CLASS DESCRIPTIONS—WEEK 3, MATHCAMP 2008

Contents

Room and Time Changes

This week, and for the rest of camp, we are back in the Psychology building (S) instead of Bio 19. And just like last week, the 9am classes meet at 10am on Tuesday, and the 10am classes don't meet on Tuesday.

9am Classes (plus 10am on Tuesday)

Fermi Problems. (∗, Miranda, Tue–Thurs)

How many ping pong tables are there in Boston? How many molecules in your body were once part of a dinosaur? How cold would the planet be without an atmosphere? How much would the Earth move if everyone lived in Venezeula? How many pizzas would it take to get them there? How many people are thinking about problems like these at this very moment? Even though these questions may seem impossible, you already know the answers! In this class, we will learn how to use the information we know, to figure out something we don't. Knowing how to reason about the size of things is important in all areas of science and applied mathematics. We will see how a little bit of common sense can lead us to insights about physics, biology, global warming, engineering, as well as many useless (but fun!) morsels for thought! On the third day, we will delve more deeply into a scientific paper about ocean circulation, and show how these methods are used in real science. This class will be based heavily on participation; bring a pencil, some paper, and your brain.

Prerequisites: None.

Homework: None.

Hilbert's Third Problem: Scissors Congruence. (∗∗, Yvonne, Fri–Sat)

If you have two *polygons* of equal area, it's always possible to cut up the first one into smaller polygons and then glue up the pieces into the second polygon.

If two *polyhedra* have equal volume, can you cut the first one up into smaller polyhedron, rearrange the pieces, and glue them up to get the second polyhedron?

The answer is no. You can't cut up a cube to get a regular tetrahedron of the same volume. Come find out the beautiful reason why, how this generalizes, and whether or not we can say the same for spherical polyhedra (and what exactly that means)!

Prerequisites: None.

Homework: None.

Related to: The Banach-Tarski Paradox (Week 1)

Linear Programming. (∗∗–∗∗∗, Marisa, week 1 of 1)
Mathcampers have been industrious as of late, and they've started a little business out of the 20 classrooms on campus selling Tension-Adjusted Rubik's Cubes (TARCs) and Projections of 4-d Polytopes in 3-space (P4P3s). There is sufficient demand for them to sell at most 70 TARCs and 60 P4P3s. Each classroom devoted to the business yields either 3 TARCs or 2 P4P3s. TARCs sell for $15 apiece, but P4P3s sell for $25. They need 3 hours to build a good TARC, but 5 hours to build a P4P3. They can muster up at most 150 worker-hours between now and the end of camp, but have to pay the campers $5 an hour to build math toys. How many TARCs and P4P3s should they make to maximize their profit? If they had a sale on Rubik's cubes and offered them for $12, would the production strategy change?

That's the starting point for a Linear Programming problem—the optimization of a function given a set of constraints. We'll be talking about algorithms for solving LP problems, duality, and sensitivity analysis. The class will conclude with the max-flow, min-cut theorem from Graph Theory—using no graph theory techniques at all!

Prerequisites: Linear algebra (Week 1)

Homework: Optional; all paper-and-pencil (not computer-based)

Related to: Linear algebra (Week 1)

Quantum Computing. (∗∗∗, Alice, week 1 of 2)
Einstein once described quantum mechanics as "real black magic calculus". A quantum computer is a computer that exploits quantum phenomena to do computation in a way that's fundamentally different than the classical computers you're used to (well, probably, anyway). If you went to Scott Aaronson's colloquium, you probably already have some idea of why this is cool. Among other things, quantum computers can factor numbers in polynomial time, which would break RSA encryption if we managed to actually build a quantum computer.

In this class, I'll talk about what quantum computers are and go over the facts of quantum mechanics that we need to understand and work with them. I'll talk about quantum circuits and quantum gates and explain major quantum algorithms like Shor's and Grover's. Other topics (the end of week two) will be determined by what people are interested in and may include things like quantum teleportation and quantum key exchange.

Prerequisites: Linear Algebra (Week 1)

Homework: Recommended

Related to: The Limits of Quantum Computers (colloquium, Week 2); Computability and Complexity (Weeks 1–3); Quantum Mechanics (Week 3)

A Random Walk in Combinatorics. ($***$, Julian Gilbey, Tue–Thurs)

We will be exploring a selection of my favourite combinatorial tidbits over these three days. From a computer algorithm which can determine the number of legal positions for a Rubik's cube, to an area of mathematics which is known for its horrendously large numbers, to the recreational problem of squaring a square, the subject is both all-encompassing and beautiful. Each day will be independent of the others.

Homework: Optional.

Prerequisites: Basic group theory up to normal subgroups.

Elliptic trigonometry. ($***$, Dave Savitt, Fri–Sat)

Here's a slightly oddball way of thinking about the sine function: starting from the point $(1, 0)$ on the unit circle, $\sin(t)$ is the vertical distance one has to travel along the circle in order to travel an arclength of t. From this point of view, the fact that $\sin(\pi/6) = \frac{1}{2}$ means that to cut the circle into 12 equal pieces, the first cut should be at $y = \frac{1}{2}$. What happens if we replace the circle with an ellipse? or a leminscate? Some pretty interesting stuff, that's what, including the startling appearance of the arithmetic-geometric mean in the arclength formula for the lemniscate.

Prerequisites: Calculus (students should be comfortable calculating arclength integrals and performing integration by substitution).

The Four Pillars of Geometry. ($*$–$***$, Moon Duchin, Tue–Fri)

I'll be following John Stillwell's very cool book by that title, where he gives four fairly independent historical perspectives on what geometry is all about that end up interlocking in surprising ways. The four days are going to be more-or-less self-contained, so you can mix and match, but they all click together pretty beautifully. First pillar: straightedge and compass. Going back to Euclid, we see that huge swaths of geometry and number theory can be phrased in terms of what you can draw with lines and circles. Galois theory! Gauss on Youtube! Second pillar: affine geometry. Hello to Descartes, and the reimagining of Euclid with linear algebra. Geometry gets a theory of motion. Third pillar: projective geometry. A svelte new axiom system comes to us from Renaissance painting, promotes infinity to geometric significance, and sponsors the mathematical career of "an invention of the devil". Fourth pillar: transformations and invariants. Geometry is all about what stays the same when things change. And somehow this leads us to the disk where circles are straight and Euclid is dead, and on to quaternions and the mystery of the surplus polychoron. Pretty pictures await.

Prerequisites: None.

Related to: What is Geometry? (Week 3); Non-Euclidean Geometry (Superclass, Week 4)

Image Manipulation with Math and Matlab. ($*$, Rebecca Saxe, Sat)

Any picture can be represented as a series of points—like a TV screen. But there's an alternative. Pictures are also a series of waves, more like sound but in two dimensions. Knowing this gives the power to mess around with it. This class will include pretty demos, and a very basic introduction to Fourier transformation.

Homework: None.

Prerequisites: None.

10am Classes (except Tuesday)

Combinatorial Game Theory. ($**$, Alfonso, week 1 of 1)

We have three berry baskets: one with four strawberries, one with three blueberries, and one with two blackberries. We take turns eating them. In your turn, you may eat as many berries as you want, as long as they are all in the same basket. Then it is my turn. The player who eats the last berry wins. Will you beat me? What if we start with different amounts of berries?

In Combinatorial Game Theory, we analyze games of no chance (no drawing cards, no throwing dice). Or goal is a strategy that allows us to win from any position. Come to this class to learn the main techniques to analyze games, and to discover why Nim (the berry game above) is the model that we can use to solve a very large family of them.

Prerequisites: None.

Homework: There are two options:

1. You can simply take this class normally. Then the homework is optional.
2. You can also take this class as the first part of a project. Then homework is required. At the end of the week I will give you an individual or group project that will consist on analyzing and solving a particular game.

Problem solving without calculus. (Mark, week 1 of 1)

This week, problem solving has no star levels; in both sessions there will be problems of a wide range of difficulty, and I hope people will be working on them both individually and in groups, while I circulate and give hints as necessary. For this session the problems will not require calculus, but as anyone who has looked at IMO (International Mathematical Olympiad) problems knows, that doesn't necessarily make them easy! Again, some will be (relatively) easy and some will be hard; more importantly, the intention is for them to be interesting, varied, and fun to solve.

Prerequisites: None

Homework: From none to a lot of (optional) work, depending on how fast you are and how interested you are in continuing with the problems outside class

11:10am Classes

Differential Equations and Mathematical Modeling. ($***$, Miranda, week 2 of 2)

This week we'll do chase-and-escape (pursuit) problems, followed by either asymptotic methods, partial differential equations, or evolutionary game theory. The topics may be somewhat independent from the first week.

Prerequisites: Calculus—talk to Miranda if you are thinking of joining.

Homework: Recommended.

$\mathbb{Q}(\sqrt{d})$ in Space. ($**$, Mira, Dave, Noah, week 1 of 2)

Our Moore method class "$\mathbb{Q}$ in Space" has an entry point for new students this week, as it gradually morphs into "$\mathbb{Q}(\sqrt{d})$ in Space". For the rest of camp, we'll be focusing on numbers of the form $a + b\sqrt{d}$, using them to answer seemingly innocent questions about integers, such as "Which square numbers which are also triangular numbers?" (The first such number is 36. Good luck finding the second!)

For people who have been in the class all along, this will be a smooth transition: you can just think of it as a continuing class (and we hope you do continue). New people may need to learn a little bit about continued fractions and take a few facts about them on faith. Note that this is a Moore Method class: we give you theorems to prove and problems to solve; you do them for homework, present your solutions in class, and critique each other's proofs. You also get to argue with us and occasionally prove us wrong. It's fun!

Homework: Required.

Prerequisites: Elementary number theory (talk to us if in doubt).

What is Geometry? Getting Over Euclid. ($**$, Nina, week 1 of 1)

Euclid's *Elements* is the mostly widely distributed textbook in the history of the world, but it has serious problems from a modern perspective. His definitions, for example: "A line is length without breadth." Or his proofs: he proves Side-Angle-Side congruence of triangles by saying that when we lay one triangle on top of another it must be so. Or most interestingly, his axioms: his infamous *parallel postulate* troubled mathematicians for thousands of years as they tried to derive it from the other axioms. They were on a fool's errand—it was later shown that you can consider reasonable "geometries" which satisfy the other axioms where the parallel postulate fails. This leads us to ask: What is (a) geometry?

We have definitions of "a group," "a topological space," "a vector space," and so on. Around 1900, Hilbert, Erlanger, and Pasch (among others) axiomatized "a geometry" in the same way: its objects are points and lines which satisfy certain axioms. In this course, depending on time and interest we'll examine

- Euclidean geometry.
- The parallel postulate and its various equivalent statements (such as the Pythagorean theorem and the fact that the sum of the angles in a triangle is always 2π).
- Hilbert's *Foundations of Geometry* and what an axiomatic approach to geometry really is.
- Should all geometries satisfy the parallel postulate?: A short introduction to Non-Euclidean geometry.
- How today's mathematicians think of geometry—in terms of transformations.

Prerequisites: None.

Homework: Recommended.

Related to: Non-Euclidean Geometry (Superclass, Week 4), The Four Pillars of Geometry (Week 3).

The Shape of Infinity. (Anti, $***$, week 1 of 1)

What does space look like at infinity? In the Reeb Foliation colloquium, we saw that adding one point at infinity makes space into a sphere. But this week, Moon Duchin will tell us about projective geometry, in which we encircle the plane which a whole 'line at infinity'. This is better because it distinguishes points at infinity in different directions, which the sphere doesn't. But there are other ways to add points at infinity, which can distinguish between points at infinity that projective geometry can't tell apart.

The process of adding points at infinity is called *compactification* (cf. last week's blurb for Point-set Topology). Our goal in this class will be to construct the 'best' compactification, called the *Stone-Čech compactification*, which adds all possible points at infinity and maintains all possible distinctions between them. Our tool will be *gauge spaces*, which are more general than metric spaces but less general than topological spaces.

Prerequisites: I will define everything needed in class, so there are no formal prerequisites. However, some familiarity with ε's and/or open sets will be very helpful; at least one week of Point-set Topology (Weeks 1–2) or Real Analysis (Weeks 1–2) should suffice.

Homework: Required.

Related to: Point-set Topology (Weeks 1–2); Real Analysis (Weeks 1–2)

Quantum Mechanics. (∗∗∗∗, Allan Adams, week 1 of 1)
TBA.

Prerequisites: Calculus; Linear Algebra (Week 1)

1:10pm Classes

Problem solving with calculus. (Mark, week 1 of 1)
This week, problem solving has no star levels; in both sessions there will be problems of a wide range of difficulty, and I hope people will be working on them both individually and in groups, while I circulate and give hints as necessary. For this session the problems will require calculus, but that doesn't necessarily make them any harder! Again, some will be (relatively) easy and some will be hard; more importantly, the intention is for them to be interesting, varied, and fun to solve.

Prerequisites: Calculus; some problems may require multivariable calculus, but if you've only seen single-variable calculus you should be OK for most of the problems

Homework: From none to a lot of (optional) work, depending on how fast you are and how interested you are in continuing with the problems outside class

Analytic Number Theory. (∗∗∗, JR, week 1 of 1)
In number theory, we study functions whose domains are the set of positive integers; some common examples include $d(n)$ (the number of divisors of n), $\sigma(n)$ (the sum of the divisors of n), $\phi(n)$ (the number of positive integers less than n that are coprime to n), and many others. The fundamental theorem of arithmetic tells us that every positive integer can be written as a product of primes. Knowing the prime factorizations of an integer allows us to evaluate many arithmetic functions on that integer. However, there is one catch: factoring a number is difficult. To make matters worse, numbers which are very close in size often have differ wildly in factorization. Is there any practical way, then, to get a handle on the growth of an arithmetic function?

One of the key ideas of analytic number theory is that we can understand arithmetic functions much better if, instead of trying to evaluate them on individual integers, we sum their values over ranges of integers. For many arithmetic functions that seem very unpredictable, we can make very precise statements about their sums. In this class, we'll discuss several techniques for finding sums of arithmetic functions. We will also discuss the prime number theorem, and its relation to the Riemann Hypothesis.

Prerequisites: Some basic number theory, including familiarity with the fundamental theorem of arithmetic. Familiarity with integral calculus in one variable is also required.

Homework: Recommended

Dynamical Systems. (∗∗∗, Mike, week 1 of 2)

Given a function f from a set X to itself, we can ask what happens if we follow the orbit of a single point x, which is the sequence:

$$x \mapsto f(x) \mapsto f(f(x)) \mapsto f(f(f(x))) \mapsto \dots.$$

The orbit might be one point that maps to itself, or a periodic cycle; it might converge to a certain point or cycle; or it might be seemingly random ("chaotic", if you're into that kind of thing). Even knowing just a little about our function, we can sometimes say a lot about its dynamics. If X is the real numbers and our function is continuous, then we'll prove the surprising fact that whenever the system contains a single period 3 orbit, the system must have orbits of all periods!

Dynamical Systems is the study of deterministic time evolution systems (that is, there is no probabilistic uncertainty involved). It often involves studying (Ordinary) Differential Equations, but in this class we will almost exclusively focus on discrete time dynamical systems instead of continuous ones. In this setting, we can focus more on the dynamics of a system without worrying about the technical details of solving complicated differential equations.

Prerequisites: Real Analysis (Week 1)

Homework: Optional

Related to: Differential Equations and Mathematical Modeling (Weeks 2–3), Real Analysis (Weeks 1–2)

How to Count Mathematical Structures. (∗–∗∗, Julian Gilbey, Mathieu Guay-Paquet, week 1 of 1)

Have you ever wondered how many ways you can compute a sum like

$$1 + 2 + 3 + 4 + 5$$

by repeatedly adding two consecutive terms? Or wanted to find a closed form for the nth Fibonacci number? Asked how many ways you can write 37 as a sum of natural numbers, when the order of terms matters, or doesn't matter, or without repeated terms, or. . . ?

In this class, we'll explore these and related questions using tools from combinatorics (also known as counting with flair). In particular, we'll work with generating functions, which give us a way to manipulate whole sequences of numbers at once.

Prerequisites: Some calculus would be useful, but not really necessary.

Homework: Optional. There will be time in class to work on various problems, and you are encouraged to continue working on them outside of class.

Colloquia

Origins of the Universe. (Allan Adams, Tue)

This colloquium will be awesome! You should all come.

Projective Planes. (Moon Duchin, Wed)

This flavor of geometry grows out of the study of perspective, like when paintings seem to have a vanishing point on the horizon. We can give a simple definition of a projective plane, but we'll find that a really rich theory results, including both finite objects with interesting combinatorics and a family of infinite planes with lots of algebraic structure. And their malformed cousins that don't fit in any classification system neatly. In the end, we'll meet an innocent-looking theorem about triangles that "detects" the presence of hidden dimensions.

Words for Numbers. (Rebecca Saxe, Thurs)

Some parts of math are admittedly hard to wrap your head around, but counting seems easy enough! All a matter of perspective. Step back a thousand years in human history, a few thousand miles across the earth, or even just back to your own toddlerhood, and suddenly it's not so simple. I'll talk about giving words to numbers.

Qualifying Quiz Presentations. (Students, Fri–Sat)

Come hear your fellow students present their solutions to the problems on the Mathcamp Qualifying Quiz! We may continue Saturday after lunch, if there isn't time to finish in the colloquium slot on Friday.

Visitor Bios

Allan Adams. (MIT—physics)

Allan Adams studies quantum versions of algebraic and differential geometry that play a fundamental role in string theory, and uses these tools to explore the fate of tachyons, moose diagrams, and other puzzles involving black holes and spacetime singularities. Allan believes that everyone should understand quantum mechanics, which is as beautiful and strange as it is true, and looks forward to discussing it at Mathcamp.

Mathieu Guay-Paquet. (University of Waterloo)

Mathieu was a camper in 02–03 and a JC in 07. He is now a grad student in combinatorics, and more specifically algebraic enumeration—in other words, he likes counting things, especially things with a lot of nice structure. He also particularly likes languages (although he can only reliably speak French, English and python) and games (to the point of carrying an emergency deck of cards).

Moon Duchin. (University of California, Davis)

Moon Duchin is interested in geometry, topology, and dynamics, in lots of different combinations. Lately she's got geometric group theory on her mind. She also thinks about philosophy, cultural studies, gender theory, what they have to say about math, and what math has to say back!

Rebecca Saxe. (MIT—cognitive science)

Rebecca Saxe studies how the human brain works. Most of her research is conducted on "normal human adults"—MIT undergraduates. But every once in a while she likes to come see some brains that are *really* working, at Mathcamp.