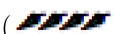
## CLASS DESCRIPTIONS — WEEK 2, MATHCAMP 2009

Contents



### 9:10 AM Classes

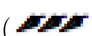
**Olympiad Problem Solving.** ( ; Dan; week 1 of 1)

This is a class for students who want to train for olympiad competitions such as USAMO, or who just like doing olympiad probems for fun. This week's topics are algebra and geometry (specifically, techniques for lazy geometers).

*Prerequisites:* Problem solving experience

*Homework:* Required

*Related to:* Olympiad Problem Solving (Week 1), Olympiad Problem Solving (Week 4)

**Infinite Trees.** ( ; Susan; week 1 of 2)

Konig's infinity lemma shows that any tree of countably infinite height has a countably infinite branch. Its obvious generalization to uncountable infinity turns out to be false: there exist trees of uncountable height whose branches are all countable. As we put more restrictions on the size of our uncountably tall trees, we need to be sneakier to prove that they exist. In fact, we'll need so much sneakiness, we'll have to go outside of normal Zermelo-Fraenkel set theory to find it! Come find out how an object can be uncountably tall. Delve into the mysteries of the Diamond Axiom. Learn how to pronounce the word "Aronszajn."

*Prerequisites:* Set Theory strongly recommended as corequisite, or talk to me.

*Homework:* Optional

**Plato's Cave and Combinatorics.** ( ; Shoe; week 1 of 2)

Do mathematical constructs exist independently of the people who study them? Is math something we discover or is it something we invent? What is the role of beauty in mathematics?

People have been pondering these questions for over two thousand years, and suprisingly (or maybe not suprisingly) we are still arguing about them today. We will ask these questions and more throughout this course, and you will get a chance to add your own opinions to the discussion. We will also study what other mathematicians and philosophers thought, including Plato, Socrates, Galileo, Kepler, and Erdos.

But wait! Are we going to actually do any **MATH**?? Of course we are! (This is Mathcamp after all) The format for a particular topic might be as follows:

- I will introduce some philosophical idea such as: The world of the senses is an illusion, there exists another world which we cannot sense. THAT world is the real world.
- We will discuss the idea.
- I will talk about what other people thought about this idea. In this case, Plato had quite a bit to say. We will read excerpts from his book *The Republic.*
- I will give examples of mathematical theorems which somehow relate to the idea. In this case we will prove the classification of the Platonic solids. We will talk about how these results either validate or oppose what we had originally thought.
- Repeat.

Mostly we will prove theorems from combinatorics. Topics will include the classification of the Platonic solids (in two different ways), an introduction to Ramsey theory, and Arrow's dictator theorem (for which he won a Nobel Prize!). If you are already familiar with these topics, this class is probably not for you. You SHOULD come to this class if (i) you are curious, (ii) you enjoy philosophical discussions and reading philosophy, and (iii) you don't mind getting your hands dirty. What I mean by (iii) is, instead of proving theorems for you, I will often give you theorems and ask you to prove them. Class time will sometimes be spent with you working individually or in groups to solve problems which I have given you.

*Prerequisites:* A thirst for knowledge

*Homework:* Optional

**Planar Graphs.** ( ; Marisa; week 2 of 4)

A continuation of *Planar Graphs* from Week 1. This week, we will be talking about $k$-connected graphs, leading up to the proof of (the hard direction of) Kuratowski's Theorem.

*Prerequisites:* If you want to join in Week 2, you should *work through the problem sets from Week 1, posted in the Main Lounge* to know what has been covered and to decide whether or not to add the class.

*Homework:* Required

**Complex Analysis.** ( ; Mark; week 1 of 2)

We'll look at the spectacular changes that occur in calculus when you allow the variable $x$ (now to be called $z = x + iy$ ) to take on complex values. As we'll see, functions that are "differentiable" in a region of the complex plane have many surprising properties. For example, they always have power series expansions, and if you know what the function is everywhere on some closed curve, then you can deduce its value anywhere inside the closed curve! This material, much of which was first developed by Cauchy (remind me to tell you, probably in the second week of the class, why the mathematician named her dog "Cauchy" ...), is not only quite beautiful, but it has important applications both in- and outside mathematics.

For instance, complex analysis was used in proving Dirichlet's famous theorem about primes in an arithmetic progression, which says that if $a$ and $b$ are positive integers with $\gcd(a,b) = 1$ , then the sequence: $a, a + b, a + 2b, a + 3b, ...$ contains infinitely many primes. As far as I know, this was the first major result in analytic number theory, the branch of number theory which uses complex analysis as a fundamental tool and which includes such key questions as the Riemann hypothesis. On the other hand, complex analysis can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow.

*Prerequisites:* Multivariable Calculus

*Homework:* Recommended

## 10:10 AM Classes

**Not That Again: Solving Problems Faster by Exploiting Shared Substructures.** ( ; *Michael Littman*; Tuesday and Wednesday)

When people use computers to solve problems, they search for mathematical structure that can quickly guide their software to the best answer—the needle in a haystack of possibilities. One powerful trick goes by the name "dynamic programming" and has been used in applications ranging from understanding language to morphing images to generating driving directions to catching homework cheaters. We'll work through some simple examples such as addingspacestostrings, figuring out how to maximize your chance of winning in blackjack, and efficiently eliminating exploding lemmings.

*Prerequisites:* None

*Homework:* None

**Combinatorial Number Theory: Subsequence Sums.** ( $\frac{1}{2}$; *Matt DeVos*; Friday-Saturday)

If you have a set of 3 integers, then some 2 element subset must sum to an even number. More generally, a classic theorem tells us that given a set of $2n-1$ integers, there exists an $n$ element subset which sums to a multiple of $n$. We will prove this happy fact together with a bunch of related results.

*Prerequisites:* Basic group theory

*Homework:* None

**How (Not) To Get Rich Quick, and Other Paradoxes.** ( ; Tim!; Thursday)

Would you like to win an infinite amount of money? Then I have a game for you! With just a few coin flips, you could win an arbitrarily large amount. But not only that: the expected value of your winnings will be infinite. It's easy! I'll give you a dollar just for showing up. Then I'll let you flip a coin. If it comes up heads, you can flip again, while if it comes up tails, the game is over. Every time you get heads, I'll double your winnings. As soon as you get tails, you keep whatever money you have and walk away. You're expected earnings are infinite, but just because you've been so nice, I'll let you play for $5. What do you say?

Seem too good to be true? We'll see what sort of shady buisness is going on here. We'll also look at two other games that seem to contradict common sense: The Monty Hall Paradox, and the Two Envelope Paradox.

*Prerequisites:* None

*Homework:* None

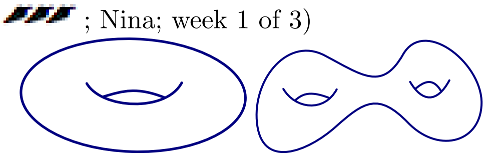
**Rubik's Cube and Other Puzzles.** ( ; *Dror Bar Natan*; week 1 of 1)

The $3 \times 3 \times 3$ Rubik cube has 43,252,003,274,489,856,000 reachable configurations; this is more than a human or a computer can crunch through in his or her lifetime. So how do we know this number is right? And how do we program a computer to solve the Rubik cube without going through all these configurations? By abstract thought alone and a powerful tool called "group theory," we will answer this question and many similar ones involving other "permutation group" puzzles, in a completely systematic manner.

*Prerequisites:* None

*Homework:* None

**Geometry, Plane and Fancy: Euclidean and Non–Euclidean Geometries.** ( ; Nina; week 1 of 3)



What is the difference between these two figures? A topologist would recognize that their genus differs (i.e. they have a different number of holes).

That's a good start, but a geometer might also recognize that one can be made into a "flat" surface, and the other into a "hyperbolic" surface. What does that mean?

A flat surface is modeled on the Euclidean plane– that is, around each point is a neighborhood where distance is measured like on the Euclidean plane. We have all the Euclidean properties we know and enjoy: triangles with angle measure $2\pi$, parallel postulate, etc.

A hyperbolic surface is modeled on the hyperbolic plane, a space that is negatively curved instead of flat. Triangles in this plane (and on our hyperbolic surfaces) have angle measure smaller than $2\pi$ among other weird negative curvature phenomena.

Throughout the class, we will use the group of *isometries* (distance–preserving maps), which are helpful in proving most of the important characteristics of plane geometries: Euclidean, Hyperbolic, Spherical.

The first week we'll start with the Euclidean plane, its isometry group, and Euclidean (or "flat") surfaces (which arise exactly as quotients by certain isometries). We'll also build up the machinery of Möbius transformations of $\mathbb{C}$ which we'll need to to explore the analogous areas of hyperbolic geometry in the second and third week of the course. The third week will end with a few days of spherical geometry.

*Prerequisites:* Linear Algebra. Helpful but not necessary: Calculus, Group Theory, Point Set Topology.

*Homework:* Required

**Integration for Ninjas.** ( ; JR; week 1 of 1)

The president has been kidnapped! You receive a ransom note from the notorious criminal mastermind, Dr. Integrus. In order to secure the return of the president, Dr. Integrus demands that you tell him the value of $\int_0^1 \frac{\log(t)^3}{1-t}\,dt$. In an attempt to find the anti-derivative of $\frac{\log(t)^3}{1-t}$, you call an emergency meeting of the best integrators in the world. They work for many days and nights, but unfortunately, no anti-derivative can be found. Just when all hope seems lost, in comes the Black Ninja. The Black Ninja informs you that, through many years of intense training, he has mastered several ancient techniques that often enable him to compute a definite integrals, despite not being able to find an anti-derivative of the function being integrated. Will the Black Ninja's ancient techniques be enough? Can the president be saved? Tune in during Week 2 to find out.

*Prerequisites:* Basic integration

*Homework:* Recommended

## 11:10 AM Classes

**Finite Fields.** ( ; Mathieu; week 1 of 2)

A field is a set like $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, where you can add, subtract, multiply, and divide (but not by zero!). In this class, we'll classify all finite fields, construct them, and prove many of their properties. It turns out that they're as nice as they could possibly be, but still pretty complicated!

*Prerequisites:* Familiarity with linear combinations and modular arithmetic

*Homework:* Recommended

**Topics in Number Theory.** ( $\frac{1}{2}$; Mark; week 1 of 1)

This class really consists of two "mini-classes" which are independent of each other, so you can take either one or both. Separate descriptions follow.

Multiplicative Functions (Wednesday and Thursday) Many number-theoretic functions, including the number of divisors and the Euler phi-function, share the property that $f(mn) = f(m)f(n)$ whenever $gcd(m, n) = 1$ . There is an interesting operation, related to multiplication of series, on the set of such *multiplicative* functions, which gives that set a nice structure. If this sounds intriguing and/or if you would like to be able to compute (by hand) the sum of the 10th powers of all the divisors of 686,000,000,000 in a minute or so, do come!

Quadratic Reciprocity (Friday and Saturday) Let $p$ and $q$ be distinct primes. What, if anything, is the relation between the answers to the two following questions:

(1) Is $p$ a square (mod $q$) ?
(2) Is $q$ a square (mod $p$) ?

In this class you'll find out; the answer is a given by an important and surprising theorem which took Gauss a year to prove (which means that it might well have taken most of us a lifetime or longer) and for which he eventually gave six different proofs. If all goes well you'll get to see all of one particularly nice proof, parts of which are due to one of Gauss's best students, Eisenstein. And next time someone asks you whether or not 101 is a square modulo 9973 , you'll be able to answer a lot more quickly!

*Prerequisites:* For days 1 and 2, some comfort with summation notation and some previous exposure to number theory - the introduction to number theory course will be more than enough. For days 3 and 4, introduction to number theory, at least through Fermat's Little Theorem.

*Homework:* Recommended

**Noncommutative Ring Theory II: Commutative Rings.** ( ; Waffle; week 1 of 1)

Last week Ari talked about noncommutative rings; this week I'll talk about a particularly nice class of examples of noncommutative rings, namely commutative rings. In particular, we'll focus on a special kind of commutative ring called a *principal ideal domain*, or PID. PIDs are rings that generally behave like the integers, and have a good notion of prime factorization and greatest common divisors. Like over semisimple rings, modules over a PID have a particularly simple structure, at least when they're finitely generated. However, while the structure of modules over a semisimple ring follows fairly directly from the definition of semisimple, the structure theorem for modules over a PID is a lot harder, and proving it will be the main goal of the class.

Since the integers are PID, this structure theorem will classify all finitely generated $\mathbb{Z}$-modules, which are just finitely generated abelian groups. In this case, the theorem says that any finitely generated abelian group is a direct sum of cyclic groups in an essentially unique way. Other examples of PIDs will have powerful applications to linear algebra.

*Prerequisites:* Noncommutative Ring Theory

*Homework:* Required

*Related to:* Noncommutative Ring Theory, Commutative Algebra, Linear Algebra

**Frog Hops: Groups and Graphs.** ( ; Yvonne; week 2 of 2)

This week, our frogs live in groups, and we exploit the frogs to understand groups and presentations of groups via Cayley graphs. You should not come to this class if you already know what a Cayley graph is, or if you are extremely familiar with presentations of groups. You *should* come to this class to –

- Write groups in terms of generators and relators
- Use Cayley graphs to express operations used to build groups, such as cartesian product, free product, and amalgamated product.

Along the way, you will find out trade secrets such as interesting infinite generating sets for $\mathbb{Z}$, the stupidest presentation of a group, and the TV antenna fractal group.

*Prerequisites:* None

*Homework:* Not required

**Knot Diagrams, Reidemeister Moves, and the Jones Polynomial.** ( ; Ari; Saturday)

Can you tell by staring at a tangled loop of string whether it can be unknotted? What about detecting whether two knots are the same? To deal with these questions, mathematicians invent knot invariants, which are combinatorial, algebraic, or topological gadgets used to distinguish knots from each other. We will learn about two-dimensional drawings of knots (and how these differ from the real thing) and define the Jones Polynomial, a powerful yet simple knot invariant that can tell apart mirror images!

*Prerequisites:* None

*Homework:* None

## 1:10 PM Classes

**Set Theory.** ( ; Waffle; week 1 of 2)

What is infinity? You probably already know something about what the "size" of an infinite set is, and how it's possible for one infinity to be bigger than another one. In this class, however, we'll go a lot further than anything you may have seen before. We'll play with infinite numbers, and we'll learn how to do arithmetic with them. In fact, infinite arithmetic turns out to be easier than ordinary arithmetic: if $\kappa$ and $\lambda$ are infinite numbers, then $\kappa + \lambda = \kappa\lambda = \max(\kappa, \lambda)$ is just whichever of $\kappa$ and $\lambda$ is larger.

In the process, we will develop a tool called *transfinite induction.* Transfinite induction is like a cross between induction and the Energizer Bunny—instead of stopping when you've gone through all the natural numbers, it just keeps going and going and going. This is needed when dealing with arbitrary infinite sets, since they're bigger than the natural numbers and just keep going and going and going. Just as ordinary induction is very useful for proving theorems about the natural numbers, transfinite induction is an essential tool for proving things about infinite sets. It also has lots of applications outside of set theory (for example, to prove the intermediate value theorem in analysis, or that any vector space has a basis in linear algebra).

In Week 3, we'll change focus a bit and look at what the universe of *all* sets looks like. We'll study in detail the Zermelo-Fraenkel Axioms for set theory, and how each one of them is important for building up all the sets we need to do math (or in some cases, we'll find the axioms are unnecessary and can be proven from the other axioms!). We'll also see how to show that we can't prove an axiom from the other axioms.

*Prerequisites:* None

*Homework:* Recommended

*Related to:* Set Theory, Infinite Trees, Martin's Axiom

*Required for:* Infinite Tress, Martin's Axiom

**Inner Products.** ( ; Alison; week 1 of 2)

One reason why linear algebra is so useful is because there are so many things you can think of as elements of a vector space, for example polynomials or functions in general – in fact, in this class, we'll even learn about a way of representing circles in the Euclidean plane by vectors. When we are thinking about vectors in $\mathbb{R}^3$, our geometrical intution helps us: we can think about the length of a vector, or the angle between two vectors, or about reflecting a vector in a plane. Can we transfer these geometric intutions to general vector spaces? Is there a meaningful way of talking about the "length" of a vector, or the "angle" between two vectors, or to say that two vectors are "perpendicular", when the vectors in question are actually functions or elements of some other abstract vector space? It turns out that, yes, we can, once we have chosen an inner product on our vector space. The most familiar example of an inner product is the dot product on $\mathbb{R}^n$, but there are many other examples, and in this class we'll see some of the many applications of the concept of an inner product to various areas of mathematics.

In the first week, we'll begin by talking about this problem: You have a point $P$ and a plane (or more generally, a hyperplane) in a vector space. How can you find the point of your plane that is closest to $P$? We'll solve this problem using inner products, and then see how it relates to least-squares approximation and Fourier series approximation.

In the second week we'll talk about some weirder inner products, ones which are not positive definite (this means that we can have vectors with negative length). We'll learn about inner products related to special relativity and hyperbolic geometry. Finally, we'll learn how to prove some classical theorems of Euclidean geometry using linear algebra.

Over the course of both weeks, we'll also talk about some of the theory of inner product spaces, including orthonormal bases, Gram-Schmidt orthogonalization, and the signature of an inner product.

*Prerequisites:* Linear algebra

*Homework:* Recommended

**Dominoes on Chessboards [MM].** ( $\frac{1}{2}$; Mira, Jonathan; week 1 of 3)

How many ways are there to tile a chessboard with $2\times 1$ dominoes? On a standard $8 \times 8$ chessboard, there are 12,988,816 tilings. On a chessboard with $M$ rows and $N$ columns, the number of tilings is $\prod_{m=1}^{M}\prod_{n=1}^{N}\left(4\cos^2\frac{m\pi}{M+1}+4\cos^2\frac{n\pi}{N+1}\right)^{1/4}$, as three physicists discovered in 1961. Where does this astounding formula come from? How did the cosines get there—what do domino tilings have to do with trigonometry? And why are physicists interested in dominoes on chessboards? Thinking about these questions will lead us into some very beautiful graph theory and linear algebra. Topology will enter the story in Week 4, when we'll ask what happens to the number of domino tilings if we wrap our chessboard on a cylinder or a torus.

*Prerequisites:* We'll build up all the graph theory we need from scratch, but you need some experience with linear algebra (linear maps, eigenvectors, diagonalization). A significant portion of the first week of the course will be devoted to building up additional linear algebra background: we will be developing the theory of determinants rigorously from scratch. In fact, you can treat the first week of this course as a continuation of the Linear Algebra course in Week 1 or a separate mini-course on determinants: you can come just for that week, even if you don't care about domino tilings. On the other hand, if you feel confident that you understand determinants (not just the usual formula for computing them, but what they mean and how they relate to permutations), you may be able to skip the first week – or at least the first few days – of the class. Talk to Mira to make sure.

*Homework:* Required. This is a Moore Method course, meaning that we'll supply the definitions and the theorems; it will be up to you to work out all the proofs and computations.

**Physics Problem Solving.** ( ; Dan; week 1 of 1)

How can we compute $\left\lfloor \frac{1}{\arctan(x)} \right\rfloor$ using two frictionless carts and a wall? What is the effective resistance between two corners of an icosahedral resistor network? How do you find the acceleration of an infinite pulley system? We will look at these and other physics problems with creative mathematical solutions.

*Prerequisites:* High school physics

*Homework:* Recommended

**Measure and Integration.** ( ; Mike; week 1 of 2)

Integrating polynomials and the exponential function all day is great, but what if you want to integrate an infinite series of functions, and they're nasty functions that have it in for you (and they've got your shoes)? In week 1 we'll learn about measures and how to integrate on an abstract measure space, derive the most important limit theorems for integrals, and take a look at Lebesgue measure, and possibly other important examples of measure spaces. In week 2 we'll study maximal functions, maximal inequalities, and applications to differentiation.

*Prerequisites:* Real Analysis

*Homework:* Required

## Colloquia

**The Hardest Math I've Ever Really Used.** (*Dror Bar Natan*, Tuesday)

What's the hardest math I've ever used in real life? Me, myself, directly - not by using a cellphone or a GPS device that somebody else designed. And in "real life" - not while studying or teaching mathematics?

I use addition and subtraction daily, adding up bills or calculating change. I use percentages often, though mostly it is just "add 15 percents." I seldom use multiplication and division: when I buy in bulk, or when I need to know how many tiles I need to replace my kitchen floor. I've used powers twice in my life, doing calculations related to mortgages. I've used a tiny bit of $2 \times 2$ linear algebra for a tiny bit of non-math-related computer graphics I've played with. And for a long time, that was all. In my talk I will tell you how recently a math topic discovered only in the 1800's made a brief and modest appearance in my non-mathematical life. There are many books devoted to that topic and a lot of active research. Yet for all I know, nobody ever needed the actual gory formulas for such a simple reason before.

**Teaching Computers to Play Games with Meaning.** (*Michael Littman*, Wednesday)

People are very good at creating programs to play word games. In Scrabble, hangman, Boggle, there are computer programs that outplay human opponents by a significant margin. But, the tables are turned when we talk about games where the meanings of the words actually matter—Trivial Pursuit, Jeopardy, crosswords puzzles. Whereas programmers can attack the first class of games purely syntactically using clever search algorithms, word meanings are much harder to tame. In this talk, I'll describe some attempts to create programs that play the second set. Statistics, probability, and modularity are three key ideas that have proven quite promising.

**How to Draw a Line.** (*Matt DeVos*, Thursday)

This talk will be an introduction to linkages. First, I will describe an amazing one which draws a straight line. Then, building on this, we will explore some deeper questions about curves and configuration spaces.

**Qualifying Quiz Presentations, Part 1.** (Your Fellow Students, Friday)

Come hear your fellow students present their solutions to the problems on the Mathcamp Qualifying Quiz! (Part 1. Part 2 is on Saturday during the digestif.)

## Digestif

**Qualifying Quiz Presentations, Part 2.** (Your Fellow Students, Saturday)

Come hear your fellow students present their solutions to the problems on the Mathcamp Qualifying Quiz! (Part 2. Part 1 is on Friday during colloquium.)

## Visitor Bios

**Dror Bar Natan.** (University of Toronto)

Dror is best known in the academic world for his work on knot theory (in particular Khovanov homology and finite type invariants), and in the popular world for his part in the collaboration debunking *The Bible Code*. Practically everything you might want to know about him can be found on his website,

*http://www.math.toronto.edu/˜drorbn/*

including the following paragraph:

"I believe math is too deep. Rather than making it deeper, a better use of my time would be to make some deep ends easier and more accessible. I believe math is too abstract, or at least appears to be too abstract, for much of what may be computed hardly ever is. Thus, whenever I can, I code. Yet I have sinned a few times and written on deep math that was not accompanied with programs. I usually work on knot theory and its surprising relationship with algebra, geometry and quantum field theory. I got my Ph.D. at Princeton, did time at Harvard, Hebrew U., Berkeley and MSRI, and I now work at the University of Toronto."

**Michael Littman.** (Rutgers University)

Michael Littman studies machine learning - a branch of computer science concerned with designing algorithms that improve with experience. His work draws on mathematics areas like computational statistics and discrete structures, which he has used to create programs to solve crossword puzzles and control curious robots.

**Matt DeVos.** (Simon Fraser University)

Matt DeVos is interested in discrete math of many different flavors. Most of his research is in graph theory, but lots of times there is algebra involved, and recently he has been doing some combinatorial number theory. He got his Ph.D. from Princeton in 2000 and is now a professor at Simon Fraser. Last year he helped to launch a wiki for unsolved math problems called the Open Problem Garden. When not doing math, he likes to ride his bicycle(s).

**Jonathan Tannenhauser.** (Wellesley College)

Jonathan Tannenhauser first visited Mathcamp in 1999. His background is in particle physics and string theory, but more recently he has become interested in the computational biology of songbird genomics.