CLASS DESCRIPTIONS — WEEK 1, MATHCAMP 2009

Contents

9:10 AM Classes

Probabilistic Reasoning in Number Theory. (; JR; week 1 of 1)

Consider the following question: what is the probability that 5,446,367 is a prime number? In some sense, probability has nothing to do with it. A number is either prime, or it's not. However, sometimes in mathematics, things that are not random can behave as if they are. Take, for example, the prime numbers. Even though a number is either prime or composite, there is a very precise mathematical sense in which the 'probability' that a number n is prime is about 1 in $\log n$. We can use similar reasoning to investigate twin primes (pairs of primes which differ by two). In this class, we will examine several open conjectures (including the twin prime conjecture, Artin's conjecture, and the Collatz conjecture), and use probabilistic reasoning to see where they come from, and why they are 'very likely' true.

Prerequisites: Some basic number theory (Chinese Remainder Theorem, Fermat's Little Theorem), calculus (differentiation and integration)

Homework: Optional

Generating Functions. (; Alison; week 1 of 1)

How many ways are there of dividing an n-sided polygon into triangles along its diagonals? Is it possible to design a nonstandard pair of dice so that the probability of rolling any given sum is exactly what it would be for a normal pair of dice, even though the dice themselves look different? What is the combinatorial significance of the coefficients in the power series expansion for $\tan x$:

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^5 + \frac{62}{2835}x^9 + \cdots ?$$

We'll explore the theory of generating functions, which provides a language that we can use to translate between algebra and combinatorics. A generating function is a power series whose coefficients encode some combinatorial information, or, in the words of Herbert Wilf, "a clothesline on which we hang up a sequence of numbers for display."

Armed with the techniques of generating functions, we'll answer the questions above, explore other questions from combinatorics, and learn how to prove some beautiful combinatorial identities.

Prerequisites: Basic calculus (differentiation, Taylor series) recommended

Homework: Recommended

Real Analysis. (; Mike; week 1 of 1)

You have a date with density. Would you like to really understand the rigorous background for calculus, differential equations, measure theory, probability, statistics, physics, analytic number theory, differential geometry, harmonic analysis, complex analysis, image processing, fluid dynamics, dynamical systems, ergodic theory, control theory, spectral theory, operator algebras, numerical analysis, and who knows what else? Real analysis provides a solid foundation for many other fields with a wealth of beautiful theorems and ideas, as well as the chance for lots of practice with logic and proofs. If you've ever wanted to learn "$\epsilon - \delta$ proofs" (or never heard of them), this is the place!

In this class we'll study suprema (least upper bounds), sequences, limits, infinite sums and products, inequalities, bounds, spaces of functions, and additional topics as time permits. Plus, there will be in-depth coverage of the exponential function. (... for reals!)

Prerequisites: None

Homework: Required

Required for: Measure and Integration, Fourier Analysis

Set Theory as a Foundation for Mathematics. (; Waffle; week 1 of 1)

What is a number? Stop and think. Do you *know*? Do they even exist (as a mathematical concept)? Fortunately for all of mathematics, the answer is "yes", and in this class we'll see why. We'll build the numbers you know and love from the natural numbers up through the reals. We'll see how even the most basic properties that you never think about can be proved (such as the fact that addition is commutative: $x + y = y + x$). You might be familiar with proof by induction (if not, you will be after TYNTK), but I bet you've never seen a real *proof* that this common technique works. We'll show that it *is* possible to rigorously prove it (once you've stated it carefully: if you have a statement $P(\cdots)$ and you want to show that P is true for each natural number n, i.e. $P(n)$, then it's sufficient to prove that $P(0)$ and $P(k) \Rightarrow P(k+1)$). However, we can't even think about proving that induction works without a very precise and rigorous definition of what numbers are in the first place!

How can we define numbers? What is there more basic than numbers that we could build numbers out of? The answer is *sets.* You might know sets as "collections of objects", but for us, those "objects" will themselves be sets. In fact, we'll be able to start from just the empty set and build up everything you know about numbers (and more!), using only the logical concept of sets. It turns out that *everything* we study in mathematics can be expressed using sets, something I hope to convince you of by the end of the week.

Prerequisites: None

Homework: Required

Related to: Set Theory, Continuum Hypothesis

Introduction to Number Theory. (; Mark; week 1 of 1)
How do you find the GCD of two large numbers without having to factor them? What is the mathematics used when you send confidential information, such as your credit card number or your low opinion of some influential person, over the Internet? By the end of the week you should definitely have the answers to these questions, and maybe some others such as: What postages can you get (and not get) if you have only 8 cent and 17 cent stamps available? What right triangles have integer sides (like the 3-4-5 triangle) but are not similar to each other? Beyond specific answers to such questions, number theory offers insight into many beautiful and subtle properties of the integers. For thousands of years professional and amateur mathematicians have been fascinated by the subject (by the way, some of the amateurs, like the 17th century lawyer Fermat and the modern-day theoretical physicist Dyson, are not to be underestimated!) and chances are that you, too, will enjoy it quite a bit.

Prerequisites: Modular arithmetic

Homework: Recommended

10:10 AM Classes

Olympiad Problem Solving. (; Dan; week 1 of 1)
This is a class for students who want to train for olympiad competitions such as USAMO, or who just like doing olympiad probems for fun. This week's topics are number theory (Tues-Thurs) and inequalities (Fri-Sat).

Prerequisites: Problem solving experience

Homework: Required

Related to: Olympiad Problem Solving (week 2), Problem Solving in Calculus (week 3), Olympiad Problem Solving (week 4)

Planar Graphs. (; Marisa; week 1 of 4)
In a Moore Method (very interactive, student-driven) fashion, this course will explore the ins and outs of Planar Graphs. The material will be split into three sections:

(1) **Meeting Planar Graphs**: We'll open this course with a week familiarizing ourselves with graphs and meeting the concept of planarity. Our vehicles will be familiar objects (like the Platonic solids), and using those objects, we'll develop some vocabulary and proof techniques common to Graph Theory. *If you have already seen some Graph Theory but do not know the proof of Kuratowski's theorem, you might elect to skip week 1 and join us in week 2. If in doubt, talk to Marisa.*

(2) **Characterizing Planar Graphs**: The core of this course will be developing the proof of Kuratowski's characterization of exactly which graphs are planar. It's a surprisingly simple and clear description, and the proof is pleasingly attainable.
(3) **Results on Planar Graphs** (Or: it's not easy being planar): "Planar" is a pretty significant constraint on a graph. In the third half of this course, we'll explore some cool theorems that illustrate just how much being planar will buy you.

Prerequisites: None

Homework: Required

Related to: Linear Programming and Network Flow, Combinatorial Geometry

Multivariable Calculus (crash course). (; Mark; week 1 of 1)

In real life, interesting quantities usually depend on several variables (such as the coordinates of a location, the time, the temperature, etc.) As a result, ordinary (single-variable) calculus isn't enough for most problems. This class will quickly take you through the basics of calculus for functions of several variables. As time permits, we'll see some cool applications in and outside math, such as:

- If you're in the desert, how should you choose what direction to go in to cool off as soon as possible?
- How large is the total area under a bell curve?
- What force fields are consistent with energy conservation?

If we're lucky, we'll get to Green's Theorem, which will be used next week in the complex analysis course, on Saturday. If not, Green's Theorem will be covered on the first day of the complex analysis course.

Prerequisites: Single-variable calculus; some previous exposure to vectors would help

Homework: Recommended

Required for: Complex Analysis

Things You Need To Know: Practice. (; Susan and Mathieu; week 1 of 1)

Do you know what a proof is? Can you write a proof by induction, or by contradiction? Do you know the pigeonhole principle, and how to use invariants? If you answered no to any of these questions, or if you just want more practice writing clear and correct proofs, this class is for you! We'll go over these topics, and pair you up with a proof advisor to give you daily feedback on the proofs you write.

Prerequisites: None

Homework: Required

Related to: TYNTK: Theory

Required for: Everything

John Conway. (- ; John Conway; week 1 of 1)
NTBA

Prerequisites: The ability to count to 10. Or maybe a PhD. Who knows?

Homework: Hopefully none, but you never can tell.

Related to: Everything. Or nothing. You decide!

Required for: Life.

11:10 AM Classes

Things You Need To Know: Theory. (; Mathieu and Susan; week 1 of 1)
This class is a companion to TYNTK: Practice, and is really five one-day classes on various topics that will be used throughout camp. You should feel free to come to any or all of these.

- **Day 1:** Complex numbers.
- **Day 2:** Combinatorics.
- **Day 3:** Infinity and cardinality.
- **Day 4:** Modular arithmetic.
- **Day 5:** Order and equivalence relations.

Prerequisites: None

Homework: Required

Related to: TYNTK: Practice

Required for: Everything

Probabilistic Learning. (; *Josh Tenenbaum*; week 1 of 1)
The goal of this class is to build simple mathematical models of how people learn about the world – primarily to understand the mathematics, but secondarily to implement these models in simple computer programs and test these models by experimenting on people. The math will mostly come from elementary probability theory, and a branch of statistics known as Bayesian statistics. The computer programs can be implemented in any high-level language that you know. Experience with programming isn't necessary for this class but it will extend what you can do. It may also be possible to extend what we do in class into a longer project that you can work on over the whole of camp.

The fundamental problem of learning we will focus on is the problem of generalization: going beyond the data you observe. When a child learns how to use a word like "dog" or "horse" from hearing the word used by their parents to refer to a few examples of dogs or horses, they learn much more than the names for these few individual dogs or horses. They acquire an ability to use that word to refer to a whole range of other animals they've never seen before: they know how to pick out a new "dog" or "horse" when they see it, and they know when a new animal is not a "dog" or a "horse." How do they do this? The human ability to generalize from very sparse data – maybe just one or a few examples – goes far beyond the capacities of conventional machine learning algorithms. We will develop some mathematics that can explain how people generalize successfully from such meager data, and show how this mathematics can be used to build quantitatively predictive models of people's behavior that we can test in simple experiments.

Bidding Games. (; *Sam Payne*; Tuesday-Wednesday)

What happens if you play a typical two-player game, like chess or hex or connect-four, but instead of alternating moves you have to bid for the right to move? Say we play bidding chess and each start with one hundred chips. If I bid nineteen for the first move and you bid twenty-three, then you give me twenty-three chips and make a move on the chessboard. Now I have 123 chips and you have 77 and we bid for the second move. By the way, you bid way too much, and now you're toast!

Prerequisites: None

Homework: Evening bidding games session

Chip Firing Games. ($\frac{1}{2}$; *Sam Payne*; Thursday-Friday)

This class will go more deeply into the mathematics of chip firing games introduced in the colloquium on sandpiles and chip firing. We will discuss the Riemann-Roch Theorem (usually presented at the end of a graduate class in algebraic geometry) through its chip-firing analogue. Along the way we will prove the Matrix Tree Theorem, which gives an amazingly beautiful way of counting the spanning trees of a graph in terms of the determinant of a matrix coming from the Laplacian of the graph.

Prerequisites: Sandpiles and Chip Firing Colloquium; some experience with determinants and finite abelian groups will be helpful

$\frac{22}{7}$, Parallelograms, and Farey Fractions. (; Noah; Saturday)

The most exciting mathematics is often the result of surprising and unexpected relationships between completely different subjects. Exploiting such a connection, mathematicians attack problems in one field using the intuitions and results from another field. Sometimes this translation will turn a difficult question into an easy one. Here we'll study one such connection between a topic in number theory known as Diophantine approximation and ordinary plane geometry. In particular we'll explain how to find really good approximations to π, like $\frac{22}{7}$ using geometric techniques and silly addition ($\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$).

Prerequisites: Knowledge of the Euclidean Algorithm (covered in Introduction to Number Theory)

Homework: None

Elliptic Curves. (; David; week 1 of 2)

Elliptic curves have their fingers in amazingly varied fields of mathematics: they stand at the intersection of number theory, complex analysis, algebraic geometry and algorithms. Andrew Wiles could not have proved Fermat's Last Theorem without elliptic curves and they remain a very active area of research: you can win a million dollars for proving the Birch and Swinnerton-Dyer conjecture.

In this two week introduction, we'll start by learning how to add points on elliptic curves: this group law forms the foundation of their importance in number theory. In the remainder of the first week we will prove some deep theorems about the points on elliptic curves that have integer and rational coordinates. We'll start off the second week with elliptic curves over finite fields, a fast method for factoring integers, and how to use elliptic curves in cryptography. I hope to finish off the course by applying complex analysis to give a completely different way of looking at elliptic curves, which will allow us to parameterize all complex elliptic curves the following week in my Modular Forms course.

Prerequisites: Formally, there are no pre-reqs, but I will be assuming various results from other areas of math. A knowledge of groups and fields will be very useful, though you can get by without it; we'll briefly use determinants and derivatives of polynomials; some complex analysis will be useful for the end of the second week but is not required; the first day of Finite Fields will help you with the first half of week 2, but is not required.

Homework: Required

Related to: Complex Analysis, Finite Fields, Modular Forms, Factoring

Required for: Modular Forms

1:10 PM Classes

Point-Set Topology. (; Shoe; week 1 of 2)

Imagine you have a hollow sphere and you cut out three holes from anywhere on the surface. Now imagine you have a pair of pants (hopefully you do). What do these two objects have in common?

Well to a topologist these are the exact same thing! Topologists study the properties of a space which are unaffected by bending, twisting, stretching, and squishing. And indeed the two objects above can be stretched or bent to look like each other, so topologically they are indistinguishable.

In this class we will introduce the idea of a topological space and look at many different examples, some which you already know and love ($\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ for example) and others which may appear to be the product of a mathematician's bad dream (for example a version of $\mathbb{R}$ in which a single sequence converges to 2 different numbers). We will look at functions between these strange spaces and study what it means to say a function is continuous in this general case. This class will give new meaning to words such as open, closed, and connected.

Prerequisites: None, but knowledge of calculus (especially continuity) will help

Homework: Required

Related to: Real Analysis, Algebraic Topology

Required for: Topics in Topology, Boolean Algebras

Noncommutative Ring Theory. (; Ari; week 1 of 1)

Rings are among the most ubiquitous algebraic structures in mathematics. Tragically, most children are taught at a young age that multiplication is commutative; that is, $x \cdot y$ and $y \cdot x$ represent the same quantity. However, many of the most important rings that you will find in nature are not commutative! We will study the basic properties of noncommutative rings and modules, culminating with the powerful Artin-Wedderburn theorem which classifies semisimple rings.

Prerequisites: Linear algebra, group theory

Homework: Yes

Related to: All future ring theory classes

Required for: Ring Theory 2

Linear Algebra. (; Mira; week 1 of 1)

Linear algebra is the area of math that deals with vectors and matrices. It is one of the most useful methods in mathematics, both within pure math (as you can see from the number of Mathcamp classes that require this one as a prerequisite) and in its applications to the real world. One could argue that most of what mathematicians (and physicists, and engineers, and economists) do with their time is try to reduce hopelessly complicated non-linear problems to linear ones that can actually be solved. Thus for many applied fields, the most important math to know is not calculus, but linear algebra.

Obviously we can't cover all of linear algebra in one week, but this class will give you a basic background, as well as a preview of some of the most important results. We're going to start out on the plane, where linear algebra springs out of geometry. We'll define linear maps and give an intuitive preview of one of the central themes of linear algebra—eigenvectors and their eigenvalues. Then we'll leave our two-dimensional pictures behind and introduce the more general concepts of vector space, linear independence, dimension, inner products, orthonormal bases, and diagonalization. (If you don't know what any of these words mean, that's great: come to the class! If you know all of them, then you probably don't need this class.)

The class will culminate in a big theorem about eigenvectors of symmetric matrices, the Spectral Theorem. This result is fundamental to a variety of applications, ranging from population genetics and image processing to personality psychology. Hopefully we'll have a chance to see at least one of these examples in this class. For another cool application of linear algebra, come to my class in Weeks 2 - 4 ("Domino tilings of a chessboard").

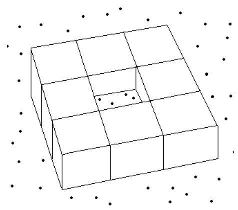
Prerequisites: None

Homework: Required, daily

Required for: Finite Fields, Domino Tilings of a Chessboard, Inner Products, Fourier Analysis, Geometry and Transformations

Polyhedra and Curvature. (; JR; week 1 of 1)

If we draw any triangle in the plane, we know that the sum of the angles will always be 180 degrees. This is not true, however, for a triangle drawn on the surface of a polyhedron. The actual sum of the angles will depend on the shape of the polyhedron. Curvature is a way of measuring how much a polyhedron differs from a flat plane. In this class, we will discuss what curvature is, how to compute it, and how it relates to other quantities (like the sum of the angles of a triangle). We will also consider three-dimensional figures such as the following:

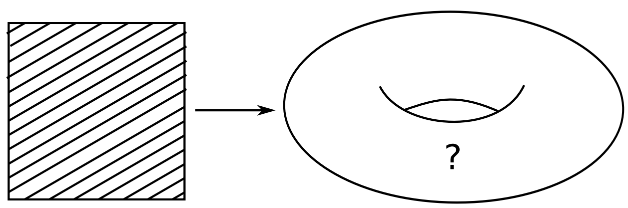


This surface, built from 32 squares, is like a polyhedron, but with a hole through the middle (this figure is topologically different from familiar polyhedra like the cube and tetrahedron). There is a cool formula relating the curvature of such a polyhedral surface to its topology (the Gauss-Bonnet Formula), which we will prove in this class.

Prerequisites: None

Homework: Recommended

Frog Hops, Paint Spots, and the Torus. (; Yvonne; week 1 of 2)



By simultaneously gluing the top to the bottom and the left side to the right side of a square, you can obtain a torus. Suppose before gluing the sides, you mark the square with parallel lines. What would these lines look like on the torus? Would they close up or not? How would you prove this? We will build up to a beautiful result concerning these toroidal geodesics, exploring related problems along the way (including one featuring frogs and paint). This class will be an excellent introduction to some of the ideas fundamental to group theory and geometric topology.

Prerequisites: None

Homework: None

Colloquia

Sandpiles and Chip Firing. (*Sam Payne*, Tuesday)

In the abelian sandpile model, grains of sand are dropped randomly on an array of piles, and when one pile gets too much higher than its neighbors it topples over, sending grains of sand to each of the adjacent piles. This simple model produces complex phenomena, similar in some ways to Conway's Game of Life, and was the first example in physics of a dynamical system with self-organized criticality. Mathematicians rediscovered a variation on this model, stacking chips on the vertices of a graph and allowing a vertex with many chips to fire on chip to each adjacent vertex. The mathematics of these chip firing games is subtle, with connections to algebraic geometry and number theory, and many tricky open problems.

The John Conway Hour. (*John Conway*, Wednesday)
NTBA

How to Avoid Being Eaten by Bears... in Space! (Ari, Thursday)
Suppose that you are being chased by a hungry bear, which can run exactly as fast as you can, in a circular arena. Many thoughts might pass through your mind, such as, "Oh, no! A bear!" or "This wouldn't have happened if I'd signed in on time" or "I wonder whether this bear will catch me in finite time, or whether I'll be able to evade it indefinitely." The third of these turns out to be an interesting math problem with an elegant and unintuitive solution. We will generalize the question and strategies for bear evasion to higher-dimensional discs populated by multiple bears.

The Probabilistic Method. (*Po-Shen Loh*, Friday)
"Leave nothing to chance." One would expect that in the rigorous environment of mathematical logic, such a statement would hold the status of doctrine. However, half a century ago, researchers discovered that to the contrary, artificially-introduced randomness could be used as a powerful tool to prove deterministic statements with absolute certainty. This revolutionized the field of discrete mathematics.

Indeed, consider the following application. Suppose one needs to show the existence of a combinatorial arrangement with certain properties. Instead of exhibiting an (often intricate) satisfying construction, consider a random arrangement, sampled from a suitable probability space. It is then enough to show that the random arrangement is suitable with probability strictly greater than zero. This alternative perspective allows one to apply results from the theory of probability, and in many cases it makes the problem substantially more tractable. In this talk, we will illustrate this technique, now known as the Probabilistic Method, through examples and applications.

Visitor Bios

John Conway. (Princeton University)
One of the most creative thinkers of our time, John Conway is known for his ground-breaking contributions to such diverse fields as knot theory, geometry of high dimensions, group theory, transfinite arithmetic, and the theory of mathematical games. Outside the mathematical community, he is perhaps best known as the inventor of the "Game of Life."

Josh Tenenbaum. (MIT)
Josh Tenenbaum is a professor in MIT's Department of Brain and Cognitive Science. In his research, he builds mathematical models of human and machine learning, reasoning, and perception. His interests also include neural networks, information theory, and statistical inference.

Nick Proudfoot. (University of Oregon)
When not starring in Indian soap operas or playing his guitar in the subway, Nick can occasionally be found in his office writing papers about representation theory, combinatorics, and algebraic geometry.

Sam Payne. (Stanford University)

Sam Payne was a four-time Mathcamp mentor (02, 03, 05, 06), teaching classes on topology, polytopes, algebraic curves, Young tableaux, and the awesomeness of Archimedes. He is currently a postdoc at Stanford, specializing in algebraic geometry by day and bidding games by night. Sam is a cross between a rock star and the Incredible Hulk.

Po-Shen Loh. (Carnegie Mellon University)

Po-Shen Loh was on the 1999 US IMO team and has taught five times at the US Math Olympiad Program. He is currently finishing up his PhD on the Probabilistic Method in Combinatorics at Princeton and will be starting a professorship at Carnegie Mellon this January.

Future Multi-Week Classes

We'll give you more information later on in camp, but here are sneak previews for a few of the upcoming classes!

Set Theory. ($\frac{1}{2}$; Waffle; Weeks 2-3)

What is infinity? You may already know some things about what the "size" of an infinite set is, and how it's possible for one infinity to be bigger than another one (if you don't know about this, you can learn about it in Things You Need To Know in Week 1, and you should, because it's really cool!). In this class, however, we'll go a lot further than anything you may have seen before. We'll play with infinite numbers, and we'll learn how to do arithmetic with them. In fact, infinite arithmetic turns out to be easier than ordinary arithmetic: if κ and λ are infinite numbers, then $\kappa + \lambda = \kappa\lambda = \max(\kappa, \lambda)$ is just whichever of κ and λ is larger.

In the process, we will develop a tool called *transfinite induction.* Transfinite induction is like a cross between induction and the Energizer Bunny—instead of stopping when you've gone through all the natural numbers, it just keeps going and going and going. This is needed when dealing with arbitrary infinite sets, since they're bigger than the natural numbers and just keep going and going and going. Just as ordinary induction is very useful for proving theorems about the natural numbers, transfinite induction is an essential tool for proving things about infinite sets. It also has lots of applications outside of set theory (for example, to prove that any vector space has a basis).

In Week 3, we'll change focus a bit and look at what the universe of *all* sets looks like. We'll study in detail the Zermelo-Fraenkel Axioms for set theory, and how each one of them is important for building up all the sets we need to do math (or in some cases, we'll find the axioms are unnecessary and can be proven from the other axioms!). We'll also see how we can show that we can't prove an axiom from the other axioms.

Prerequisites: None

Homework: Recommended

Related to: Set Theory as a Foundation for Mathematics, Infinite Trees, Continuum Hypothesis

Required for: Infinite Trees, Continuum Hypothesis

Complex Analysis. (; Mark; weeks 2 and 3)

We'll look at the spectacular changes that occur in calculus when you allow the variable x (now to be called $z = x + iy$) to take on complex values. As we'll see, functions that are "differentiable" in a region of the complex plane have many surprising properties. For example, they always have power series expansions, and if you know what the function is everywhere on some closed curve, then you can deduce its value anywhere inside the closed curve! This material, much of which was first developed by Cauchy (remind me to tell you, probably in the second week of the class, why the mathematician named her dog "Cauchy" ...), is not only quite beautiful, but it has important applications both in- and outside mathematics. For instance, complex analysis was used in proving Dirichlet's famous theorem about primes in an arithmetic progression, which says that if a and b are positive integers with $\gcd(a, b) = 1$, then the sequence $a, a + b, a + 2b, a + 3b, ...$ contains infinitely many primes. As far as I know, this was the first major result in analytic number theory, the branch of number theory which uses complex analysis as a fundamental tool and which includes such key questions as the Riemann hypothesis. On the other hand, complex analysis can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow.

Prerequisites: Multivariable Calculus

Homework: Recommended

Finite Fields. (; Mathieu; weeks 2 and 3)

A field is a set like $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, where you can add, subtract, multiply, and divide (but not by zero!). In this class, we'll classify all finite fields, construct them, and prove many of their properties. It turns out that they're as nice as they could possibly be, but still pretty complicated!

Prerequisites: Familiarity with linear combinations and modular arithmetic

Homework: Recommended.

Plato's Cave and Combinatorics. (; Shoe; weeks 2 and 3)

Plato believed that people were trapped in the world of perception, that there was another world, more real than the one we see around us, a world which exists outside our senses. Join us as we explore this idea, find out what Plato was talking about, and figure out how to escape from "the cave."

This class will blend mathematics and philosophy as we try to understand what math really is, and why we do it.

Prerequisites: none

Homework: optional

Domino Tilings of a Chessboard? ($\frac{1}{2}$; Mira and co-teachers; weeks 2 - 4)

Innocent Question: How many ways are there to cover an $M \times N$ chessboard with non-overlapping dominoes?

Answer:

$$\prod_{m=1}^{M}\prod_{n=1}^{N}(4\cos^2\frac{m\pi}{M+1}+4\cos^2\frac{m\pi}{N+1})^{1/4}$$

"Wait," you say, "**that's insane!!!** That doesn't even look like an integer!" You're right – but it's true. The proof uses graph theory and linear algebra in a really beautiful and unusual way. Look for more details in the Week 2 class descriptions.

Prerequisites: Linear Algebra

Homework: Moore Method

Combinatorial Geometry. ($\frac{1}{2}$; Dan; weeks 3 and 4)

Combinatorial geometry is the study of a large number of points (or lines or circles etc.) We will look at questions like: (1) Given a finite set of points in the plane, not all collinear, is there a line that passes through exactly two of them? Can we find many lines that pass through exactly two of them? (2) What is the minimum number of slopes determined by n points, not all collinear? (3) If we have a set of n points and a set of m lines, how many point-line incidences can there be? (4) If we can draw a graph in the plane so that every pair of edges cross an even number of times, can we draw it in the plane so that no edges cross?

In answering these questions we will need to use only a few elementary facts from geometry, but that does not mean they are easy! Many of the proofs display a great deal of ingenuity—they are "from the Book" as Paul Erdös would say.

Homework will be a large part of this class since much of the fun of combinatorial geometry lies in finding your own creative solutions to the problems.

Prerequisites: Basic graph theory. In particular you should be familiar with Euler's formula $F + V - E = 2$.

Homework: Required.

Related to: Planar Graphs (weeks 1-4)

Modular Forms. (; David; week 3 marathon)

Modular forms play a crucial role in many of the advances in algebraic number theory over the past thirty years: the proof of the Fermat's Last Theorem, the solution of the Congruent Number problem, the proof of the Taniyama-Shimura conjecture and the recent proof of the Sato-Tate conjecture. The proofs of these theorems are far beyond the scope of this class, but I hope to de-mystify the concept of a modular form. We'll learn what modular forms are and how to work with them explicitly using Fourier analysis. We'll find amazing relationships between divisor functions, and try to get an idea of why these objects play such a central role in modern number theory.

This course is a marathon, meaning that we meet all day, every day of week 3. Class will be a mixture of lecture and time to work on problems. We will be using Sage (wide-ranging mathematical software that includes a lot of functions for computing with modular forms) for parts of the course. If you want to take this course (whether or not you're unsure about prerequisites), please come talk to me early in camp: this will give me an idea of who's interested, and you a chance to study any prerequisites that you're missing.

Prerequisites: Elliptic Curves, first week of Complex Analysis, Linear Algebra (basics + eigenvalues from the course, plus determinants and matrices over $\mathbb{Z}/N\mathbb{Z}$), group theory ("Finite index subgroup of $SL_2(\mathbb{Z})$" shouldn't scare you, and you should be comfortable with group actions)

Homework: Marathon

Related to: Elliptic Curves, Fourier Analysis