of temporal flux and 9-categorical dynamics. He is not formally affiliated with any academic institution, but instead his research is privately funded. He has some history with Mathcamp but has never quite managed to visit before, and he's extremely excited to finally have a chance to see it in all of its glory.