CLASS DESCRIPTIONS—WEEK 2, MATHCAMP 2007

Contents

Important Notes

- Not all classes continuing from Week 1 are listed below, only those that are specifically inviting new students to enter.
- Many continuing classes are changing their meeting time and/or room; check the schedule.
- 9am classes *only* now last 60 minutes, although they don't meet on Tuesday (when we have assembly instead). The second and third classes of the day now begin at 10:10am and 11:10am.

New Classes at 9am

Intro to Graph Theory. (∗; Sarah, Leigh; week 1 of 1)

For the purposes of this class, a tree is a graph and a plot of $y = x^2$ is not. Many branches of mathematics seem to like to take normal words and give them new meanings, but graph theory is probably the branch that does this the most enthusiastically. For instance, it makes perfect sense for a graph theorist to assert that all stars are trees, but that there are many trees that are not stars. Kites on the other hand are not trees but, fortunately, it is impossible for a kite to be stuck in a tree. After this class, the previous two sentences should seem like perfectly reasonable statements. We will start with the definition of a graph and then go on to explore trees, Hamiltonian and Eulerian circuits, planarity, matchings and coloring. Along the way we will learn how many people you need at a party to guarantee that there will either be three mutual friends or three mutual strangers, whether it is possible for the JCs to close all of the lounges in the Leonard, Sturtevant, Taylor complex using each main staircase exactly once if they start and end at the office, and why we might need more colors to properly color a map if our world were a donut rather than a sphere.

Prerequisites: None.

Homework: Optional (but fun!).

Required for: The Rubik's cube and other puzzles (week 4; recommended)

Related to: Basic combinatorics (Week 1); Geometric graph theory (Week 2)

Quadratic Reciprocity. ($**$–$***$; Mark; Wed, Thu)

Let p and q be distinct primes. What, if anything, is the relation between the answers to the following two questions?

- Question 1: "Is q a square modulo p?"
- Question 2: "Is p a square modulo q?"

In this class you'll find out; the relation is an important and surprising result which took Gauss a year to prove, and for which he eventually gave six different proofs. If all goes well you'll get to see one particularly nice proof, part of which is due to one of Gauss's best students, Eisenstein. And next time someone asks you whether 101 is a square mod 9973, you'll be able to answer a lot more quickly, whether you're a Cyborg or a Luddite.

Prerequisites: Some intro number theory, specifically Fermat's little theorem.

Homework: Recommended.

Related to: Intro number theory (Weeks 1–2); Multiplicative functions (Week 2); Unique factorization (Weeks 2–3)

Multiplicative Functions. ($**$–$***$; Mark; Fri, Sat)

Many number-theoretic functions, including the Euler phi-function (which is not a prerequisite) and the number of divisors, share the property that

$$f(mn) = f(m)f(n) \text{ whenever } gcd(m, n) = 1.$$

There is an interesting operation, related to multiplication of series, on the set of such *multiplicative* functions, which gives that set a nice structure. If this sounds like fun and/or if you'd like to be able to find the sum of the 10th powers of all the divisors of 686,000,000,000 (by hand) in a minute or so, do come to this class.

Prerequisites: No fear of summation notation, and a bit of number theory experience. The first week of intro number theory should be plenty.

Homework: Optional, or possibly none.

Related to: Intro number theory (Weeks 1–2); Quadratic reciprocity (Week 2); Unique factorization (Weeks 2–3)

Intro to Cryptography. ($***$; Marisa; week 1 of 3)

Cryptography has been around in some form for as long as there have been secrets. We've come a long way since the field's inception. One of the earliest encryption systems went like this: take a reasonably short message and an inconspicuous messenger, shave the messenger's head, and tattoo "invade on July 16th" to the messenger's head. Once the hair grows back, the messenger travels to the recipient, who shaves his or her head to retrieve the message. Said system had a few obvious drawbacks.

The advent of computing in the 20th century has forced a dramatic and rapid change in the techniques, and cryptography is now a dynamic area of math research. In this class, we're going to study public key and symmetric key encryption using a smidgen of group theory and number theory. We'll start with an important question (that is surprisingly hard to answer): what is a good definition of a *secure* cryptosystem? And the natural follow-up, of course, will be whether or not we can build one. I think we can!

Prerequisites: You need to know what a cyclic group is.

Homework: Recommended.

Related to: Zero-knowledge proofs (Week 3); Information and Coding Theory (Weeks 1–2, 4)

The Mathematical Grammar of Quantum Mechanics. (∗∗∗–∗∗∗∗; *Allan Adams*; week 1 of 1)

The world is so deeply weird, so staggeringly strange, that people studying it one hundred years ago invented a whole new language to describe it's lemony ways: Quantum Mechanics. After introducing a couple of the most counterintuitive aspects of reality in Tuesday's colloquium, this class will develop the mathematical grammar of quantum mechanics (vector spaces, Hilbert spaces, operators and matrices, a dash of group theory, the path integral and more) and use it to prove some astonishing theorems about the real world, from a derivation of the Principle of Least Action to the classical proof, and quantum mechanical violation, of Bell's inequality (which translates into English as, roughly, "the number of tall bald men in any room must be greater than or equal to zero"—hardly the sort of thing you'd expect to be false!).

Weds and Thurs. The Mathematical Grammar of QM and the Proof, and Violation, of Bell's Inequality (∗∗∗). *Prerequisites:* Largely self-contained, but some familiarity with vectors and matrices will help.

Fri and Sat. Feynman's Path Integral, the Method of Stationary Phase, and the Calculus of Variations (∗∗∗∗). *Prerequisites*: Basic calculus needed, some basic physics helpful (e.g. $F = ma$, momentum, energy).

Related to: Quantum Mechanics through Pictures (Thurs–Sat)

New Classes at 10:10am

Understanding Infinity. (∗; Dan, Ryan; week 1 of 1)

It seems crazy that you could talk about infinity in any meaningful way. But, in fact, the study of infinite sets is very precise, and absolutely gorgeous. We can come up with ways to distinguish different "sizes" of infinity and study how they interact. In this class, we'll see how to compare the sizes of the natural numbers, the rational numbers, and the reals. (Two of them are "equal" in size, but one is larger!) Then we'll move on to even larger sizes of infinity—those you can't even visualize—and we'll see how to compare them.

Towards the end of the class, we'll extend our work to ordinals and cardinals, which are different ways of enlarging the natural numbers to encompass infinite sets.

Prerequisites: None.

Homework: Required, but very interesting!

Groups and Symmetry. (∗∗; Leigh; week 2 of 2)

The emphasis for Week 2 of G&S is really groups. Before we put symmetry aside (until the next week), we'll discuss direct products and how they give us the *full* symmetry group of (most of) the platonic solids (rotations and reflections as opposed to rotations only). Then we'll move on to the statements and proofs of some lovely classical results in group theory. For instance:

(1) The order of an element divides the order of the group. But...
(2) Having n divide $|G|$ is no guarantee of the existence of an element of order n...unless n happens to be prime.
(3) If p is an odd prime, then any group of order $2p$ is cyclic or dihedral.

Along the way, we'll define equivalence classes, conjugations (not of verbs), partitions, cosets, and a few other things. We might scratch the surface of group actions.

Prerequisites: Week 1 of G&S or familiarity with the following:

(1) The groups $\mathbb{Z}_n$, D_n, S_n, A_n
(2) The words *order, abelian, cyclic, subgroup, generator*

Homework: Optional.

Required for: Group actions (Week 3); Rubik's cube and other puzzles (Week 4); other classes (see Week 1 description)

Point Set Topology 2—all about compactness. ($***$; Nina; week 2 of 2)

This is an entry point to "Point Set Topology 1" which was taught this past week.

Just a few reasons why compact spaces are cool:

(1) Not every continuous function has a maximum or a minimum. How do we guarantee it will? The answer is compactness.
(2) Is the interval $(0, 1)$ closed? The answer is sometimes yes and sometimes no. (See if you know why.) Isn't there a more morally decent concept of closed? The answer is compactness.
(3) Can you think of a sequence which doesn't converge? $1, 2, 3, 4, 5, \ldots$. Can you think of a *bounded* sequence which doesn't converge? $0, 1, 0, 1, \ldots$. Can you think of a bounded sequence *without* a convergent subsequence? I think not. Why? The answer is compactness.

Given any reasonable topological space, we can make it compact by adding a single point. How? Come find out.

Homework: Recommended.

Prerequisites: "Point Set Topology 1", or knowing the definition of topology, open sets, closed sets, limit points, continuous maps, homeomorphism, connectivity.

Related to: Low-dimensional topology (Week 2); Orientable and non-orientable spaces (Week 2); Metric spaces (Week 3); Algebraic topology (Week 4)

New Classes at 11:10am

Orientable and Non-Orientable Spaces. ($*$; *Erica Flapan*; Tues, Wed)

What would it be like to live in a space where you could go for a walk and come back as your mirror image? Such a space is said to be non-orientable. We consider various types of 2 and 3-dimensional non-orientable spaces. We say a surface only has one side if there is some way for a 3-dimensional creature to go for a walk along the surface and come back to its original position but on the opposite side of the surface. For example, a Möbius strip in $\mathbb{R}^3$ is non-orientable and has only one side. We will give examples of non-orientable surfaces with two sides, and orientable surfaces with only one side. Then we discuss when the properties of non-orientable and one-sided are equivalent.

Prerequisites: None.

Homework: Strongly recommended.

Related to: Low-dimensional topology (Week 2); Point-Set Topology (Weeks 1–2); The Hairy Ball Theorem (Week 3)

Two Simple Statistical Ideas and One Beautiful Complicated One. ($*$; *Rebecca Saxe*; Wed)

In science, a standard scenario is: you have two collections of something, and you want to know whether the two are the same or different on some dimension. How can you tell? Statistics! The key thing about a statistic, for a scientist, is not the equation that computes it, but the intuition to understand which one to use and when. This class will cover three basic intuitions—two simple ones and one beautiful complicated one—that will come in very handy in the rest of your lives.

Prerequisites: None.

Homework: None.

Linear Algebra. ($**$; Marisa, Mira; week 2 of 2)

Linear algebra continues! If you know about linear transformations, linear independence, basis, and dimension, but don't know about orthogonal projections, diagonalization, and eigenvectors of symmetric matrices (all related, all very powerful stuff), come join us.

Prerequisites: Linear Algebra Week 1, or equivalent.

Homework: Required, daily.

Required for: Lots of things; see Week 1 description.

Unique Factorization [By Discovery]. ($**$; Noah, David; week 1 of 2)

It may seem obvious to you that every integer can be factored uniquely into primes. After all, you learned how to factor numbers like 45 and 64 years ago. Yet if this fact were true in a more general setting Fermat's Last Theorem would have been proved 370 years ago. The quest to understand the breakdown of unique factorization has spurred the development of vast areas of mathematics.

For example, if one works with numbers of the form $a + b\sqrt{-5}$, where a and b are integers, then one can factor 6 in two distinct ways: $6 = 2 \cdot 3$ and $6 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$.

We will learn how to formulate the statement of unique factorization precisely, prove it in some cases where it does hold, and discover examples where it breaks down. The exploration will be done primarily by you: Noah and David will be available to guide you and steer you in the right direction, but you will be the ones to actually prove the results both inside and outside of class.

Prerequisites: Some number theory.

Homework: Required.

Related to: Intro Number Theory (Weeks 1–2); Multiplicative functions (Week 2); Quadratic reciprocity (Week 2)

Quantum Mechanics through Pictures. ($*$–$***$; *Theo Johnson-Freyd*; Thurs, Fri, Sat)

My hope in this course is to provide some intuition for quantum mechanics. The course is structured not as a single coherent story, but as three independent (but related!) lectures: come to any subset.

Day 1. Quantum Mechanics in Pictures. ($*$)

A graphical notation system, developed by Penrose and others, generalizes Feynman diagrams for use in almost any theory, and allows almost everything to be said via "picture proof". I'll present the axioms of quantum mechanics, including why Scotty can "beam" himself from the Enterprise to the Excelsior, but why he cannot clone himself into two perfect copies.

Day 2. A Finite Toy Model for Quantum Mechanics. (∗∗)

Heisenberg famously observed that it's impossible to simultaneously know with complete certainty the position and momentum of an electron: if you measure one, you lose the information about the other. So let's take that as our axiom of an extremely simplified "toy" physics: the maximum knowledge you can have about a system is equal to the amount you don't know. We'll get some (but not all!) quantum mechanical effects, including some of the most bizarre.

Day 3. What is electric charge? (∗∗∗)

Why does electric charge come in discrete units? I'll explain what physicists mean by "symmetry", and how electric charge is really just another kind of momentum. For this talk, it will be helpful, but certainly not required, for you to have seen some linear algebra and/or some group theory (homomorphisms, group actions).

Prerequisites: None.

Homework: None.

Related to: Quantum mechanics (Week 2)

Crash Course in Mathematical Logic. (∗∗∗; Anti; Tue)

We spend a lot of time at Mathcamp proving things, but just what is a proof? Can we formalize the notion of proof, and prove theorems *about* proofs? You may have heard, for instance, that some statements like the "Continuum Hypothesis" are *provably unprovable.* Such "unprovability" theorems belong to the realm of mathematical logic.

This one-day class will be an introduction to first-order mathematical logic, which (in its various forms) is sufficient to formalize nearly all mathematical practice. We'll discuss proof systems and truth in models and the resulting dichotomy between syntax and semantics, which is an important thing for every mathematician to be familiar with. Later this week, Kenny will discuss some provably unprovable statements and their philosophical consequences.

Prerequisites: None.

Required for: Philosophy in mathematics (Week 2)

Homework: Recommended.

Philosophy in Mathematics: The Role and Nature of Proof. (∗∗∗; Kenny Easwaran; Thu, Fri, Sat)

How can philosophical thinking help us in doing mathematics?

In the 19th century, as the standards of rigor in mathematics increased, an important philosophical question started to rise along with it, about just what the role of proof is in mathematics, and what should count as a proof. This question was at first addressed primarily by philosophers, establishing the field of mathematical logic. But by 1930, this work had led to serious mathematical results that shattered pre-existing confidence in the idea that mathematical truth just is the same thing as provability.

In this class I will outline the proof of Gödel's two Incompleteness Theorems (as well as his Completeness Theorem), show how they arose from the philosophical work of the period, and show some consequences that they do (and don't!) have for mathematics at large.

Prerequisites: Mathematical logic.

Homework: Recommended.

Complex Analysis. ($***$; Mark; week 1 of 2)

We'll look at the spectacular changes that occur when you are doing calculus and you allow the variable x (now to be called $z = x + iy$) to take on complex values. As we'll see, functions that are "differentiable" in a region of the complex plane have many surprising properties. For example, they always have power series expansions, and if you know what the function is everywhere on some closed curve, then you can deduce what it is anywhere inside the closed curve! This material, much of which was developed by Cauchy (remind me to tell you, probably in week 2, why the mathematician named his dog "Cauchy"...), is not only quite beautiful, but it has important applications both in- and outside mathematics.

For example, complex analysis was used in proving Dirichlet's famous theorem about primes in an arithmetic progression: If a and b are positive integers with $gcd(a, b) = 1$, then the sequence

$$a, a + b, a + 2b, a + 3b, \ldots$$

contains infinitely many primes. This was, as far as I know, the first major result in analytic number theory, the branch of number theory which uses complex analysis as an essential tool and which includes such key questions as the Riemann hypothesis. On the other hand, complex analysis can also be used to solve applied problems involving heat conduction, electrostatic potential, and fluid flow.

Prerequisites: Multivariable calculus.

Required for: Elliptic functions (Week 4), Klein's quartic (Week 5).

Homework: Recommended.

Geometric Graph Theory. ($****$; M@; week 1 of 1)

Topological graph theory is the study of graphs, such as planar graphs, defined by topological constraints, such as edges not crossing. Geometric graph theory is the study of graphs defined by geometric constraints, such as edges having prescribed lengths. Here are two motivating problems which we will discuss in this course.

(1) How many colors are needed to color the plane so that no two points at unit distance are the same color? This is a very famous unsolved problem which has been around for about 50 years, but it seems to be extremely hard. All that is known is that at least 4 colors are necessary, but 7 are sufficient. The fact that the answer to this innocent sounding question is known to be $4, 5, 6$, or 7, but nobody seems to be able to say more than that, is frustrating and hilarious.

(2) Is it true that every bounded set of points in $\mathbb{R}^d$ can be partitioned into $d + 1$ sets of strictly smaller diameter? This was known as Borsuk's Conjecture for 60 years. It is true for $d = 2, 3$ and it seems that it was generally thought to be true for all d. Then Jeff Kahn and Gil Kalai destroyed all hopes of proving it with a spectacular counterexample in $\mathbb{R}^{1325}$. Their paper is only 3 pages long, and even contains an awesome literary quote. We will discuss their construction and related ideas in detail. (By the way, this problem is still wide open when $d = 4$, so there is still plenty of work to be done here too.)

It may not be obvious what these problems have to do with graph theory, and with each other, but you will find out in this class.

Prerequisites: None.

Homework: Optional.

Related to: Intro graph theory (Week 2)

New Classes at 1:10pm

Intro to Number Theory. ($**$; David; week 2 of 2)

The Cyborg Number Theory class will continue. In the first week we covered divisibility and gcds, Euclid's algorithm, modular arithmetic and modular exponentiation, Fermat's little theorem and primes and primality testing. The second week will include more exploration of primes (including implementing primality testing and exploring the distributions of primes using SAGE), factoring and the Chinese remainder theorem. If you would like to join the class, or if you took Noah's version and would enjoy coming for another week, talk to me.

Low-Dimensional Topology. ($**$; Dan, Noah; week 1 of 1)

Suppose one day you wake up to find yourself in "surfaceland." You are two dimensional, and can only move in two directions. You begin to wonder what your world would look like from the outside. Is it a sphere like the surface of the earth? Might you be on the surface of a doughnut? Maybe something more exotic like a Möbius strip? In this class you'll learn all the possible surface worlds that you might be on, and how to tell them apart without leaving the surface. We'll spend one day defining surfaces, finding examples, and describing surfaces by gluing together triangles. The second day we'll discuss how to distinguish different surfaces using an invariant called "Euler characteristic." The third day we'll classify all surfaces.

For the last two days, we're going to explore an application of Euler characteristic to sensor networks. Suppose you have a bunch of autonomous sensors, each of which has a range in which they can scan, and the sensors each detect targets in their range. You want to know how many targets there are, but some targets might be detected by multiple sensors. So how do you avoid overcounting? The answer turns out to be an amazing application of Euler characteristic, and we'll see how to solve this seemingly complex problem with a surprisingly simple and beautiful method called "integrating with respect to Euler characteristic." (No calculus required.)

Prerequisites: None.

Homework: Optional.

Related to: Point-set topology (Weeks 1–2); Orientable and non-orientable spaces (Week 2); The Hairy Ball Theorem (Week 3); Musical orbifolds (Week 4)

Colloquia (4-5 pm)

Pad Thai with Electrons. (*Allan Adams*; Tuesday)

The world is so deeply weird, so staggeringly strange, that people studying it one hundred years ago invented a whole new language to describe it's lemony ways: Quantum Mechanics. In this colloquium, we'll introduce a couple of the most counterintuitive aspects of reality.

Required for: The Mathematical Grammar of Quantum Mechanics, Wed–Sat.

Symmetries of Flexible Molecules. (*Erica Flapan*; Wednesday)

Mirror image symmetry plays an important role in predicting the behavior of molecules. Recently, knots and links and other non-planar molecules have been synthesized which are large enough that they no longer have the rigidity that is characteristic of small molecules. In order to understand the symmetries of such molecules we need to understand their deformations. In this talk we will discuss how topology can be used to help us analyze the symmetries of flexible molecules.

The Mysterious Numbers of Professor Hensel. (*Fernando Gouvea*; Thursday)
What if we started from our normal way of writing down whole numbers, like 12 or 345, and then decided to allow the numbers to grow infinite in the *wrong* direction? Can we make sense of ...373737373737373737? Do such numbers behave properly? Are they good for anything?

How to Use Mathematical Models to Understand the Mind. (*Rebecca Saxe*; Friday)
Blue is more similar to violet than to orange, and Hawaii is more similar to Cuba than to Iowa (at least in some respects). But how do we all know this? Somehow, our minds calculate a metric of similarity. In psychology, there was a famous, decades long fight about this calculation. Then a couple of years ago, that fight was conclusively resolved by an elegant mathematical model—created by Mathcamp's very own Josh Tenenbaum. Come hear how.

Digestif (2–2:30 Saturday)

My other favorite mathematical magic trick. (Mira; Saturday)
Last summer, I gave a Saturday colloquium called "My favorite mathematical magic trick"—but now I have another one. Come and find out what it is!

Visitor Bios

Allan Adams. (MIT)
Allan Adams studies quantum versions of algebraic and differential geometry that play a fundamental role in string theory, and uses these tools to explore the fate of tachyons, moose diagrams, and other puzzles involving black holes and spacetime singularities. Dr. Adams believes that everyone should understand quantum mechanics, which is as beautiful and strange as it is true, and looks forward to discussing it at Mathcamp.

Kenny Easwaran. (Berkeley)
Kenny Easwaran has just finished his fifth year in the PhD program in Logic and the Methodology of Science at Berkeley. He started out aiming to work in mathematical logic, and especially set theory, but over time has moved towards philosophy, in particular the philosophy of mathematics, and also the application of probability to epistemology (the study of knowledge). He has been a mentor at Mathcamp in '02, '03, '04, and '06, and was a camper back in '98. His hair is not currently red or blue.

Erica Flapan. (Pomona College)
Erica Flapan is interested in topology and in using it to understand the shape of our universe. Her book *When Topology Meets Chemistry* shows how topology plays a role in analyzing molecular symmetries. Most recently, she has been doing research on how knots and links can occur in DNA as a result of recombination. At Mathcamp, she would like to expose students to topology and some of its exciting applications.

Fernando Gouvea. (Colby College)
Fernando Gouvea started his mathematical career as a number theorist, but is slowly morphing into a historian of mathematics. He is the author (or co-author) of four (or five, depending on how you count) books, and he was co-editor of a fifth (or sixth). He is also the editor of FOCUS, the news magazine of the Mathematical Association of America (MAA), and of MAA Reviews, the MAA's online book review service. He has a wide range of interests, and is fond of describing

himself as “Christian, orthodox, Brazilian, American, conservative, husband, father, member of a Lutheran church, Sunday School teacher, choir director, editor, author, dog owner, bibliophile, science fiction fan, wine geek, adoptive Mainer, historian wannabe, and the proud possessor of a graying scraggly beard.”

Theo Johnson-Freyd. (Berkeley)

Theo Johnson-Freyd recently graduated from Stanford University with a major in Mathematics, and also dabbles in dance and theoretical physics. His ambition is to simultaneously revolutionize calculus education and string theory, by successfully marrying nonstandard analysis and noncommutative geometry, about which he knows very little. He also enjoys baking, ballroom, and long walks by the lake.

Rebecca Saxe. (MIT)

Rebecca Saxe studies the neural and psychological basis of social cognition, in the Brain and Cognitive Science department at MIT. In this work, she asks: Do we have special mechanisms, designed by evolution for recognising and/or reasoning about other minds, or does social cognition share the general-purpose machinery we use for recognising chairs and reasoning about falling apples? How and why does the human brain succeed so easily where computers and logicians fail?