CLASS DESCRIPTIONS—WEEK 4, MATHCAMP 2008

WARNING! This week was so huge that we had to switch the hour and room coordinates on the schedule. Read it carefully.

Contents

Introducing... Superclasses!

What is ... a Superclass?

A superclass is super because it is 2-3 hours a day. They are designed for you to learn super-lots.

In a superclass, you will hear some math from your instructor, but you will also be doing a lot of work. You can expect to spend much of time working on problems and questions. As for out of class time, *you should read the blurbs to find out homework requirements! They are all different.*

We are offering six superclasses this week as well as some regular classes. Choose wisely and have fun!

The superclasses are:

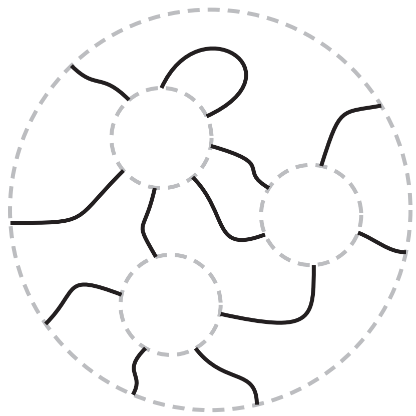
- Planar Algebras, Noah, ∗∗∗-∗∗∗∗
- The Road Coloring Problem, Greg Budzban, ∗∗-∗∗∗
- DNA Topology, Javier Arsuaga and Julian G, ∗∗∗

- Algebraic Topology, JR, ∗∗∗∗
- Hyperbolic Geometry, Yvonne and Nina, ∗∗∗
- Set Theory, Susan, ∗∗

9 – 10:50 AM, 1 – 2 PM

Planar Algebras. (∗∗∗–∗∗∗∗, Noah, Superclass)

In ordinary algebra you consider products like $x \cdot y \cdot z$. These kinds of products are 1-dimensional. You can multiply on the left, or on the right, but you can't multiply on top or on bottom. In a Planar Algebra there are a lot more multiplications. For example here's one way to multiply three elements in a Planar Algebra:



The goal of this class is to prove the Jones-Ocneanu classification of positive definite Planar Algebras with $d < 2$ (which is a "quantum analogue of the McKay correspondence"). There will be lots and lots of pictures. This class is an exciting opportunity to learn material that is accessible, but which is still cutting edge research. There aren't any textbooks on Planar Algebras written yet, and with any luck by the end of the class we'll reach material in a paper I'm writing that's still unfinished.

Homework: Homework is required, but you should be able to finish most of it during scheduled class time.

Prerequisites: Linear algebra. I will be using inner products a lot. You should be comfortable with what it means for a bilinear form to positive definite, and with what a perpendicular complement is.

Related to: Group Theory, Knot Theory, 2-Categories

The Road Coloring Problem. (∗∗–∗∗∗, Greg Budzban, Superclass)

Imagine a city where the following is true. Whenever you need to know how to get somewhere, you call an information hotline, and tell them your desired destination. They then give you directions, *without asking you where you are*. No matter where you start from, the directions work.

The Road Coloring Problem asks: how can we construct a city so that this happens?

This question was posed in the 1970's and solved only in 2007.

Come to this Superclass to find out more about it, and some of the approaches to this only recently-solved problem!

NOTE: If you attend this Superclass, you'll also get to go to the class by Mira on Two Games and a Code on Tuesday.

Prerequisites: Knowing how to multiply matrices.

Related to: Computability and Complexity, Finite Automata, Graph Theory

Homework: Recommended; required on Friday

Two Games and a Code. ($**$, Mira, Tuesday 10-12)

Game 1: 20 questions with a liar. I'm thinking of a number from 1 to 16. You are trying to guess my number by asking me Yes/No questions about it. The catch is that, for exactly *one* of your questions, I am allowed (but not required) to lie. How many questions do you need to ask if you want to be sure of guessing my number? (It's pretty easy to see that you can always do it with 9 questions – but can you do better?)

Game 2: A game show with hats. This game is played by a team of 7 players. Each person is randomly assigned a red or blue hat. Everyone can see everyone else's hats, but not their own. However, the hat assignments are independent, so seeing everyone else's colors gives a player no information about her own.

At a signal from the host, all the players simultaneously either guess their own hat color or say "pass". The team wins if there is *at least* one correct guess and *no* incorrect guesses (passes are OK). What guessing strategy should the players adopt to maximize their probability of winning? (It's pretty easy to see how they can play to win with probability 1/2 – but can they do better? Try it for 3 players first.)

What do these problems have to do with each other? It turns out that the perfect strategies for both games involve a beautiful mathematical structure called the *Hamming Code*, invented by Richard Hamming in 1947 to "correct" computer errors caused by (literal) bugs. The Hamming Code seems to pop up everywhere, from digital communication to projective geometry. (It made an incognito appearance in Moon's colloquium last Thursday.) In this class, we'll solve both of the above games (and play them!). We'll also talk a little more generally about error-correcting codes.

Prerequisites: None. If you are familiar with the Hamming code, you should figure out how to play these games on your own (a fun challenge!) and skip the class.

Homework: None.

DNA Topology. ($***$, Javier Arsuaga, Julian Gilbey, Superclass)

Molecular biology and genetics are in great need of new mathematics. It is expected that the development of these mathematical techniques will have a tremendous impact not only in basic biology but also in the study of a number of diseases. In this course we will use knot theory and basic statistical physics to study the 3D structure of chromosomes in different organisms. This is a challenging problem (for instance the human genome needs to be condensed 10,000 times to fit in the nucleus of a cell) that remains mostly unexplored. This will be a highly interdisciplinary (and unusual) course where biological and mathematical concepts will be weaved with the final goal of answering a biological question. In particular, students will learn about some of the problems in DNA packing in certain viruses and in trypanosomes (the bugs that transmit the sleeping sickness disease), the mathematics needed to answer these problems and a highly sophisticated computer software called KnotPlot. Students will work in groups, obtain new results and give a presentation on their findings.

Prerequisites: Knot theory, basic understanding of $\mathbb{R}^3$, vector products, probability and averages.

Homework: Required.

11 – 12 PM, 2 – 3:50PM

Algebraic Topology. ($\ast\ast\ast\ast$, JR, Superclass)

What's the difference between a sphere and a torus? Simply put, the torus has a hole in the middle, but the sphere does not. Trouble is, we can only see the hole in the torus when we look at it from the outside. What if we were two-dimensional beings living on the surface of the torus. How could we find the hole? One way would be to take a walk, trailing a string behind us. After walking around the torus, we would reach our starting point. When we tried to pull the rope tight, however, we would be unable to. This would tell us that there was a hole somewhere that our string was caught around. The study of algebraic topology starts off by asking which loops in a given space can be pulled tight.

Did you know that any map from the closed unit disc to itself has a fixed point? Or that at any point in time, there are two locations that are diametrically opposite on the earth's surface with the same temperature AND the same barometric pressure? Or that every polynomial has a root in the complex numbers? These are all facts which can be proven using the fundamental group, an object central to algebraic topology. In this class, you will learn about the fundamental group, and about covering spaces, which are essential tools for studying the fundamental group.

Prerequisites: Point-set topology, group theory.

Homework: Recommended

Hyperbolic Geometry. ($***$, Nina and Yvonne, Superclass)

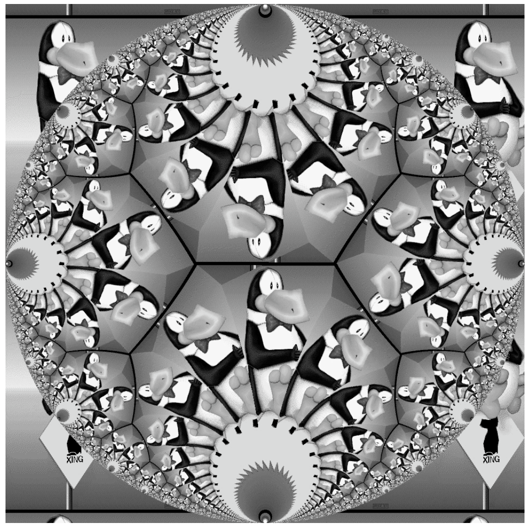


FIGURE 1. Hyperbolic Penguins!

If you want to know what these penguins are doing, you should come to this superclass. If you aren't convinced yet, you should read the rest of this blurb.

The tale of hyperbolic space is one of mystery and intrigue, inspired by attempts to prove Euclid's parallel postulate. Moon Duchin's colloquium showed how projective geometry is a world without parallel lines. We'll show how hyperbolic geometry also breaks the axiom ... in the opposite way, by having "more" parallel lines than in Euclidean space.

Here are some interesting and bizarre properties of hyperbolic space: triangles with infinite radius and finite area, triangles whose angles sum to less than π (sometimes even zero!), and everywhere you stand (if you live in hyperbolic space) is a saddle point.

We'll be studying the fascinating space through its metric and rich group of isometries. We'll see how to visualize groups, what a "hyperbolic group" is, and how the geometry of hyperbolic space gives rise to interesting group theory.

Finally, if you're still not convinced: one of the candidates for the shape of the universe is hyperbolic.

Homework: None

Prerequisites: linear algebra, group theory, complex numbers, geometric curiosity

Related to: All of Moon's classes, What is Geometry, Group Theory, SL(2,$\mathbb{Z}$), Isometries of Euclidean Space.

Set Theory. ($**$, Susan, Superclass)

We hear the word "set" in mathematics all the time. The real numbers, the rational numbers, the integers, open sets, closed sets, sets with sets of sets as their elements. There are sets all over the place. But what, exactly, is a set? Perhaps a better question is: what isn't? What about the set of all sets? The set of all sets that do not contain themselves as elements?

It turns out we need to be careful when we talk about sets. In this class we will learn which sets are valid and which aren't. We'll also learn how to use sets to study the behavior of sets that are really big. We'll define the ordinal numbers and the cardinal numbers and learn how to do arithmetic with infinity. We'll find out that sometimes it's possible to use induction on an uncountable set. Come prepared to build mathematics from the ground up.

Prerequisites: None

Homework: None

Related to: Transfinite Induction

9AM CLASSES

Field Extensions and the Galois Correspondence. (∗∗∗∗, Mark, week 1 of 1)

Can we be sure that the complex numbers exist, and that their arithmetic is consistent, without going through endless checking? How do we know which regular n-gons can be constructed with only the "Euclidean tools" of straightedge and compass, and how do we know that the classical Greek construction problems, such as squaring the circle and trisecting "the" angle (well, most angles) are unsolvable? What number systems exist between the rationals and the field we get by including the cube root of 2 and a square root of -3 along with the rationals? Why isn't there an analog of the quadratic formula for 5th degree polynomial equations, even though there are analogs for degree 3 and degree 4 ? In this class we'll develop many of the mathematical tools needed to answer these questions, although there is basically no hope of getting far enough in a week to really get to the bottom of the last question. Warning: Even if the prerequisites seem modest, the star level reflects the pace I intend to go at, in hopes of at least showing you, and perhaps proving, the Fundamental Theorem of Galois Theory by the end of the class, with various stops for views from high points along the way. Perhaps we should all sign a pledge about getting enough sleep?

Prerequisites: Linear algebra, some group theory, and basic ideas about fields and polynomials (in one variable). Some previous work with factor groups or factor rings would be helpful, because these potentially confusing objects will be introduced at ∗∗∗∗pace.

Homework: optional

Related to: Group theory, p–adics.

Basic Graph Theory. (∗–∗∗, Marisa, week 1 of 1)

A classic problem by Bondy and Murty asks the following:

A certain bridge club has a special rule to the effect that four members may play together only if no two of them have previously partnered one another. At one meeting, fourteen members—each of whom has previously partnered five others—turn up. Three games are played, and then proceedings come to a halt because of the club rule. Just as the members are preparing to leave, a new member, unknown to any of them, arrives. Show that at least one more game can now be played.

One natural way to frame this problem mathematically is in the language of graph theory. We'll spend this week proving this and other similarly cool facts about counting (graph theory is a branch of combinatorics!) and modeling (graph theory is a simple way to describe networks of people or places).

Prerequisites: None

Homework: Recommended

Related to: Graphs on Surfaces (The content of these classes will be disjoint.)

10AM CLASSES

Advanced Problem Solving—Linear and Abstract Algebra. (∗∗∗∗, David Patrick, Tue – Thr)

We'll look at some interesting (I hope!) problems involving what professional mathematicians would simply call "algebra", but what you might call "linear algebra" or "abstract algebra". The three days are essentially independent of each other.

Tuesday: problems involving linear algebra (which you can read to mean matrices)

Wednesday: problems involving group theory (though don't be too put off by this: if you know the basic definition of a group, then you should be fine)

Thursday: a potpourri of harder algebra problems

The problems will mostly be taken from the Putnam Examination, an annual contest for college undergraduates. If things go well, then on Thursday we may take a look at what is the hardest Putnam problem ever: evaluating the determinant of a particularly evil matrix.

Prerequisites: All that's essentially required is that you know how to multiply matrices and calculate a determinant. We'll cover other topics as needed, but the more of the following words that you know, the more comfortable you'll be:

Linear algebra: vector, inverse, transpose, singular, rank, basis, (linear) independence, diagonalization, eigenvalue, eigenvector, trace, null space

Abstract algebra: group, subgroup, order, identity element, inverse, coset, commutative, associative, ring, ideal, unit

Homework: None required, though there will be extra problems to work on between classes if you so desire.

Advanced Problem Solving — Geometry. (∗∗∗∗, Ivan Matic, Fri – Sat)

Geometry! Have you ever been scared by those obscure olympiad–type geometry problems? In Ivan's class, problems are scared by you.

Abelian Groups (Jubilation!) (∗∗–∗∗∗, Mark, week 1 of 1)

It commutes again! (If the reference seems mysterious, ask a Contrapositone or wait for the talent show.) Seriously, in this class we'll be able to give a complete description of all Abelian (commutative) groups that are "finitely generated" (you'll find out what that means). By a major theorem of Mordell, that includes the groups you get by taking the rational points on any nonsingular cubic curve with rational coefficients, and as time permits we'll talk about this application. However, it's fine if you don't know, or think you know, what the technical terms (except "groups" and "rational") in the previous sentence mean. If you enjoy "classifications" that tell you exactly what mathematical objects of a certain kind exist, then you should like this class.

Prerequisites: Some group theory (week 1 of Intro Group Theory will do)

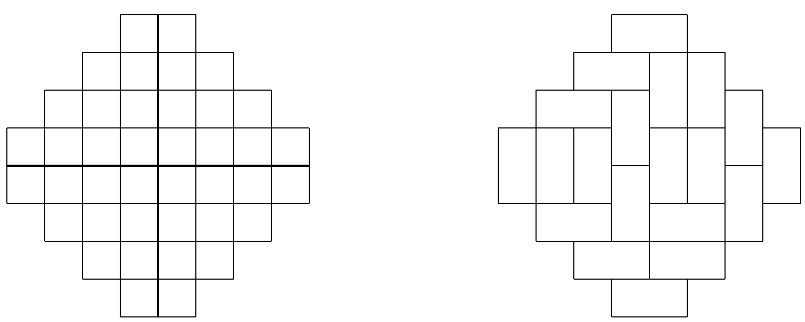
Homework: optional

Related to: Intro Number Theory, in particular the Chinese Remainder Theorem; Cubic Curves.

11AM CLASSES

Aztec diamonds. (∗∗∗, Julian, Weds, Thurs)

An Aztec diamond is a diamond shape made of squares, as shown in the left hand diagram for a diamond of order 4. It can be covered in dominoes as shown in the right hand diagram.



How many ways are there of covering such a shape with dominoes? And what would a random such tiling look like on average? These seemingly hard questions have been studied intensively in the past twenty years and have revealed some beautiful mathematics (as well as pictures).

In this class, we will explore some of the hidden secrets of the Aztecs; there will be a few proofs or sketch proofs, but the emphasis will be on the known results.

Prerequisites: None.

Homework: Optional.

Almost All. (∗∗∗∗, Kenny, Fri – Sat)

Intuitively, the boundary of an open set in $\mathbb{R}^2$ is tiny, while almost all of the plane lies in the interior or exterior of the set. In fact, Rene Baire showed about a hundred years ago that no countable union of boundaries (which he called a ”meager” set) can contain any disc.

At about the same time, Emile Borel and Henri Lebesgue worked out a way to generalize the notion of area to sets that aren't open or closed. They showed that no collection of sets with measure 0 can contain any disc. Thus, intuitively, a set whose complement is measure 0 contains almost all of the plane.

In this class I will discuss Baire's categorization of sets and prove his theorem, and discuss Borel and Lebesgue measure, and show that they contradict each other in terms of which sets contain ”almost all” the plane. In particular, there is a meager set whose complement has measure 0.

Homework: None.

Related to: Measure Theory, Real Analysis.

The Redfield-Pólya Theorem. (∗∗, Alfonso, week 1 of 1)

Can you solve any of the following problems in under two minutes using just pen and paper?

- How many ways are there to colour a cube if we need to paint each face with one of red, blue, or yellow? (Two cubes are the same if you can obtain one from the other via a rotation.) Of those, how many do not use any colour more than three times?
- How many different necklaces with 20 beads can you make, if we have white and black beads? How many of them have an even number of beads of each colour?
- How many different graphs with seven vertices are there? Of them, how many with each number of edges?

The answer to the first question on each problem is given easily by the Burnside lemma, a cute little theorem in group theory. The answer to the second question on each problem is given by the Redfiled-Pólya theorem, a brutally powerful result in group theory which, undeservedly, is not very well known. I will reveal both for you.

Prerequisites: You need to understand what a group is. If you know what an action is, that will be useful, but is not required.

Homework: Recommended, but it won't be much.

Related to: Group Theory (Weeks 1 and 2), combinatorics.

Math of Music. ($*$, Miranda, week 1 of 1)

Math is fun and beautiful. Music is fun and beautiful. Let's do both! There are a number of aspects of music theory and perception that can be understood using a little bit of mathematics. We will talk about things like:

- Why does a clarinet's 'A' sound different from an oboe's 'A'?
- Why do grand pianos sound better than upright pianos? And why are even the grandest of the grand always a little bit out of tune?
- Consonance and dissonance - why do some pairs of notes sound great together, and others sound terrible?
- Why did music that Mozart wrote sound different to him than it does to us?
- Why is it impossible to construct a scale where all of the 'perfect' fifths are perfect?
- Why are there twelve notes in a scale?
- Musical perception - why and when do we hear notes that don't really exist?
- Musical paradoxes - how to construct a scale that always goes up, yet repeats itself infinitely many times (like Escher's staircase).

And we won't just talk - we'll listen to these things too!

Prerequisites: It will help if you know a little bit about music theory, such as what is the note a perfect fifth above C. Other than that, none!

Homework: None.

Intro to Problem Solving—Combinatorics. ($*$–$**$, David Patrick, Tue – Thr)

Combinatorics is for many people the hardest arena of problem solving. I think that the (perhaps surprising) reason is that there aren't really any formulas to memorize. Rather, you need to learn techniques and the right way of thinking about counting problems. Tuesday's class will be a quick coverage of some common basic counting techniques; Wednesday and Thursday will be spent looking at more difficult problems.

Prerequisites: None for Tuesday; for Weds and Thurs, you should either attend Tuesday's class or be somewhat familiar with "basic" counting, meaning most or all the following concepts: combinations, permutations, "balls and urns" problems, distinguishability vs. indistinguishability, using symmetry to solve counting problems, inclusion/exclusion.

Homework: None required. Tuesday's class will have practice problems handed out. On Wednesday I'll hand out a problem set. We'll do some of the problems in class on Wednesday, you can think about the rest of them between classes, and on Thursday we'll discuss the remainder of them.

Intro to Problem Solving — Geometry. ($**$, Ivan Matic, Fri – Sat)

Introduction to the interesting tricks used in solving problems in geometry. The emphasis will be on non-computational methods in solving geometry problems. We will always try to look for miracles inside pictures and seek for hidden properties of the geometrical objects involved. We will learn how to use congruences, similarities, circles, areas, and geometric transformations in solving problems.

Colloquia

The road coloring problem. (Greg Budzban, Tuesday)

You should come.

Some applications of knot theory to molecular biology. (Javier Arsuaga, Wednesday)

Knot theory studies simple closed curves in space. Despite its traditional classification as a "pure math" subject, knot theory has been extensively applied in physics, chemistry and molecular biology. In this talk I will review some of the current applications of knot theory to molecular biology, in particular to the structure and maintenance of chromosomes. A chromosome consists of proteins and a single DNA molecule, which is usually very long and that is folded into a very small volume (e.g. the nucleus of the cell). This spatial confinement is related to the formation of knots (or links if several molecules are involved) which can be informative of the chromosome organization. Noteworthy is the example of DNA knots found in certain viruses that are unveiling new properties of the DNA organization inside the virus. During this presentation I will also introduce some of the experimental, mathematical and computational tools commonly used to study DNA knots.

Noncommutative Algebraic Geometry. (David Patrick, Thursday)

You've already done some algebraic geometry in your life. If you've ever graphed a polynomial on the Cartesian plane, that's algebraic geometry! For example, the graph of the polynomial $x^2 + y^2 - 1$ (algebra!) is a circle (geometry!) centered at $(0,0)$ with radius 1, and more generally we can graph $f(x,y) = 0$ on the plane, where $f(x,y)$ is any polynomial in x and y. But what if the Cartesian plane is "noncommutative"? Meaning, what if xy is not the same thing as yx?

I'll talk about:

(1) Some examples of noncommutative algebra
(2) A brief overview of "classical" algebraic geometry
(3) How we might mix the two to get "noncommutative algebraic geometry"
(4) Why should we care?

Puzzles. (Ivan Matic, Friday)

Puzzles are math related problems with elegant, short, unexpected, and undeserved solutions. Very few people hate them. Puzzles that I will talk about will be math games with one or more players. Warning: sum of the puzzles will be about prisoners and wardens, and wardens will sometimes violate Rule 1. They won't divide by zero, however.

Visitor Bios

Javier Arsuaga (San Francisco State University). Javier is a mathematician who uses knot theory to study DNA. Most of the time, DNA is a long and skinny linear molecule, but under certain experimental conditions it becomes circular and knotted. At Mathcamp, Javier will show students how DNA knots can be used to study chromosome organization in some viruses, as well as to understand the action of certain enzymes. Their course will be highly interdisciplinary, combining molecular biology, random knot theory and the physics of polymers. (Don't worry, Javier will teach you all the background you need to know!)

Greg Budzban (Southern Illinois University). Greg Budzban is interested in modeling various processes in the world as stochastic discrete dynamical systems. His main research involves studying these models using algebraic structures, especially transformation semigroups. He is particularly interested in problems that can be modeled using automata, and will discuss one such problem called the Road Coloring Problem that was recently solved after about 40 years of work. He previously was principal investigator (PI) at Martin Marietta Orlando Aerospace for stochastic computer vision models, and is currently co-PI for an NSF curriculum development grant with the Algebra Project. This grant is helping to design a challenging high-school curriculum for underserved minority students centered around mathematics problems with current research interest.

Kenny Easwaran (University of Southern California). Kenny just got his PhD from the program in Logic and the Methodology of Science at Berkeley, and will soon be starting as a professor in the philosophy department at the University of Southern California. He was a camper in 1998, a JC in 1999, and a mentor in 2002, 2003, 2004, and 2006. His mathematical interests are primarily in set theory and logic, and the interpretation of probability. He enjoys hair-dyeing, puzzles, board games, and both electronic and classical music.

Ivan Matic (University of California at Berkeley). Ivan is a graduate student at the Math Department, UC Berkeley. His area of interests are mathematical analysis, partial differential equations, and probability. He very much likes problem solving. He is the co-author of the website www.imomath.com and the book "The IMO Compendium".

Dave Patrick (Art of Problem Solving). Dave Patrick is a textbook writer and instructor at Art of Problem Solving (AoPS). He is the author of two of AoPS's textbooks (and is currently working on a third), and has taught problem-solving courses for AoPS at all levels from MATHCOUNTS to the Putnam and most everything in between. In his past life doing mathematics research (at MIT and the University of Washington), Dave studied noncommutative algebraic geometry, in particular the classification and structure of noncommutative ruled surfaces.