CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2008

Contents

Room and Time Changes

This week we are not in the Psychology building (S) at all. We still have all our classrooms in the Physics building (H), and in addition we have the rooms **Biology 19** (across from the Physics building, in the basement) and **GCC-D** (in the Gray Campus Center). Assembly and colloquia will be in Vollum Lecture Hall, where the beginning of camp assembly was. The TAU board will be in the foyer of the library building.

Also, this week the morning class blocks start at 9:00am, 10:00am, and 11:10am, so that 10:00am classes are 60 minutes long. Note that 9:00am classes meet at 10:00am on Tuesday, and 10:00am classes do not meet on Tuesday. Classes continuing from Week 1 generally have no description, unless they have an "entry point" in this week.

9am Classes (plus 10am on Tuesday)

Math of Juggling. (∗∗, Anti, Tues–Thurs)

Perhaps it's not surprising that juggling is a very mathematical activity, since it is are intimately dependent on patterns and mathematics is (among other things) the study of patterns. Moreover, just like in mathematics, understanding juggling patterns allows us to not only describe the ones we already know, but to invent new ones! In this class, we will make up a notation for juggling patterns and use it for both of these purposes. Moreover, we'll prove some theorems about how to find and count juggling patterns of different types.

Prerequisites: If you are already familiar with notation for juggling patterns, do *not* come to this class.

Homework: Optional.

Introduction to Tropical Geometry and Phylogenetics. ($**$, Linda, Fri)

Tropical Geometry is a relatively new area in mathematics, involving algebraic geometry over the tropical semiring, where addition and multiplication are replaced by minimum and addition. The study of tropical objects has connections to many parts of mathematics (symplectic geometry, complex analysis, and dynamical systems, as well as combinatorics, computer algebra, and statistical models). In this lecture, we will focus on a connection between the tropical Grassmannian and its relationship to biological questions in phylogenetics.

Prerequisites: None.

Homework: None.

Introduction to Algebraic Statistics. ($**$, Linda, Sat)

Algebraic Statistics is concerned with problems that lie at the intersection of algebra, geometry, combinatorics, and statistics. Methods from algebra and geometry can be used to make statistical inferences; many statistical models for discrete random variables can be represented by classical algebraic varieties. We will give a few examples of these connections and touch upon applications to mathematical biology.

Prerequisites: None.

Homework: None.

Differential Equations and Mathematical Modeling. ($**$–$***$, Miranda, week 1 of 2)

You probably know how to solve the equation $(f(x))^2 + x^3 = 7$ for $f(x)$, but how would you solve an equation like $(f'(x))^2 + f(x) + x^6 = 1$? This is an example of a *differential equation*, which is an equation involving the derivatives of a function f which we must solve for f. This kind of equation arises all the time in modeling situations (as well as in pure mathematics), but they are usually very hard, if not impossible, to solve analytically.

Fortunately, we don't have to actually solve the equation in order to obtain useful information about its solution. There are lots of clever mathematical things we can do to turn the equation into something meaningful. In this class, we will look at how to set up a differential equation to model different kinds of situations, and see what we can do with it afterwards. We'll look at applications like population dynamics, chase-and-escape problems, ecology, how planets move, traffic flow, and how to model a game of evolutionary rock-paper-scissors. The kind of math we'll learn in the first week includes linearization, phase-plane analysis, stability, perturbation theory, bifurcations, non-dimensionalization, singular perturbations, as well as how to think about a physical situation in terms of a differential equation. Some of the topics we could do in the second week are asymptotics, boundary layers, multi-scale analysis, partial differential equations, diffusions, shock waves, Fourier analysis, wave theory (the actual topics depend on what people in the class are interested in.)

We will make reference to some more difficult calculus once in a while for those who have seen it, but for the most part, you should be able to follow the class if you know only basic calculus, such as having a good feel for what a derivative is and some of its geometric properties, the chain rule, and integration by parts.

Prerequisites: Linear algebra, a bit of calculus

Homework: Recommended.

Artificial Intelligence. (∗∗∗, Noah Goodman, Tues–Thurs)

What would it take to write a computer program as smart as you are? In this class we'll explore some of the abilities that an AI would need to have to be worth its name (i.e. Intelligent). We'll specifically be looking into planning, reasoning, and learning, and some of the techniques that have been proposed to achieve them. We'll especially focus on ways that an AI can cope with uncertainty, in order to dominate the world.

Prerequisites: Some familiarity with computer programming and/or probability will enhance your enjoyment of this class, though it's not necessary.

Probability and the mind, continued. (Josh Tenenbaum, Fri)

This class will follow up on my colloquium, talking about some additional aspects of human cognition that can be explained using probability theory. Topics that might be covered, in addition to those mentioned above, are cognitive development—how children learn about the basic structure of their world—and social cognition—how we interpret the actions of other people by inferring their hidden mental states (beliefs, desires, goals, emotions).

Open Q&A session about the Brain/Mind. (Josh Tenenbaum, Sat)

I will take questions about any aspect of the brain/mind that you're interested in, and perhaps also tie up some loose ends from my previous classes. Come prepared with the questions you're most excited to learn about, or even better, email me your ideas in advance: `jbt@mit.edu`.

Point-set Topology. (∗∗∗∗, JR, week 2 of 2)

A topological space can be a dangerous place. Take the open interval $(0, 1)$, for example. Suppose you decide to take a walk one day. First, you walk to $\frac{1}{2}$, then to $\frac{1}{3}$, then on to $\frac{1}{4}$, and so on. At the end of your walk, you find yourself at 0. When you look down, you realize you are no longer standing on the space, and you fall and break your ankle.

The closed interval $[0, 1]$ is much safer. It's impossible to fall off the edge of this space. The difference is that the closed interval $[0, 1]$ is compact, whereas the open interval $(0, 1)$ is not. Compact spaces have many important properties. For instance, all continuous, real-valued functions defined on them are bounded. In this class, you'll learn what makes a space compact, what properties you can deduce about a compact space, and even how to turn your favorite non-compact space into a compact one.

Prerequisites: The material covered in the first week of point-set topology, including topological spaces, subspace, product, and quotient topologies, and continuous functions.

Homework: Recommended.

Required for: Algebraic Topology (week 4)

10am Classes (except Tuesday)

Sequences and Series. (∗∗, Alice, week 1 of 1)

Does this sum converge?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

How about this one?

$$1 + \frac{1}{2} + \frac{1}{3} + ... = \sum_{n=1}^{\infty} \frac{1}{n}$$

How about this?

$$\sum_{n=1}^{\infty} \left(\frac{1+i}{2}\right)^n$$

The first two you've probably seen in school, but you probably haven't seen the third (it does converge—try to figure out what the limit is). In this course we'll study sequences and series, both real and complex. We will figure out a meaningful notion of the limit of a sequence, and of a series. If there's time, we'll also look at continuity of functions and power series. By the end, you should understand many of the building blocks of analysis.

This course will be largely proof-based: For each theorem I'll either prove it or provide you with a proof for reference. You will also prove a number of theorems on the homework. If you've just learned about proofs in Things You Need to Know, this course will give you a chance to practice your skills.

Prerequisites: Summation notation, a little set notation, complex numbers. If you took TYNTK, you should be fine.

Homework: Daily, Required.

Applications of Linear Algebra. (∗∗, Miranda and Mira, week 1 of 1)

In this class, we'll focus on a powerful linear algebra technique called *singular value decomposition* (a.k.a. SVD). Youb can think of SVD as a way of finding eigenvectors for matrices that don't have any! (Actually, you find the eigenvectors for a different matrix, which is closely related to the original one.)

Applications of SVD are everywhere. After explaining the theory, we'll look at examples from (some subset of) climate data, image processing (using pictures of past Mathcampers!), genetics, web search, and statistics. In the homework assignments, you'll get to try some of these applications for yourself: expect to spend at least some of TAU working with Matlab in the computer lab. (Don't worry, we'll teach you the basics of Matlab first.)

Homework: Highly recommended (and fun!)

Prerequisites: Enough linear algebra to understand the statement of the Spectral Theorem: "Let A be an $n \times n$ matrix. A is symmetric iff there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A". The Mathcamp linear algebra class from Week 1 will cover all the necessary prerequisites.

Cubic curves. (∗∗–∗∗∗, Mark, week 1 of 1)

A curve in the x, y-plane is called a cubic curve if it is given by a polynomial equation $f(x, y) = 0$ of degree 3. Compared to conic sections (of degree 2), at first sight cubic curves are unpleasantly diverse and complicated; Newton distinguished more than 70 different types of them, and later Plücker made a more refined classification into over 200 types. However, as we'll see, by using complex numbers and points at infinity we can bring considerable order into the chaos, and cubic curves have many elegant and excellent properties. One of those properties, concerning intersections, will allow us to prove a beautiful theorem of Pascal about hexagons and conic sections, and it will also allow us to define a group structure on any cubic curve. (Well, we may have to leave off a "bad" point first; but a cubic curve can have only one such point, and most cubic curves don't have any.)

Cubic curves without "bad" points are known as elliptic curves, and they are quite important in number theory, for example in the proof of "Fermat's Last Theorem"; however, in this week's class the point of view will be mostly geometric, and no knowledge of number theory (or even of groups) is required. With any luck, along the way you'll pick up some nice ideas that extend beyond cubic curves, such as how to deal with points at infinity (using "homogeneous coordinates"), what to expect from intersections, and where to look for "bad" points and for inflection points.

Prerequisites: Some calculus would help

Homework: Optional

p-adics. (∗∗∗∗, David, Greg, week 1 of 2)

So you know something about the p-adics, they're alright, yeah sure fine. But what're they good for, really? We'll spend the first week of this class learning about the p-adics. We'll see a number of different ways to define them. We'll look at how one can tell with a glance if a polynomial with p-adic coefficients is irreducible or not, and determine something about its roots. We'll determine the structure of p-adic units using the p-adic exponential and logarithm, and apply the result to determine the structure of the group of numbers relatively prime to any integer n. We'll consider extensions of p-adic integers and study how these canonically decompose.

In the second week, we'll apply p-adics to the outside world. We'll work to prove the Hasse principle: if a quadratic equation (in any number of variables) has a solution mod p for every p, and a solution over the reals, then it has a solution in the rationals.

Prerequisites: Very comfortable with groups, rings and fields

Homework: Required

11:10am Classes

The History of Fermat's Last Theorem. (∗–∗∗, Holly Swisher, Tues–Fri)

Fermat's last theorem is one of the simplest theorems to state, yet it brought on a multi-millenial saga as people searched for a proof. This course will tell the story from the beginning, starting with Pythagorean triples dating back to 1500 BC, through the eventual success of Andrew Wiles, et. al. in the 1990's. Along the way we will learn about Fermat's and Euler's proofs of small cases, Sophie Germain's contributions, the concept of unique factorization, and finally a tiny glimpse into the mysterious world of elliptic curves and modular forms. Furthermore, we will discuss the intrigue and controversy that surrounded this problem from it's beginning.

How Big Is Infinity? (∗∗, Anti, Sat)

In "TYNTK: Infinity", you proved that there are more real numbers than there are natural numbers. This may lead you to ask questions such as the following:

- Are there infinite sets larger than the real numbers?
- Are there sets larger than the natural numbers, but smaller than the real numbers?
- Is there a limit to how large infinite sets can be?

My answers to these questions are "Yes", "We don't know", and "It depends on what you mean", respectively. In this class we'll prove the first answer, discuss our lack of knowledge of the second (which, surprisingly enough, can be stated in a very precise sense), and argue about the third.

Prerequisites: None, except "TYNTK: Infinity" or equivalent.

Homework: None.

Related to: Set Theory (Superclass, Week 4)

Topics in Group Theory. (∗∗–∗∗∗, Nina, week 2 of 2)

The keyword for this class is "symmetry." We'll use group theory to explore the symmetry of certain **polytopes (∗∗)** (days 1 and 2), **tilings and patterns (∗∗–∗∗∗)** in the plane (days 3 and 4), and **infinite graphs (∗∗∗)** (day 5). Instead of looking at algebraic representations of groups, we want to see how they exist naturally in geometry. The three parts of the week will not depend on each other; feel free to drop in for portions that interest you.

Prerequisites: Group Theory (week 1 is more than enough): definition of group, symmetric group, notation for symmetric group, dihedral group. Linear Algebra is helpful for Tilings and Patterns, though not required (the week 1 class is more than enough): linear transformations of $\mathbf{R}^2$.

Homework: Optional.

Related to: Platonic Solids. Day 5 may be related to Non-Euclidean Geometry (Superclass, Week 4)

The Complexity of Boolean Functions. (∗∗–∗∗∗, Scott Aaronson, week 1 of 1)

Boolean functions are just any functions mapping input bits to output bits. This course will start with some embarrassingly concrete questions about these functions—how do you compose them from elementary "gates" (like AND, OR, and NOT)? How do you measure their complexity?—and will end with some of the frontier issues of theoretical computer science, including P versus NP, the power of randomness, and the power of quantum computing.

Along the way, we'll see (time permitting):

- Claude Shannon's famous "counting argument"
- Measures of the complexity of Boolean functions: decision-tree complexity, certificate complexity, and block sensitivity
- Upper and lower bounds on decision-tree complexity for AND/OR trees
- Quantum query complexity: the core of quantum algorithms
- The connections between Boolean functions and real polynomials
- Circuit lower bounds as an approach to the P vs. NP question

This course complements other theoretical CS courses at Mathcamp: overlap is minimal, and students can take any subset of the courses depending on interest.

Related to: Finite Automata (Week 1); Computability and Complexity (Weeks 1–3); The limits of Quantum Computers (Colloquium, this week)

Advanced Problem Solving, or, How To Change Coins, M&M's, or Chicken Nuggests—the Linear Diophantine Problem of Frobenius. (∗∗∗∗, Matt Beck, week 1 of 1)

How many ways are there to change 42 cents? How many ways will there be when all the pennies are gone? How about if nickels were worth four cents?

More generally, suppose we have coins of denominations $a_1, a_2, \ldots a_d$. Can one find a formula for the number $c(n)$ of ways to change n cents? A seemingly easier question is: can you change n cents, using only our coins? Depending on the culinary preference of the audience, we may state these questions in terms of bags of M&M's or boxes of Chicken Nuggets ("Can you buy Chicken Nuggets so that our 34 friends get exactly one each?").

We will see that if $a_1, a_2, \ldots, a_d$ do not have any common factors then we can be certain that we can change n, provided n is large enough. A natural task then is to find the largest integer that cannot be changed. This problem, often called the linear Diophantine problem of Frobenius, is solved in closed form for $d = 2$, in somewhat-closed form for $d = 3$, and wide open for $d > 3$.

We will use the magic of generating functions to discuss the $d = 2$ case of this classical problem, which will exhibit its connections to Number Theory, Algebra, and Combinatorics.

1:10pm Classes

Intro to Problem Solving, or, Some of My Favorite Number Theory Problems. (∗∗, Matt Beck, week 1 of 1)

We'll discuss (and work on!) some more and some less famous problems in number theory, with appearances by perfect numbers, lockers, a strange currency system, and lots of primes.

Knots, Labelings, and Algebra. (∗∗, Susan, week 2 of 3)

In week one of Knots, Labelings, and Algebra, we developed the machinery necessary to discuss knot invariants and constructed our first labeling scheme. This week we will be labeling our knots with the integers mod a prime number p. This will allow us to study our knots using tools from linear algebra.

If you have had a basic course in knot theory but want to know more, this is an ideal time to jump in. Also join us next week, when we will be labeling knots with group elements!

Prerequisites: Knot Theory week 1 or other introduction to knot theory. Specifically, come and talk to me at TAU if you have never seen tricolorability of knots.

Homework: Recommended

Seminar Class

Isometries of $\mathbb{R}^n$. (∗∗, Dan, week 1 of 1)

Consider the following amazing theorem: any function $\mathbb{R}^n \to \mathbb{R}^n$ that preserves distance is a composition of at most $n + 1$ reflections.

It's pretty awesome. It tells us that every *isometry* (that's a map that preserves distance) is generated by reflections, which is kind of surprising when you think about it. It works for every dimension, so it includes not just rotations and translations but isometries in $\mathbb{R}^4$, $\mathbb{R}^5$, and more.

The goal of this class is for you to prove the theorem. A small group of us will gather, and I'll provide hints if you need them, but the insights and progress will all be a result of your work together. I won't lecture: you'll come up with the big ideas and see them go into action. This is your chance to do big, interesting theorems in a group together.

Once you've proved the theorem, we'll start to explore isometries of other interesting spaces.

How to sign up for this class: I don't want to run a class of more than around 8–10 people. If you'd like to sign up, then you need to put your name on the Academic Announcements board on the back wall of the main lounge. Based on signups, I'll schedule the class at a time that is most convenient for everyone. If lots of people sign up, I will try to run multiple sessions of the class. The deadline for signing up is Saturday at midnight.

Prerequisites: None. Group theory might help you to understand some of the later material on a deeper level.

Homework: None.

Cutting Cake. (Matt De Vos, Tues)

This talk is an introduction to the subject of fair division—the problem of dividing a resource, say cake, among a group of people in a fair manner. We will start out by discussing some different notions of "fair" and then we will learn a beautiful procedure for fair division based on some combinatorial topology.

The Limits of Quantum Computers. (Scott Aaronson, Weds)

In the popular imagination, quantum computers would be almost magical devices, able to "solve impossible problems in an instant" by trying exponentially many solutions in parallel. In this talk, I'll describe results in quantum computing theory that directly challenge this view. For example, at least in the "black-box model" that we know how to analyze, quantum computers would need exponential time to break cryptographic hash functions or find local optima, just as classical computers would. I'll also describe how studying the limitations of quantum computers can lead to new insights into classical computation as well as physics.

Related to: Boolean functions (Week 2); Computability and complexity (Weeks 1–3); Quantum Computation (Weeks 3–4)

Probability and the Mind. (Josh Tenenbaum, Thurs)

Suppose that we view the human mind as a computer, a system for processing information and solving problems. One of the most important and impressive kinds of information-processing problems that the mind solves is inductive inference, or "going beyond the data given". Our visual system can infer the three-dimensional structure of our local environment, recognizing what objects are where, given only a single two-dimensional image on our retina. When we notice coincidences or patterns in our experience, these are also inductive inferences: we can infer the existence of a hidden causal connection from observing suspicious coincidences in sequences of events; we can learn the meaning of a new word by detecting a pattern of commonalities among objects that the word is used to refer to. I will talk about how the mind solves these problems of inductive inference, using some of the basic mathematics of probability theory. The mind's computations be modeled as a kind of "intuitive statistics", finding the causal explanations most likely to have produced the data we observe through our senses.

Discreet Volume Computations for Polytopes: An Invitation to Ehrhart Theory. (Matt Beck, Fri)

Our goal is to compute the volume of certain easy (and fun!) geometric objects, called polytopes, which are fundamental in many areas of mathematics. Although polytopes have an easy description, e.g., using a linear system of equalities and inequalities, volume computation is hard even for these basic objects. Our approach is to compute the *discrete volume* of a polytope P, namely, the number of grid points that lie inside P, given a fixed grid in Euclidean space such as the set of all integer points. A theory initiated by Ehrhart implies that the discrete volume of a polytope has some remarkable properties. We will exemplify Ehrhart theory with the help of several families of polytopes whose discrete volumes are connected with some of our friends in various mathematical areas, such as binomial coefficients, Eulerian, Stirling, and Bernoulli numbers.

Visitor Bios

Scott Aaronson. (MIT—computer science)

Scott Aaronson studies the mathematics of computation. He is interested in the limitations of quantum computers ("what we can't do with computers we don't have"), and more generally, in fundamental limits on what can efficiently be computed in the physical world. He is the creator of Complexity Zoo, an online encyclopedia of over 460 classes of computational problems—so you can ask him not only about the famous ones like P and NP, but also about MA_EXP, NISZK, coC=P, and QMA/qpoly. He was a camper at Mathcamp'96, where a talk by Richard Karp gave him his first exposure to theoretical computer science. He is thrilled to be back!

Matthias Beck. (San Francisco State University)

Matt's research is in discrete geometry and number theory. Particular interests include counting integer points (think of those as "grid points" of your favorite lattice) in polyhedra (in many dimensions) and the application of these enumeration functions to various combinatorial and number-theoretic topics and problems. Matt is also the co-director of the San Francisco Math Circle and has worked with many students and teachers at various other Math Circles in the Bay Area.

Linda Chen. (Ohio State University)

Linda Chen, a former Mathcamp mentor, studies algebraic geometry and algebraic combinatorics. Some of her work involves Schubert calculus, quot schemes, Fulton-MacPherson spaces, the moduli space of curves and its generalizations, quantum cohomology, toric varieties, Hilbert schemes of points on surfaces, equivariant cohomology and GKM spaces, orbifold cohomology and Gromov-Witten theory of stacks, and other aspects of enumerative and combinatorial geometry.

Matt De Vos. (Simon Fraser University)

Matt De Vos is interested in discrete math of many different flavors. Most of his research is in graph theory, but lots of times there is algebra involved, and recently he has been doing some combinatorial number theory. He got his Ph.D. from Princeton in 2000 and is now a professor at Simon Fraser. Last year he helped to launch a wiki for unsolved math problems called the Open Problem Garden. When not doing math, he likes to ride his bicycle(s).

Noah Goodman. (MIT—cognitive science)

Noah Goodman, a former Mathcamp mentor, approaches the study of mind with a combination of formal (mathematical) analysis, philosophical orientation, and empirical grounding. His research focuses on concepts and causality: what is the nature of causal and conceptual knowledge? How do we acquire this knowledge, and how do we use it?

Holly Swisher. (Oregon State University)

Holly Swisher, a former Mathcamp mentor, works in the areas of partitions and modular forms. In particular, many combinatorially interesting sequences (like many relating to partitions for example) appear as the Fourier coefficients of an analytic function called a modular form. Using the theory of modular forms on such a function can thus be used to get interesting arithmetic information about the original sequence we care about. It's a useful trick!

Josh Tenenbaum. (MIT—cognitive science)

Josh Tenenbaum is a professor in MIT's Department of Brain and Cognitive Science. In his research, he builds mathematical models of human and machine learning, reasoning, and perception. His interests also include neural networks, information theory, and statistical inference.