PROPOSED CLASSES FOR WEEK 5

Contents

Proposed New Classes

The Axiom of Choice. (∗∗∗–∗∗∗∗; Alfonso, Anti, Ari; 4 days)

The axiom of choice says that given any set of nonempty sets, there is a function which chooses one element from each set. In this class we'll try to answer three natural questions one might ask upon hearing this for the first time: Why is that an axiom? What is it good for? And why is it at all controversial?

In the first two days (∗∗∗), we'll explain why we need axioms to define what sets are and how they behave, give the basic axioms of Zermelo-Frankel set theory, and talk about several useful statements which are equivalent to the axiom of choice. On the third day (∗∗∗∗), we'll explain one of the counterintuitive consequences of the axiom of choice which make it controversial: the Banach-Tarski paradox, which states that a solid ball can be cut into five pieces which can then be rearranged to form two solid balls of the same size. Finally, on the fourth day (∗∗∗∗), we'll explain why Choice has to be an axiom: we can prove that we *can't* prove, or disprove, the axiom of choice from the other axioms of set theory.

Bayesian Statistics; or, Don't Listen to Anything They Teach You in School! (∗∗; Mira; 4 days)

Statistics is the science of analyzing data in the presence of uncertainty or with incomplete information. Since there is little in life that is certain, and information is always scarce, we humans can't go a day without doing some kind of statistics – in our routine cognitive functions, in science, in the political arena, etc.

When you start studying it in school, statistics at first looks a lot like math. Yet if you've ever taken a statistics class, you might have felt your inner mathematician getting increasingly annoyed – and with good reason! Many mathematicians shun statistics as not being a legitimate branch of mathematics. Among the general public too, statistics has a bad rep. Try to imagine the famous quote by Mark Twain ("There are three kinds of lies: lies, damned lies, and statistics") applied to mathematics. Impossible! So what's the difference?

The difference is that, unlike math, statistics (as it is usually done) appears to be just a scrapbook of arbitrary tests and procedures, with no basic underlying principle. This attitude makes it easy to come up with a variety of "lies", through either negligence or malice. But there are serious problems with statistics even when it is done carefully and honestly. For instance, if you look at what some of the standard statistical tests are actually measuring, you will find that most of them are not asking the questions that they're supposed to be answering. Instead they're measuring something related, but different and much more convoluted.

Why is statistics so screwed up? There are interesting historical and philosophical reasons, and we'll discuss them. The good news is: there is an alternative. There is a way of doing statistics which is really math, which doesn't substitute artificial questions for the questions you actually want answered, and which makes perfect sense every step of the way. We only have four days, so we may not get to a lot of the technical stuff. Still I hope to give you a sense of how Bayesian statistics works and to convince you that it's the way to go.

Prerequisites: Basic probability theory, calculus. If you don't know calculus, you should still be able to get something out of the course, as long you're willing to tune out for a few difficult stretches.

Homework: Optional.

Related to: Information and Coding Theory.

Bijective proofs. ($*$–$**$; Julian; 2–4 days)

There are many, many examples in combinatorics of collections which have the same size as different collections. For example, the number of labelled trees on n vertices is n^{n-2}, which is the same as the number of $(n-2)$-letter words with an n-letter alphabet. There is a way of connecting these two situations to give a bijective correspondence between words and trees, although it is not straightforward.

This is a hands-on group-work class where we will be attempting to construct bijections between different combinatorial structures. We will explore as many examples as time provides.

The proposed class on parking functions and priority queues is closely related to this proposal.

Prerequisites: None.

Big numbers! ($*$; Julian; 1 day)

Back by popular demand, a repeat of the class from Mathcamp 2006. We'll be learning a proof of van der Waeden's theorem: if we colour the whole numbers with c colours, there are monochromatic arithmetic progressions of arbitrary (finite) length.

Big warning: this class will involve big numbers. I mean, really huge numbers. Astonishingly collosal numbers. In fact, numbers so large that this class comes with a health warning: Do *not* attend this class if you are of a nervous disposition or are scared of large numbers!

Prerequisites: None.

Celtic Knots. ($**$; Noah; 2 days)

The first day you'll learn how to draw fancy celtic knots. This process only works because of a certain theorem in graph theory. We'll spend the second day proving this theorem and explaining why it applies to celtic knots.

Combinatorial Homotopy Theory. ($****$; M@; 4 days)

In this course we will discuss the topology of combinatorially defined spaces, especially simplicial complexes and posets. We will emphasize hands-on techniques for computing topological facts about them – in particular simplicial collapses and anticollapses, nerve lemmas, the Borsuk-Ulam theorem, and perhaps discrete Morse theory.

One question we will address by the end of the week is the following. If you partition the k-subsets of an n-set into $n-2k+1$ families, at least one of the families contains a disjoint pair. (Note that $k = 1$ is one version of the pigeonhole principle, so this can be viewed as a generalization.) This was proved by Lovász in 1979, and his method is topological. We will sketch his proof and talk a little bit about recent work in this area.

Prerequisites: Basic knowledge of point set topology, graph theory, and groups.

Homework: None.

Related to: Zoology of Polytopes, Point Set Topology, Geometric Graph Theory, Algebraic Topology, Hyperplane Arrangements.

Cubic Curves and Related Things. ($**$–$***$; Mark; 3–4 days)

If you went to the "Moonshine" talk, you heard an elliptic curve defined as the complex numbers modulo a lattice. From an entirely different point of view, an elliptic curve can be defined as a cubic curve without "bad" (singular) points; a cubic curve is just a curve given by a cubic polynomial equation $f(x, y) = 0$. If this class happens, we'll see that such a curve has a "natural" group structure—if you include a point at infinity (and we'll talk about how to do that). A theorem we use for this will also show the existence of a beautiful geometric configuration ("Pascal's hexagon").

Prerequisites: Some multivariable calculus and some basic group theory would be helpful.

Dan's Inner Monologue. ($**$–$***$; Dan; 4 days)

This will be a stream-of-consciousness class about whatever topic seems interesting at the time. It'll be like a discussion over a napkin at lunch—jumping around, doing different things all the time, but always aiming for something interesting, some cool result you might not expect. Each day, I'll take a prompt on some mathematics and start talking about it. The topic and pace will vary by day. We'll keep things interesting and unexpected for the entire week.

Disclaimer: this class is a big experiment (and, secretly, a way for me to avoid doing preparation work). It could turn out to be really interesting and amazing, but it could also be a total flop when I fail to come up with interesting things to talk about. You are welcome to come, but you are guaranteed absolutely nothing!

Prerequisites: None.

Homework: None.

Elliptic Curves. ($***$–$****$; David; 1–4 days)

One can define an elliptic curve in a number of ways. Perhaps the most elementary is as the set of points (x, y) with $y^2 + a_1xy + a + 3y = x^3 + a_2x^2 + a_4x + a_6$, where the a's are such that the resulting curve has no singularities. Behind this simple definition, however, lies a huge amount of amazing mathematics. The set of points on an elliptic curve, for example, form an abelian group, and one can use this group in all kinds of applications to number theory and cryptography.

In the first day of this class we will define elliptic curves and the group structure on them in a few different ways. We will consider the problem of finding points on the curve with rational coordinates and see how we can use the addition law to get new rational points using old ones. We will also consider elliptic curves over the complex numbers and see another characterization of them in that way. This class will be at a $***$ pace.

Starting on the second day we will speed up and start covering different topics in the theory of elliptic curves, possibly including the Birch and Swinnerton-Dyer conjecture (one of the Clay million dollar problems), modular forms, elliptic curves over finite fields, isogenies of elliptic curves and height functions on points of the elliptic curve.

Prerequisites: Group theory. After the first day, complex analysis and finite fields.

Euclid. ($*$; Mark; 1 day)
What did Euclid actually do, and what did the mathematics he wrote down actually look like? This class is intended to give you a feel for the answers.

Prerequisites: none

The Feeling of Power. ($*$; Anti; 1 day)
Suppose that you are marooned on a desert island without a calculator. You invent a clever scheme to escape which involves building a rocket ship out of coconut husks, but in order to calculate the right fuel ratio you end up needing to take a square root. Fortunately, you went to this class, so you know a way to take square roots manually to arbitrary decimal places, like long division, without using a calculator. In fact, you'll still be okay even if you need to take a cube root!

Fermat. ($*$–$**$; Mark; 1 day)
So you've heard of Fermat's Last Theorem, and maybe you know how to prove his Little Theorem, but do you know how many of his number-theoretic proofs survive? And did you know that although he was of an earlier generation than Newton and Leibniz, he had already developed some "calculus" methods? If you're curious about all this, do come to this class (if it happens).

Prerequisites: None.

The Field With One Element. ($****$; Anti; 2–3 days)
A field is defined to be a ring in which you can divide by anything nonzero *and* in which $1 \neq 0$. There is no argument; that is the definition. In particular, every field has at least two distinct elements, namely 0 and 1. In this class I'll try to convince you that there is nevertheless something which deserves to be called 'the field with one element', but it is *not* what you would get by naively dropping the requirement $1 \neq 0$ in the definition of a field. Instead, it is obtained by generalizing the notion of 'ring'. Amazingly, it turns out that *doing linear algebra over the field with one element is the same as doing combinatorics with finite sets.* This idea also has fascinating connections with quantum gravity... but we probably won't have time to talk about them.

Prerequisites: Linear algebra; some ring theory.

Related to: All abstract algebra; p-adic numbers; Projective geometry (the first 2 days of "Infinity and the Design of Experiments").

Fractional Graph Theory. ($***$; Ari; 2 days)
Who made the rule that graph invariants such as chromatic number must always be integers? Join the fractional resistance! Find out how to solve Mathcamp's scheduling problems by chopping up class periods!

Prerequisites: Linear Algebra, some graph theory.

Frobenius Algebras and 2D Topological Quantum Field Theories. ($****$; Ari; 4 days)
To a physicist, a topological quantum field theory (TQFT) is a background-free quantum theory with no local degrees of freedom. To a mathematician who knows no physics, a TQFT is simply a particularly nice functor from a topological category to an algebraic category. We will classify such functors and show that they are equivalent (in a categorical sense) to Frobenius algebras.

Prerequisites: Linear Algebra, Category Theory, some topology.

Fun with Fibonacci. (∗; Julian; 1 day)

We know about writing numbers in different bases. But what would base Fibonacci look like? And what is the equivalent of 'multiplying by 10' in this new context? We'll explore a fascinating listing of the integers in a Fibonacci-based table which has many curious and surprising properties, first introduced by Conway, and explored by the Queen Mary, University of London's Combinatorics Study Group.

Prerequisites: None.

Graphs on Surfaces. (∗∗–∗∗∗; Marisa, M@; 4 days)

Take five points (vertices) on the plane and connect them with curves (edges) so that each vertex is connected to every other vertex. Move the vertices around until they're in a position to give you the least possible number of edge crossings. Can you get the number down to zero, or is there always a crossing? The answer comes in the form of Kuratowski's Theorem. Would it help to draw them on the torus instead of the plane? What about on the projective plane?

Topological Graph Theory asks questions of embedding graphs on surfaces. This week, we'll talk about embeddings, minimum number of crossings, and colorings. We'll even move off of surfaces and into three dimensional space: John Conway proved that given any embedding of the complete graph K_6 in three-dimensional space, some pair of triangles forms a link. Recently, Robertson and Seymour completely characterized linklessly embeddable graphs—graphs which can be embedded in $\mathbb{R}^3$ with no pair of cycles linking. The answer turns out to be a 3-dimensional analogue of Kuratowski's Theorem.

Prerequisites: If you don't already know what a graph is, see the instructors.

Related to: Intro Graph Theory, Geometric Graph Theory, Point Set Topology, Algebraic Topology.

Hidden Markov Models. (∗∗∗; Mira; 4 days)

Say you're a computer scientist, trying to teach a machine to recognize human handwriting. Or maybe you're a climatologist, trying to make sense of weather patterns. Or a geneticist, trying to gather meaningful information from a strand of DNA.

In all cases, you observe a sequence of chaotic-looking data. You believe the sequence follows some relatively simple laws (English spelling; global climate trends; the genetic code), and you either want to learn more about these laws or to use what you already know about them (as in handwriting recognition). The problem is that these laws often apply not to the phenomena that you can actually observe, but to some "hidden states" of the system to which you have no direct access. For instance, the rules of English spelling are formulated in terms of letters of the alphabet – but what you (or the machine) actually observe are just squiggles on a page. If you can identify these squiggles as letters, you're done; if not, how are the rules of spelling going to help you?

Hidden Markov Models (HMMs) provide a powerful mathematical framework for uncovering the "hidden" structure of observations. In addition to the examples above, HMMs are used in economics, computer speech recognition, animal behavior, computer vision, and psychology. In this class, we'll develop the three most important HMM algorithms and apply them to some simple examples (hopefully using a computer, if I get my act together in time).

Homework: Optional.

Prerequisites: Basic probability.

Related to: Information and Coding Theory (especially Week 4).

How a Mathematician Reads a Newspaper. (∗; Julian; 1 day)

(A repeat of the class from Mathcamp 2006.)

Maths is full of hypotheses, theorems and logical arguments. What happens when we apply our thinking to a piece of text, say from a newspaper? You will need an open mind, and a willingness to explore a text logically! This session will be heavily based on work by Bandler and Grinder, who developed a model for analysing text based on Chomsky's theory of transformational grammar.

Prerequisites: None.

How to Win the Finite Group Game. (∗∗∗; Dave; 4 days)

> "Modular group of order 16!"
> "Oh yeah? Quasihedral group of order 16!"

You may have seen people playing the finite group game. In this game, we take turns naming finite groups. If I can name a group of smaller order than the one you named, or prove that the group you named is isomorphic to one that was named previously, you lose a point. We continue until you have so many negative points that you don't want to play anymore. Trash talking is encouraged.

Want to know how to win the finite group game? I will teach you.

Prerequisites: You definitely need to know what a group is. The more group theory you know, the better.

Infinity and Experiments. (∗; Mira; 3–4 days)

First, infinity. You may have learned about cardinals and ordinals, but did you know that you can actually visualize infinity geometrically? Projective geometry is a way of introducing infinity to the plane. Once you put in an appropriate set of coordinates, you can "rotate" the whole picture, so that what used to be infinity is now smack in the middle of your graph. In this way, you can actually *see infinity*!

Now, experiments. A few months ago, my friend Rebecca (yes, the one who was here two weeks ago) asked me to help her analyze her data. Rebecca does neuroimaging. She had scanned the brains of about 22 people, but the phenomenon she was looking for cannot be observed in any one person; to see it, you have to take the scans of about 15 people and average them. So Rebecca wanted to look at many different subsets of (about) 15 out of (about) 22 subjects. To avoid bias, she wanted to select these subsets so that each subject's scan was used equally often, and each pair of subjects appeared together equally often. She asked me how to do this, and after some thought I realized, "Aha! Just use the finite projective plane over the field of order 4."

What does the projective plane that people invented in order to incorporate infinity into geometry have to do with Rebecca's problem? Come and find out! We'll spend the first two days on the basics of projective geometry and the last day or two on its relationship to combinatorial design theory (the area of math that Rebecca's problem comes from).

Related to: Algebraic Geometry.

Prerequisites: None. In particular, you do *not* need to know what a field is.

Homework: None.

Kakeya Needle Problem. (∗–∗∗; Julian, Yvonne; 2 days)

In 1917, Japanese mathematician S. Kakeya proposed a problem: What is the smallest area through which a needle of length one can be rotated 360 degrees? Clearly a circle of diameter length – area $\pi/4$ – would do the trick: just pivot the needle about its centre. But we can do better than that: if we take an equilateral triangle of altitude 1 (and hence side length $2/\sqrt{3}$), we can slide the line segment up one side of the triangle, rotate it in the vertex, slide it down the next vertex, and so forth, until the needle is fully rotated. This triangle has area $1/\sqrt{3}$, which is less than $pi/4$. In fact, a shape called a deltoid, which looks like a triangle whose edges are curved inward, does the trick – and its area is just $\pi/8$ – less than $1/\sqrt{3}$, and just half the area of the circle. Can we do better? In 1928 the mathematician A. S. Besicovich came up with the unexpected answer to Kakeya's problem: no matter how small an area you choose, it's possible to rotate a needle of length one through that a shape with area. More recently, even this solution was improved! Using fractals, one can construct a *zero area* region in which to rotate the needle. We will discuss both these results in our class.

Prerequisites: High school geometry.

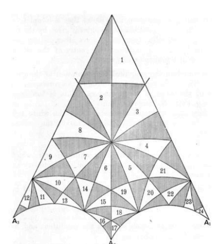
The Klein Quartic. (∗∗∗∗; Dave, Noah, Yvonne; 4 days)

Dave: Functions describe the world. In particular,

$$x^3y + y^3z + z^3x = 0$$

describes a fascinating highly symmetric complex surface.

Noah: Speaking of things with a lot of symmetry, I found this pretty picture for $PSL_2(\mathbb{F}_7)$ acting on a surface!



Yvonne: You call that symmetry? You know what has a lot of symmetry. Hyperbolic space, that's what. Taking quotients of hyperbolic space is the best way to find surfaces with lots of symmetry.

Student: What's going on? Why are you talking about 3 totally different subjects? Aren't those three surfaces totally different? One is algebraic, one topological, and one geometric!

Dave: Actually, they're all the same!

Student, post-class: I see the light.

Prerequisites: Group Theory required, we also recommend that you know at least one of Algebraic Geometry, Hyperbolic Geometry, and Low-Dimensional Topology.

Lie Algebras and Their Representations. (∗∗∗∗; Noah; 4 days)

This will be a rapid introduction to Lie algebra theory. The goal will be to understand the representation theory of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$.

The Magic of Determinants. (∗∗–∗∗∗; Mark; 3 days)

Determinants, which are numbers associated to square matrices, have many useful properties; for example, they are needed to give a general formula for the inverse of a matrix. Unfortunately, determinants are often defined in a very *ad hoc* way (using Laplace expansion) which may obscure what is really happening. This class would give a "better" theoretical framework as well as some geometric intuition, and I'd try to at least give an outline of the proofs of all the main computational properties of determinants, such as the Laplace expansion. No previous knowledge of determinants required.

Prerequisites: Matrix multiplication.

Mathematics and Education. (∗; Dan; 1–4 days)

"The Math Wars" have been raging—a disagreement between mathematicians and educators about how it's best to teach mathematics. "Standards" for mathematics education have been published, biased web pages have been launched (with such amusing names as `mathematicallycorrect.com` and `mathematicallysane.com`), and books and articles are everywhere. Meanwhile, teachers suffer from a lack of training and confusion over this new "fad," which they are certain will not stick.

We're going to learn about the different emphases in mathematics education today and study some of the (skimpy) research that has been done about issues in math education. We won't talk just about the "Math Wars;" we'll also talk about standardized testing and other issues in math education.

Prerequisites: None.

Homework: Required. (Reading non-math articles to discuss in class.)

Modern Algebraic Geometry. (∗∗∗∗; Dave; 4 days)

Want some more algebraic geometry? Then this is the class for you.

The topics that we covered in weeks 1 and 2 of algebraic geometry are often referred to as "classical algebraic geometry". The subject was completely revolutionized in the mid-twentieth century by Grothendieck's idea of sheaves and schemes, which combine topology and ring theory in an important and powerful way. Basically, the idea is this: we saw that there is a correspondence between rings and geometric objects, but what if we thought of the ring itself as a geometric object? We will study these objects in all of their awesomeness.

Prerequisites: It would be helpful to know a little topology, and of course to have been in Algebraic Geometry.

Multiplication is Easier than You Thought. (∗; Julian; 1 day)

How easy is it to multiply big numbers? How slow would it to be to multiply two 1000-digits numbers? With the use of large primes in cryptography, it becomes really important to be able to do these calculations quickly. In this class, we will give an overview of some techniques for fast computer-based multiplication.

Prerequisites: None.

Odd Numbers in Pascal's Triangle (By Discovery). ($*$; Alfonso; 1 day)

Which numbers in Pascal's triangle are odd? The answer is nice, but the proof is brutally nice. And you will figure it out yourselves.

Prerequisites: Do not come to this class if you already know the answer!

Related to: Combinatorics.

p-Adic Extensions. ($****$; David; 1–4 days)

In Fernando's class you gained some intuition for the p-adics and learned about their structure as a metric space and how to "do calculus" with them. In this class we will approach p-adics from a more algebraic standpoint and consider field extensions of $\mathbb{Q}_p$. Such field extensions are intimately related to field extensions of $\mathbb{Q}$, but are simpler from an algebraic standpoint because the "integral elements" have only one prime. We will discuss these field extensions from the standpoint of completion of global number fields as well as on their own. We will consider their Galois groups and see what we can learn about classifying extensions. And we will take the whole machinery of the p-adics into a more general setting.

If you want to take this class without having taken From Greece to Galois, talk to me.

Prerequisites: p-adics, From Greece to Galois.

Parking functions and priority queues. ($*$; Julian; 1 day)

How do you park a lot of cars in a one-way street? And what do you do when you have too many jobs to do? And what's the one got to do with the other? This was one part of my Ph.D. thesis.

Prerequisites: None, except a willingness to take part and have some fun.

Party! ($**$; Dave; 2 days)

When I go to parties, I have a question that I like to ask the other party-goers: "What's the minimum number of people who could attend this party that would guarantee the existence of a group of 3 people, all of whom know each other, or all of whom do not know each other?" This might be why I don't get invited to a lot of parties.

This problem is not very difficult, but its simplest generalizations have defied all reasonable approaches. In fact, the n-person party problem remains unsolved for all $n \geq 5$. Come learn about it.

Prerequisites: None.

Quadratic Imaginary Number Theory. ($***$; Noah; 4 days)

In this class we'll study the properties of rings like the Gaussian Integers (numbers of the form $a + bi$ for integer a and b). Interesting applications include which integers can be written as $x^2 + y^2$, which integers can be written as $x^3 + y^3$, why 25 and 27 are the only square and cube that are two apart, and why $x^2 - x + 41$ produces primes for any small integer value of x.

Quadratic Sines: Reading a Mathematical Paper. ($****$; Julian; 2 days)

Which angles are rational multiples of π and have sines which are of the form $a_0 + a_1\sqrt{n_1} + \cdots + a_k\sqrt{n_k}$ for some rational a_0, a_1, $\ldots$, a_k and whole numbers n_1, $\ldots$, n_k? Join us for a pleasant stroll through an application of Galois theory as we read a mathematical paper which answers this question.

Prerequisites: Galois theory (and you really need it!)

Quaternions: Everything You Always Wanted to Know (But Were Too Afraid to Ask). (∗∗∗∗; Nina; 3 days)

The quaternions ($\mathbb{H}$) can be thought of as a 4-dimensional generalization of the complex numbers. Unlike the complex numbers, however, they are not commutative.

We'll look at the quaternions from algebraic, geometric, and combinatorial perspectives.

- To start off with we'll be learning some of the algebraic structure of the quaternions, such as the rules for multiplication and how to embed them in the algebra of 2×2 complex matrices.
- Next we'll explore some geometric properties of the quaternions, such as how three-dimensional sphere, which is topologically the unit-quaternions, inherits a group structure from the quaternions. (Note: this makes it a rather special sphere, along with $S^1 \subset \mathbb{C}$ and $S^7 \subset \mathbb{O}$, the "octonians", an 8-dimensional generalization of $\mathbb{H}$.) We have a (2-fold covering) map from the 3-sphere onto the group of rotations in $\mathbb{R}^3$ (called $SO(3)$). This allows us to use the quaternions to ease calculations with rotations of $\mathbb{R}^3$, helping greatly in fields like computer graphics.
- Finally, we'll explore some combinatorial properties of the quaternions—looking at a special 4-dimensional polytope, called the 24-cell, which is the convex hull of a certain subgroup of the unit quaternions. We'll learn some techniques for visualizing 4-polytopes in general, and construct a ZOMEtool version of the 24-cell.

Prerequisites: Linear algebra.

Homework: Optional.

Related to: Groups and Symmetry, Linear Algebra, Zoology of Polytopes.

The Redfield-Polya theorem II—The Real Deal. (∗∗∗; Alfonso; 1–2 days)

If you took M@'s "Enumeration celebration", or Leigh's "Groups and actions", or my class in Week 5 at MC2006, you are probably familiar with the Burnside Lemma (aka the Class Lemma, aka Redfield-Polya theorem, aka many other things): If G is a group acting on a set X, then the number of orbits of the action is given by

$$\frac{1}{|G|} \sum_{g \in G} |Fix_X(g)| \ . \tag{1}$$

But that is not the full version of Redfield-Polya! There is more. For instance, how many different necklaces can you build with 20 beads out of very large supplies of red, green, and blue beads? With the help of (1), it will take you less than 5 minutes to calculate that the answer is 87230157. But what if I ask you to tell me how many such necklaces are there with R red beads, B blue beads, and G green beads, for each value of R, B, and G? You can still answer it in less than 5 minutes.

Prerequisites: You need to understand Equation (1) and its proof.

Related to: Enumeration Celebration, Groups and Symmetry, Groups and Actions, The Redfield Polya Theorem (MC2006).

Representations of the Symmetric Group. (∗∗∗∗; Noah; 4 days)
There is a classic description of the irreducible representations of the symmetric group in terms of combinatorial objects called tableaux. Typically it is difficult to see where these diagrams come from in terms of the representation theory. In this class we'd work through Vershik and Okunkov's paper on a new approach to the representation theory of S_n. In this approach tableaux pop out naturally.

Prerequisites: Representation Theory.

The Riemann Hypothesis. (∗∗∗; Mark; 2–3 days)
What is this problem that some people claim is the most important open question in pure mathematics, anyway? By the end of this class, you'll know, and you'll have seen a variety of other cool things, such as the probability that a "random" positive integer is not divisible by a perfect square (beyond 1) and the reason that 691/2730 is a useful and interesting number.

Prerequisites: Single-variable calculus, including infinite series.

Seventy-Three Proofs of the Pythagorean Theorem. (∗; Nina; 1–2 days)
We'll get through as many as we can—focusing on the prettiest, the awesomest, the most-surprising, the most general, and the most Xtreme. There will be lots of pretty pictures.

Prerequisites: None.

Homework: None.

Related to: What's So Special About Euclidean Geometry Anyways?

Space-filling Curves. (∗∗; Nina; 1–3 days)
The (magical) Cantor surjection theorem states that, "There is a continuous surjection from the Cantor set onto any other compact metric space." From this we can construct a space-filling curve: a continuous, surjective map from the interval $[0, 1] \subset \mathbb{R}$ onto any compact metric space.

This is all nice and good, but very abstract. If we let our compact metric space be $[0, 1]^2 \subset \mathbb{R}^2$, we can see some specific examples of these curves. In this class we'll look at some subset of: the Hilbert curve, the Sierpinksi curve, the Dragon curve, and the Peano curve, with special attention to their fractal-like iterative constructions.

Note: We will not prove the Cantor surjection theorem in this class, nor presume knowledge of the definition of compactness. We will work with specific examples of curves $f : [0, 1] \to [0, 1]^2$, outside the context of abstract metric spaces.

Prerequisites: Know the definition of the standard Cantor Set in $\mathbb{R}$, as seen in classes such as Real Analysis, Metric Spaces, or Measure Theory. A handout will be available if you want to take this class, but haven't seen the Cantor set before.

Related to: Real Analysis, Metric Spaces, Measure Theory.

Spectral Sequences. (∗∗∗∗; David; 1-4 days)

In algebraic topology we learned about long exact sequences, which gives us a tool for computing homology and cohomology in terms of subspaces: if $A \subset X$ then we have

$$\cdots \to H^n(A) \to H^n(X) \to H^n(X,A) \to H^{n+1}(A) \to H^{n+1}(X) \to H^{n+1}(X,A) \to \cdots.$$

When we work with higher homotopy groups, we can get a long exact sequence out of something called a fibration, which is a pair of maps $F \to E$ and $E \to B$ that satisfy the homotopy lifting property. But the relationship among the cohomology and homology theories of E, B and F is much more subtle: there is a spectral sequence relating them.

So what is a spectral sequence? It is an incredible algebraic object, with infinitely many "pages", each of which has a group at every point (p,q), with maps between them. In this class we will define spectral sequences and see how they arise, use them to compute various cohomology groups, and discuss how they show up outside of algebraic topology.

If you came to algebraic topology, this will class will be a faster continuation of similar themes. If you didn't, and you want to be kicked in the face by some hard math, feel free to come as well.

Prerequisites: Algebraic Topology.

Tantalizing, Terrific, and even Toroidal Tilings. (∗∗; Leigh; 3–4 days)

There's a lot of mathematics connected with the study of tilings. The plan in this course is to dip into some of the cool ideas about this subject. In particular, we'll look the stucture of aperiodic (i.e. not wallpaper) tilings (e.g. the famous Penrose tilings). Then we'll play with polyominoes (tiling by domino-like shapes) problems. We might take a quick peek with tilings of surfaces like tori. Or you might like to see (some of) the 14 proofs that tiling a rectangle by smaller rectangles which each have a integer side-length requires that the larger rectangle also has an integer side-length. It's week 5, so you should let me know if there's a particular aspect of tiling you'd like to learn more about.

Prerequisites: None.

Transformations of the Plane. (∗–∗∗; Mark; 1–3 days)

We'll look at rotations, reflections, and their compositions from a geometric point of view. If the class runs longer, we'll start thinking of the plane as the set of complex numbers, and look at "linear fractional transformations", which send z to $\frac{az+b}{cz+d}$, and their interesting properties.

Prerequisites: Basic linear algebra would help.

Unsolvability of the Quintic Polynomial Equation. (∗∗∗∗; Alfonso; 4 days)

Once we have proven the Fundamental Theorem of Galois Theory and once we have used it to construct a regular 17-gon, we can move to the harder proof that a general quintic polynomial equation is not solvable using radicals. The result is quite elegant:

> "Let $p(x) \in \mathbb{Q}[x]$ be an irreducible polynomial, and let L be its splitting field. Then the equation $p(x) = 0$ can be solved with radicals if and only if the Galois group $\mathrm{Gal}(L:Q)$ is solvable."

Of course, I need to tell you what a solvable group is, but let's leave that for class.

This class may be lecture or Moore Method. You may vote for one or both of them in the survey (if you would be interested in either format), as at most one of them will make it to the schedule.

Prerequisites: You need to understand the statement of the fundamental theorem of Galois theory. If you think you know enough abstract algebra but have never seen this theorem, talk to me.

Related to: From Greece to Galois.

What are the Real Numbers? (∗∗–∗∗∗; Anti; 2–4 days)
So you think you know what the real numbers are? Are they Cauchy sequences? Dedekind cuts? How about almost linear functions on the natural numbers? Special games born on day ω? Points of the locale presented by rational intervals? Elements of a terminal coalgebra? This class will be a whirlwind tour and comparison of as many definitions of the real numbers as we can possibly get though.

Related to: Real Analysis.

Proposed Continuation Classes

More Combinatorial Game Theory. (∗∗; Alfonso; 2–4 days)
On week 1 we studied the main tools to analyze *impartial* combinatorial games. In this class we will study *partisan* games, in which we allow the two players to have *different* legal moves. Examples are chess, go, or checkers. Examples that we may actually study include hackenbush, domineering, amazons, or clobber. Surreal numbers may make a guest apperance.

Prerequisites: None. There will be small portions of the class when we will make reference to week 1, but most of the class will be understandable without it.

Related to: Combinatorial Game Theory.

More Elliptic Functions, Modular Functions, . . . (∗∗∗∗; Mark; 1–4 days)
A continuation of the class from week 4 (see last week's blurb). Since I have no clear idea how far we'll get this week, it's hard to predict how far a continuation could go. . . but if we don't do it this week, I'd definitely want to prove the formula about sums of powers of divisors that was featured in this week's blurb.

Prerequisites: Elliptic Functions

More Problem Solving. (∗∗–∗∗∗∗; Bogdan; 4 days)

(1) Intro Problem Solving—the Integer and Fraction Part of a Real Number (∗∗)
(2) Olympiad Problem Solving—Complex Numbers in Geometry (∗∗∗∗)
(3) Olympiad Problem Solving—Calculus (∗∗∗∗)
(4) Olympiad Problem Solving—Abstract Algebra (∗∗∗∗)

More Representation Theory. (∗∗∗∗; Mark; 2–4 days)
A continuation of the class from week 3, with more actual computations of character tables as well as the missing proof that the irreducible characters form a basis of the space of class functions. If time permits, we'll talk about a way of getting from a representation of any subgroup to a representation of the original group, and about the elegant and useful Frobenius Reciprocity Theorem, which arises in this context.

Prerequisites: Representation Theory, or willingness to take quite a few results on faith.