CLASS DESCRIPTIONS — WEEK 3, MATHCAMP 2009

Contents

9:10 AM Classes

Commutative Algebra. ($\frac{1}{2}$; Mark; week 1 of 1)

In its classical form, algebraic geometry is the study of sets in n-dimensional space that can be described by polynomial equations (in several variables). Modern algebraic geometry relies heavily on commutative algebra. In this class, we'll start by introducing some basic algebraic ideas (such as prime and maximal ideals and quotient rings) and then see, although maybe not always with complete proof, how geometric objects correspond to some of these algebraic concepts and how some geometric ideas, such as the idea of dimension, can be studied using the language of algebra. If time permits, a bit of point-set topology (on a certain set of ideals!) will show up as well, but this will be developed as needed.

Prerequisites: Nothing specific that you couldn't catch up on as necessary, but if you've never seen any abstract algebra (groups, rings, or fields) at all, the class will almost certainly move too fast for you.

Homework: Recommended

Relativity and Minkowski Geometry. (; Dan; Saturday) Normally we think of distances as being positive real numbers. But why not do geometry with imaginary distances as well? In 1907 Hermann Minkowski realized that imaginary disances are useful for describing Einstein's special theory of relativity. We will see how relativistic phenonema like time dilation and redshifting have a really simple description in terms of Minkowski geometry, even though they may be difficult to understand intuitively.

Prerequisites: None

Homework: None

DNA Topology Hour 1. (; *Javier Arsuaga, Mariel Vazquez*; week 1 of 1)

Molecular biology and genetics are in great need of new mathematics. It is expected that the development of these mathematical techniques will have a tremendous impact not only in basic biology but also in the study of a number of diseases. In this course we will use knot theory and basic statistical physics to study the 3D structure of chromosomes in different organisms. This is a challenging problem (for instance the human genome needs to be condensed 10,000 times to fit in the nucleus of a cell) that remains mostly unexplored. This will be a highly interdisciplinary (and unusual) course where biological and mathematical concepts will be weaved with the final goal of answering a biological question. In particular, students will learn about some of the problems in DNA packing in certain viruses and in trypanosomes (the bugs that transmit the sleeping sickness disease), the mathematics needed to answer these problems and a highly sophisticated computer software called KnotPlot. Students will work in groups, obtain new results and give a presentation on their findings.

Prerequisites: Basic understanding of $\mathbb{R}^3$, vector products, probability and averages, and DNA Topology Hour 2 as corequisite

Homework: Required

Are Musical Compositions Really Math? (Part the First). (; Andre; Saturday)

When people talk about the connection between music and math, they usually mean the physics of sound: sound waves, tuning, and the overtone series. In this class, we will instead examine mathematical structure present in the notes, harmonies, and rhythms of actual pieces of music. Mathematical methods have been used by music theorists to analyze "ugly" (atonal/serialist) classical music for over 50 years, but only in the last 20 have they been used to discuss "pretty" (tonal) music as well. We'll discuss both, and learn what pitch-class and rhythm-class sets, generalized interval systems, canonical groups, and transformation graphs are; along the way, well be able to formalize how the 1st and 3rd movements of Beethoven's 1st symphony start the same way and why some atonal music ought to sound good to us. Philosophical questions about the role of math in understanding music will also be raised. Warning: if you are uncomfortable with proof by singing, this may not be the class for you.

Prerequisites: Group Theory and basic music theory (scales, chords, intervals) helpful but not required

Homework: None

Group Theory. (; JR; week 1 of 1)

Groups are found everywhere in mathematics. From the isometry group of a metric space to the transformation group of a Rubik's cube to the group of invertible linear maps on a vector space, odds are you've already run into examples of groups in your mathematical studies, probably more than once.

This class is an introduction to the abstract theory of groups. Topics will include group actions, homomorphisms, and how to build new groups from old (including quotient groups, the direct product, the semi-direct product, and the free product). We will also discuss free groups and group presentations.

Prerequisites: This class assumes you have seen some examples of groups, and have a little experience working with them. For example, having taken either 'Rubik's cube' or "Geometry, Plane and Fancy" is sufficient.

Homework: Recommended

Required for: Algebraic Topology

Rational Trigonometry and Universal Geometry. (; Julian; Tuesday-Wednesday)

Know the intersecting chords theorem? (See below if you don't.) Nice result. At least on the Euclidean plane, where we measure distances with real numbers. But what would happen if we were to measure "distances" with integers modulo 7, or the complex numbers? Can we still do geometry? What would the statement of the theorem even mean? And could it still, in some sense, be true?

And is there a way of doing some trigonometry (triangle measuring questions) in this bizarre setting?

Come and learn how to do trigonometry without a calculator, and then how it can be extended to wild and wonderful scenarios!

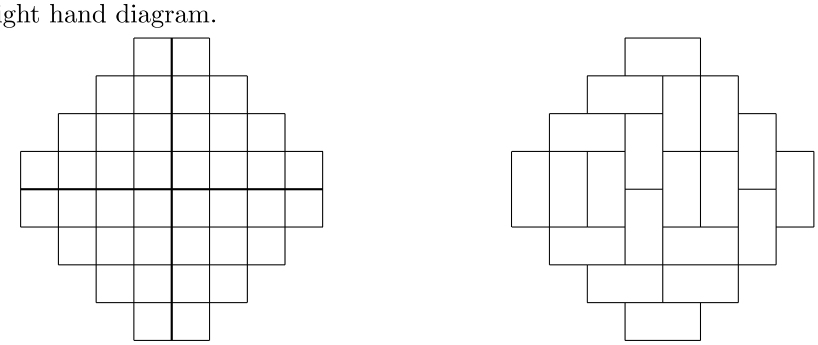
(The intersecting chords theorem: if AB and CD are two chords of a circle which intersect at P, then AP.PB = CP.PD.)

Prerequisites: Modular arithmetic; complex numbers are useful but not essential

Homework: Optional

Aztec diamonds. (; Julian; two days)

An Aztec diamond is a diamond shape made of squares, as shown in the left hand diagram for a diamond of order 4. It can be covered in dominoes as shown in the right hand diagram.



How many ways are there of covering such a shape with dominoes? And what would a random such tiling look like on average? These seemingly hard questions have been studied intensively in the past twenty years and have revealed some beautiful mathematics (as well as pictures).

In this class, we will explore some of the hidden secrets of the Aztecs; there will be a few proofs or sketch proofs, but the emphasis will be on the results.

Prerequisites: None

Homework: Optional

Related to: Dominoes on Chessboards

10:10 AM Classes

Combinatorial Geometry. ($\frac{1}{2}$; Dan; week 1 of 2)

Combinatorial geometry is the study of a large number of points (or lines or circles etc.) We will look at questions like: (1) Given a finite set of points in the plane, not all collinear, is there a line that passes through exactly two of them? Can we find many lines that pass through exactly two of them? (2) What is the minimum number of slopes determined by n points, not all collinear? (3) If we have a set of n points and a set of m lines, how many point-line incidences can there be? (4) If we can draw a graph in the plane so that every pair of edges cross an even number of times, can we draw it in the plane so that no edges cross?

In answering these questions we will need to use only a few elementary facts from geometry but that does not mean they are easy! Many of the proofs display a great deal of ingenuity—they are "from the Book" as Paul Erdös would say.

Homework will be a large part of this class since much of the fun of combinatorial geometry lies in finding your own creative solutions to the problems.

Prerequisites: Basic graph theory; in particular you should be familiar with Euler's formula $F + V - E = 2$

Homework: Required

Related to: Planar Graphs (weeks 1-4), The Probabilistic Method (week 1)

Combinatorics of Permutations. (; Alison, Mathieu; week 1 of 1)

What is a permutation? It can be viewed as an ordering of n objects, or as an action on an n-element set, or as an element of the symmetric group S_n... it can also be represented in various ways, as a collection of disjoint cycles, or as a pattern of dots in a matrix, for example. Using these many points of view, we will explore various combinatorial questions about permutations, such as:

- What does it mean to say that a permutation is even or odd?
- How many permutations have no increasing subsequence of length 3?
- What is the probability that a random permutation has no fixed points?
- What are the different types (conjugacy classes) of permutations?
- How many times do you have to swap pairs of elements to put permuted elements back in increasing order? (There are two answers!)
- How many cycles does a random permutation have?

If any of these questions sound intriguing to you, come find out the answers, and learn the relevant tools along the way!

Prerequisites: None

Homework: Optional

DNA Topology Hour 2. (; *Javier Arsuaga, Mariel Vazquex*; Tuesday-Friday)

A continuation of DNA Topology Hour 1.

Prerequisites: DNA Topology Hour 1 as corequisite

Homework: Required

Are Musical Compositions Really Math? (Part the Second). (; Andre; Saturday)

A continuation of Are Musical Compositions Really Math? (Part the First).

Prerequisites: Are Musical Compositions Really Math? (Part the First)

Homework: None

Introduction to Computational Linguistics. (; *Catherine Havasi*; week 1 of 1)

In this class, we'll cover the basics of computational linguistics, which is a branch of both artificial intelligence and linguistics. We'll talk about some machine learning techniques we can use to make computers understand and reason with language. We'll look at how we can model decision making with the ID3 algorithm. Then, we'll look at Markov chains, which we can use directly to generate text that imitates Shakespeare or your favorite staff member, or indirectly, by finding Hidden Markov Models that we can use to describe linguistic phenomena. We'll see how we can make computers look smart, by clustering related words and phrases and inferring things from these clusters.

We'll cover some of the basics of machine learning, linear algebra, and matrix decompositions. Along the way, I'll show you the fascinating world of language and teach some basic linguistics.

Geometry, Plane and Fancy: Euclidean and Non–Euclidean Geometries. (; Nina; week 2 of 3)

Have you ever thought about what it would be like to play baseball in the hyperbolic plane? Hyperbolic sports are hard. For starters, you would need 10^{94} outfielders to play baseball. And if Tiger Woods were to try his hand at a 300 foot put and be off by one degree, the ball would end up 10^{100} feet away from the hole. How absurd! How hyperbolic!

This week of the class we'll define the hyperbolic plane and find its group of isometries. Occasionally relying on its group of isometries, we'll be able to prove some famous phenomena of hyperbolic space: triangles with small angle measure, the area formula for triangles, and how the area and circumference of a circle grow exponentially with respect to the radius (leading to our strange sports phenomena).

You can enter the class at this point if you are comfortable with the definition of an isometry, elementary group theory, and complex analysis through Möbius transformations. If you want the definitions and problems from last week, let your academic advisor know and I can post them for you on the math memo board.

The third week of this course will cover [1] hyperbolic surfaces and some spherical geometry.

Prerequisites: Linear Algebra, Group Theory, Complex Numbers through Möbius Transformations

Homework: Required

Related to: Topology, Linear Algebra, Group Theory, Surfaces, Complex Analysis, Inner Products

[1] pun intended $\ddot\smile$

11:10 AM Classes

Problem Solving in Calculus. (; *Dave Patrick*; week 1 of 1)
We'll look a variety of calculus problems that are more difficult than the routine AP-style problem. This does not mean "calculate this hard integral" sort of problems (a la the MIT Integration Bee); rather, this means problems that require you to apply calculus concepts in a nontrivial way. Many of the problems that we look at will be taken from the Putnam Competition and other undergraduate-level contests.

Prerequisites: Single-variable calculus (derivatives and integrals)

Homework: Required

1:10 PM Classes

Measure and Integration. (; Mike; week 2 of 2)
In Week 1 we extended our notion of size by repeatedly doing the stupidest thing possible ... and it worked! It's a cold and stormy world out there, with continuous, nowhere differentiable functions lurking, and functions with singularities at every rational number peering out from the sewers, but the notion of the measure of a set lets us integrate essentially any function that can be written down (without the axiom of choice), as well as integrate in very abstract contexts, like on fractals, abstract probability spaces, or the space of continuous functions.

This week we'll continue to see that lots of functions aren't so bad after all, and investigate various ways of measuring size, distortion, and average behaviour. With integration out of the way, it's on to differentiation! Topics will include function spaces, points of density, maximal functions, the Lebesgue differentiation theorem, and differentiability of increasing functions.

Prerequisites: Real Analysis required, Measure and Integration (week 1) recommended

Homework: Required

Problem Solving in Combinatorics. (; *Dave Patrick*; week 1 of 1)
Combinatorics is for many people the hardest arena of problem solving. I think that the (perhaps surprising) reason for this is that there aren't really any formulas to memorize. Rather, you need to learn techniques and the right way of thinking about counting problems. The first class will be a quick coverage/review of some common basic counting techniques, along with some practice exercises. The rest of the week will be spent looking at some more difficult counting and probability problems.

Prerequisites: None

Homework: Required

Set Theory. (; Waffle; week 2 of 2)

This week, we'll study the Zermelo-Fraenkel Axioms of set theory and the overall structure of the set-theoretic universe. For (almost) every axiom, we'll show either that we can prove it from the other axioms or that we can't prove it from the other axioms and the universe would in some way be incomplete without it. In the latter case, we'll build little toy models of set theory in which all the axioms hold but one. For example, you can create a model of set theory which is perfect except that uncountable sets do not exist. We'll also learn a little bit about "large cardinals" and see how if only we could be sure that really really really really big infinite sets existed, then we could prove that set theory is consistent.

If you weren't in Set Theory last week, you can still join this week if you are familiar with the theory of ordinals and cardinals. If you want to join, please talk to me beforehand to make sure you know everything you need to know.

Prerequisites: Set Theory last week, or familiarity with ordinals and cardinals

Homework: Recommended

Related to: Set Theory as a Foundation for Mathematics, Infinite Trees, Continuum Hypothesis

Colloquia

DNA Topology. (*Mariel Vazquez*, Tuesday)

Knot theory studies simple closed curves in space. Despite its traditional classification as a "pure math" subject, knot theory has been extensively applied in physics, chemistry and molecular biology. In this talk I will review some of the current applications of knot theory to molecular biology, in particular to the structure and maintenance of chromosomes. A chromosome consists of proteins and a single DNA molecule, which is usually very long and that is folded into a very small volume (e.g. the nucleus of the cell). This spatial confinement is related to the formation of knots (or links if several molecules are involved) which can be informative of the chromosome organization. Noteworthy is the example of DNA knots found in certain viruses that are unveiling new properties of the DNA organization inside the virus. During this presentation I will also introduce some of the experimental, mathematical and computational tools commonly used to study DNA knots.

Projective Surfaces (And How To Blow Them Up). (*Dave Patrick*, Wednesday)

The classification of complex projective surfaces was one of the major achievements of algebraic geometry in the early 20th century. I'll talk about the projective plane, what a projective surface is, what it means to "blow up," what we really mean by "classification," and finally some of the ideas behind the classification itself. I'll also mention how we might apply this classification to noncommutative surfaces (where $xy \neq yx$).

More DNA Topology. (*Javier Arsuaga*, Thursday)

Giving a Computer Common Sense. (*Catherine Havasi*, Friday)

We know a lot of things that computers don't. We know, for example, that when we go into a cafe, we know we can buy coffee there; we know our computer mouse won't chomp on our fingers; and we know that when you drop something it will fall down. We even know that we don't need to say obvious facts like those in a typical conversation. But when we try to express this body of common-sense knowledge to a computer, it can be vague, inconsistent, and incomplete.

How do we teach a computer to draw sensible conclusions from this kind of noisy information? I will present one method, used by the Open Mind Common Sense project, that uses a vector space to represent the meanings of things. We'll then learn the intuition behind singular value decomposition and how we can use linear algebra to model the way we think and learn about the world.

Digestif

P, NP, and Proofs. (Dan Zaharopol, Saturday)

The P vs. NP problem is a million-dollar millenium question in mathematics about the power of computation. The question is open, so we (probably!) won't prove it here. What we will do is get an idea of the many different kinds of attacks that mathematicians can bring to bear on open problems, and the fruitful mathematics that results. At the end, we'll also explore briefly how this question about computers also says a bit about the difference in difficulty between **discovering** a solution for yourself vs. merely being **given** that solution, a surprising connection between a pure mathematical question and a real-world philosophical question.

Visitor Bios

Javier Arsuaga and Mariel Vazquez. (San Francisco State University)

Javier and Mariel are mathematicians who use knot theory to study DNA. Most of the time, DNA is a long and skinny linear molecule, but under certain experimental conditions it becomes circular and knotted. At Mathcamp, Javier and Mariel will show students how DNA knots can be used to study chromosome organization in some viruses, as well as to understand the action of certain enzymes. Their course will be highly interdisciplinary, combining molecular biology, random knot theory and the physics of polymers. (Dont worry, Javier and Mariel will teach you all the background you need to know!)

Julian Gilbey. (Watford Grammar School for Girls)

Julian is returning to Mathcamp this summer after an 11-month hiatus. Since last summer, he has started learning about Rasch Analysis (feel free to ask him what this is!), as well as getting married. He will be joined at camp in Week 4 by his wife, Debbie, who is a librarian with a musicology background. This is probably the first honeymoon ever at Mathcamp! (They are going off somewhere a little more traditional after they leave the camp.) Julian's mathematical interests are fairly random, and include combinatorics, algebra, and other interesting miscellanea. He is currently teaching high-schoolers (6th through 12th graders) near London, England.

Catherine Havasi. (Brandeis)

Catherine Havasi has been collecting common sense from the internet since 1999. She uses that knowledge, plus a bunch of machine learning methods, to model how people think about the world. Her interests are artificial intelligence, specifically singular value decomposition, semantics, machine reading and ontologies. By the time you meet her, she will have started her post-doc at the MIT Media Lab. You may have already met her at Mystery Hunt where she is often wearing a funny hat.

Dave Patrick. (Art of Problem Solving)

Dave Patrick is a textbook writer and instructor at Art of Problem Solving (AoPS). He is the author of two of AoPS's textbooks (and is currently working on a third), and has taught problem-solving courses for AoPS at all levels from MATHCOUNTS to the Putnam and most everything in between. In his past life doing mathematics research (at MIT and the University of Washington), Dave studied noncommutative algebraic geometry, in particular the classification and structure of noncommutative ruled surfaces.