Class proposals—Week 5, Mathcamp 2006

This handout contains descriptions of all the classes we could teach in Week 5. Please, use the page at the end to vote for the classes you would like to see in Week 5. Deadline to vote is Thursday at sign–in. Do not eat this handout!

Unless otherwise indicated, Week 5 classes have no homework.

Contents

What is *your* axiom of choice? (Colloquium performed by the Staff)

The Axiom of Choice, the Well-Ordering Principle, Zorn's Lemma, and many other statements are often invoked in a mysterious ritual to tame the beasts of the infinite. However, like all black magic, there are occasional unfortunate side-effects, in this case taking the form of unmeasurable sets and associated consequences.

In this lively production, the staff will illustrate many of the connections between these important statements so that you can be more clear on what is going on. Each of these statements has a slightly different flavor, and is useful in different mathematical contexts, which we will demonstrate.

Alfonso's classes

The Burnside lemma and the Redfield–Polya theorem (∗∗–∗∗∗, Alfonso, 2 days)

How many ways are there to colour the faces of a cube with red, blue and yellow? (Two cubes will be the same if one is a rotation of the other). What about an icosahedron with up to seven different colours? How many different necklaces can you make with 20 pebbles if you have white and black pebbles? If you want to be able to solve any of these problems in less than a minute with just pencil and paper, come learn the oh–so–powerful, but surprisingly not well enough known Redfield–Polyak theorem.

Prerequisites: Knowing what a group is.

Related to: groups and actions, combinatorics.

Constructibility of polygons ($****$, Alfonso, 4 days)

Do you know how to construct a regular hexagon with straightedge and compass? How about a regular heptagon? If they taught you how to do it in school, they lied to you! It is impossible to construct a regular 7-, 9-, or 11-gon with straightedge and compass, but it is possible to construct a regular 17-gon. In this class we will find out exactly which regular polygons are constructible, we will derive why, and we will (hopefully) construct a 17-gon. The final result (related to the Galois correspondence!) is one of the most beautiful pieces of math that I know!

Prerequisites: complex numbers, groups and fields.

The feeling of power ($*$, Alfonso, 1 day)

Back in my day, before there were computers, calculators, or telephones, we all learned in school easy and fast algorithms to take square roots and even cube roots with pencil and paper, just like long division. Come learn how! (And yes, I am stealing Anti's gig).

Prerequisites: None

Lorentz transformations; or, Relativistic kinematics ($***$, Alfonso, 1–2 days)

You may have, at some point, encountered an expression like

$$x' = \frac{x - vt}{\sqrt{1 - (v/c)^2}}$$

This is the expression for the Lorentz transformation, from which we can derive relativistic kinematics, and conclude that distances contract and time dilates.

With just linear algebra, this seemingly scary formula can be entirely derived from Einstein's two relativity principles:

(1) The speed of light is constant.
(2) There is no privileged point or privileged direction in space.

So we'll do it!

Prerequisites: linear algebra, tolerance for calculations.

More VMTS ($***$, Anti and Alfonso, 4 days)

(See Anti's classes).

Anti's classes

The Anti-class ($***$, Anti, 2 days)

If successful, this class will be the opposite of an ordinary class: you will come out of it knowing *less* than you did when you went in.

You mean you want to know what math we'll actually do in this class? You're no fun at all. (-: The first day we'll discuss Gödel's Incompleteness Theorem, which states that in any mathematical system, there are always statements which can be neither proven nor disproven. Then the second day I'll use this result to argue that most mathematicians' notion of 'truth' is fundamentally flawed, and that perhaps everyone should become a constructivist. (For even more radical points of view, go to Kenny's and my class on "How much infinity is there?")

Related to: "Classical logic".

How much infinity is there? ($***$, Anti and Kenny, 1 day)

(See Kenny's classes)

Math of Juggling (∗∗, Anti, 2–3 days)

Perhaps it's not surprising that juggling, like solving a Rubik's cube, is a very mathematical activity, since both are intimately dependent on patterns. And just like in mathematics, noticing a pattern allows us to not only describe existing types of juggling, but invent new kinds! In this class, we'll learn the 'siteswap' notation for simple juggling patterns and use it to invent a 4-ball juggling pattern in which balls cross back and forth from hand to hand. We'll prove some theorems about how to find and count simple juggling patterns, and if there's time, we'll move on to more complicated types of patterns.

Antirequisite: If you are already familiar with the 'siteswap' notation, you probably should not take this class.

More VMTS (∗∗∗, Anti and Alfonso, 4 days)

By the end of week 4 of "Vectors and matrices, tensors and spinors", we will have defined and talked about rotations and spinors in three dimensions. But the world we live in is not just three-dimensional; it also has another dimension called 'time', which behaves somewhat differently from the first three dimensions. In Week 5 of the class, if it happens, we'll talk about the geometry of four-dimensional 'space-time'.

It turns out that 'rotations' in space-time include the usual sort of 3-dimensional rotation, but also transformations called 'boosts' which change your point of view on the world from that of a stationary observer to that of a moving observer. This is the essence of Einstein's theory of Special Relativity, which implies, among other things, that when you move very quickly, objects become shorter and time flows more slowly. We'll derive these facts, discuss some associated 'paradoxes', and (if there's time) investigate the appropriate notion of 'spinor' in 4-dimensional space-time.

Homework: optional.

Prerequisites: Weeks 1–4 of "Vectors and matrices, tensors and spinors".

Related to: linear algebra, non–Euclidean geometry.

N-category theory (∗–∗∗∗∗∗, Anti, 1 day (Staff Research Colloquium Series))

You may have heard that 'category theory' is a scary subject which studies highly abstract and incomprehensible objects. At the beginning of this talk, I'll try to convince you that actually, you already know what categories are, and you've been using them your whole life. For example, sets are categories. The real numbers are a category. Any group is a category. Topologies and metric spaces are also categories. Even the most fundamental notions in mathematics—namely, 'true' and 'false'—are categories.

Most of these 'friendly' categories are very 'low-dimensional': we call them n-categories, where n is -1, 0, or at most $\frac{1}{2}$. When most mathematicians think of category theory, they think of 1-categories, which are a bit more difficult, although nowadays they are used nearly everywhere in mathematics. Fewer mathematicians are conversant with 2-categories, although they are gaining in prominence, while 3-categories are still considered pretty exotic.

As yet, there is no consensus even on the *definition* of n-category for $n > 3$. And yet, we already know a lot about them: the field of n-categories is filled with theorems in search of a definition. As we go on, I'll describe what an n-category 'should' be, in intuitive terms, then discuss some of these hypotheses and why they 'should' be true. We define an 'n-person' to be a person who gets a headache when thinking about $(n+1)$-categories. By the end of this talk, I hope to convince you that there is no need for anyone to be an n-person for any finite value of n.

Prerequisites: hah!

Related to: category theory, algebraic topology.

The Yoneda lemma (∗∗∗∗, Anti, 4 days)

In the depths of abstract nonsense, far beneath the waves of meaningful mathematics, lurks the mysterious Yoneda Lemma, its tentacles stretching far and wide to control the activities of mathematicians everywhere. This week, join me on a danger-filled expedition to smoke this beast from its lair. We'll need special tanks of air and pressure suits, so that we are not suffocated or squished into jelly by the tons of abstraction pressing down from above us, and powerful weapons to protect ourselves from the monsters found in the waters of category theory, such as representable functors (both covariant and contravariant) and the ever-enigmatic natural transformations.

Prerequisites: You must sign a waiver absolving Mathcamp of responsibility if you drown in abstraction or are eaten by a functor.

Ari's classes

The book stacking problem (∗, Ari, 1 day)

Suppose you have a large number of identical rectangular books. You can easily stack one on the a table so that it hangs a bit over the edge. You can also stack two so that the overhang is slightly larger. Given enough books, can you stack them such that the top book is no longer over the table at all? To answer this question, we'll use a bit of basic physics and a famous mathematical fact that you may already have encountered. (I'm not telling you what it is, because that would ruin the surprise.) If I can locate a supply of appropriate objects, we may attempt a physical demonstration.

Prerequisites: none.

Fractional graph theory (∗∗∗, Ari, 2 days)

Why is it that graph invariants (such as chromatic number, maximum clique, and maximum matching) are always integers? End the tyranny! Join the fractional resistance!

Prerequisites: basic linear algebra.

Related to: graph theory.

Noncommutative ring theory (∗∗∗∗, Ari, 4 days)

Ring theory is a central part of abstract algebra with applications in topology, geometry, logic, and many other fields. In this class, we will examine a few fundamental properties of rings, including various types of finiteness (such as ascending and descending chain conditions), and then prove the powerful Artin-Wedderburn Theorem, which fully explains the structure of semisimple rings. This has strong connections to representation theory, which we will explore if we have time.

[Some people will tell you that commutative rings are easier to understand than their noncommutative counterparts. Needless to say, these people are wrong. There is nothing more confusing and unintuitive than commutative multiplication.]

Prerequisites: abstract algebra, linear algebra.

Homework: daily.

Uniform convergence (∗∗∗, Ari, 4 days)

We have some notion of what it means for a sequence of real numbers to converge. What does it mean for a sequence of functions to converge? We will find an appropriate definition, prove some stuff about it, and construct some really bizarre consequences: an everywhere continuous, nowhere differentiable function, and a continuous surjection from a closed interval to the unit square.

Prerequisites: none.

D.A.'s classes

Grassmannians and combinatorics (∗∗∗, D.A., 1–2 days)

Just as points on a circle can correspond to lines through the origin, a Grassmannian parametrizes subspaces of a vector space. Grassmannians can be described fairly easily in terms of matrices, but they're also related to the combinatorics of symmetric functions in a surprising way. We'll spend some time playing with Grassmannians, taking them apart, and looking for combinatorial facts about them.

Prerequisites: linear algebra.

Related to: symmetric functions, projective geometry, commutative algebra.

Pascal's theorem and cubic curves ($**$–$***$, D.A., 2 days)

Pappus' theorem concerns a certain configuration of nine lines and nine points. Pascal's theorem is about two sets of three lines, a circle and a line, and nine points. We'll discuss proofs of these theorems, and learn something a little deeper about the role of the number 3.

Prerequisites: some projective geometry.

Related to: projective geometry, number theory

Dan's classes

A topological smorgasbord ($**$, Dan, 4 days)

We'll use and study topology in all kinds of strange and exciting ways. Along the way, we'll encounter unusual spaces such as various dimensions of projective space, higher-dimensional spheres, and higher-dimensional tori. Then we'll get to describe the *space* of all rotations of $\mathbb{R}^3$! Afterwards we'll use what we've developed to either classify all surfaces, or describe the Hopf fibration of the 3-sphere, depending on what you all want to do.

This class will be a wild tour in all kinds of crazy topology and should give you a great intuition for just what you can do when you think about strange ways to bend space.

Prerequisites: complex numbers and trigonometry, especially $e^{i\theta} = \cos\theta + i\sin\theta$.

Differential topology ($****$, Dan, 4 days)

We'll take calculus and vastly generalize it to the world of *manifolds*. Manifolds are topological spaces that have just barely enough structure on them to understand what tangent vectors are, what derivatives ought to be (in terms of tangent vectors!), and what it means for a function to be *smooth*. We'll work with these manifolds, discovering all sorts of strange things about what you can do with them.

Prerequisites: Calculus, preferably (but not necessarily) multivariable; also, ideally, some topology or metric spaces, but you can get away without that if you're willing to ignore some details.

More theoretical CS ($***$, Dan, 4 days)

We'll keep on studying crazy complexity classes. I'll let you guys choose some of what we want to do, but possible destinations include some of the amazing things you can do with only logarithmic space, or exploring the polynomial hierarchy, an infinite ascending chain of complexity classes that divide up the space between P and PSPACE.

Prerequisites: Theoretical computer science.

Quantum computation ($***$, Dan, 4 days)

What can you do with a quantum computer that you can't do with a normal computer (at least efficiently)? We'll study this question and look at the complexity of questions that quantum computers can answer efficiently.

Prerequisites: At least one week of theoretical CS; more would be helpful.

Ellen's classes

Buffon's needle problem; or, How NOT to approximate π ($*$, Ellen, 1–2 days)

Buffon's Needle Problem is a lovely little probabilistic question about tossing needles around. We'll look at this problem, a cute little proof of it, and how this problem can be used to approximate π (badly).

Prerequisites: Enough knowledge of probability to know what expectation is.

Related to: probability

Lattices ($**$, Ellen, 1–4 days).

Lattices are at the intersection of metric geometry, linear algebra, group theory, and just about every other fun kind of math you can think of. In this class we will (of course) learn what lattices are. Additional topics may include (but are not necessarily limited to) lattice applications to number theory, sphere packings, and efficient methods to produce a good basis for a lattice.

Prerequisites: linear algebra.

Related to: groups and actions, linear algebra, polytopes.

Scissors congruence ($*$, Ellen, 1–2 days)

Say I have a triangle of area 1 and a square of area 1. I want to cut the triangle and move the pieces so that they now form the square. Can I do it? What if I have a tetrahedron of volume 1 and a cube of volume 1? In this lecture, we will show why the answer to the first is yes and to the second is no.

Prerequisites: Probably should know what a vector space is.

Type 2 constants; or, Why forcing symmetry can only hurt you ($**$, Ellen, 2 days)

I have a convex set which is centrally symmetric, which essentially means that it is as symmetric as you could possibly want. I want to approximate this set by a projection of a polytope (a polytope is a higher dimensional analogue of a polygon). It seems reasonable that my polytope should also be centrally symmetric, no? Well, in mathematics often what seems reasonable is false, as is the case here. In this course we will look at type 2 constants (constants associated to any centrally symmetric convex body) and use them to show that it's a really bad idea to project centrally symmetric polytopes.

Prerequisites: Probably should know what a vector space is, and enough about probability to know what expectation is.

Related to: Ellen's colloquium, polytopes.

Julian's classes

Big numbers! ($*$, Julian, 1 day)

Some numbers in math(s) are big. Some are very big. And some are just scarily big. This session is not for anyone who is of a weak disposition! In 1927, van der Waerden came up with a really beautiful result showing that total disorder is impossible. We will learn a beautiful proof of his result due to Graham and Rothschild, but you are warned: the numbers involved are HUGE!

Prerequisites: No fear of really big numbers.

Generalised Riemann integration ($****$, Julian, 3 days)

The first fundamental theorem of calculus essentially says that if you differentiate a function F to get f, and then integrate f, you get back to F (modulo a constant). Unfortunately, there are some functions whose derivatives can't be integrated (can you find an example?). In this class, we will learn about a remarkably simple generalisation of the Riemann integral which makes this fundamental theorem always true. (It is also capable of integrating more functions than even the Lebesgue integral!)

Prerequisites: $\epsilon - \delta$ stuff, definition of (regular) Riemann integral.

How a mathematician reads a newspaper ($*$, Julian, 1 day)

Math(s) is full of hypotheses, theorems and logical arguments. What happens when we apply our thinking to a piece of text, say from a newspaper? You will need an open mind, and a willingness to explore a text logically! This session will be heavily based on work by Bandler and Grinder, who developed a model for analysing text based on Chomsky's theory of transformational grammar.

Prerequisites: An open mind.

Kenny's classes

How much infinity is there? (∗∗∗, Anti and Kenny, 1 day)

> "I protest against the use of infinite magnitude as something accomplished, which is never permissible in mathematics. Infinity is merely a figure of speech, the true meaning being a limit." – Carl Friedrich Gauss

Do you believe that infinite sets exist? Nowadays, few mathematicians object to infinite sets, but until quite recently (the last century or so) many mathematicians subscribed to the point of view articulated above by Gauss. In fact, as you probably know by now, it is generally accepted that there are many different sizes of infinity. In practice, most mathematicians stick with a relatively few sizes of infinity, but some set theorists believe in the existence of infinite sets far larger than anything most of us can comprehend. Do you agree? Come watch Anti and Kenny argue about whether large cardinals exist, whether any infinite sets exist—and even whether large finite numbers exist! Is there a finite number that deserves to be called $10^{10^{100}}$? (Keep in mind that the number of atoms in the entire universe is estimated to be about 10^{80}.) Maybe. . . but maybe not! Come find out why.

Probabilistic proofs (∗∗∗, Kenny, 1 day)
In the early '80s, Michael Rabin gave a quick method for deciding whether a number was prime. However, his method only gave certainty if the number was composite - if it's prime, then it only gives probabilistic certainty, not absolute certainty. So mathematicians have rejected this method of proof. I will discuss the controversy here, and try to argue that in fact probabilistic proof may be just as good as standard proof, and may even be better in some cases.

Proof and conjecture in mathematics (∗∗∗, Kenny, 1 day)
Which is more important in mathematics - proving things or making conjectures? How do conjectures lead to the development of a new subject area? How do conjectures compare to scientific theories? And how do they compare to unprovable statements? And what do computer proofs and probabilistic proofs do to the role of proof in mathematics?

Proof and Purity (∗, Kenny and Moon, 1 day)
We'll look at some of the elements that go into successful mathematical explanations, and see whether they're "merely" aesthetic, or essential to mathematics.

Is it preferable to have pure arguments? (Say, a purely number–theoretic proof of a number–theoretic theorem?) Atle Selberg won the Fields Medal in 1950, and Paul Erdős first attracted notoriety, for providing such a proof for the Prime Number Theorem, which had already been proven in 1896 using complex analysis, by Hadamard and de la Vallée Poussin. Despite the extreme importance attached to these new, "elementary" proofs, the old ones are generally considered much more elegant and illuminating.

What is meant by calling a proof "beautiful", "elegant", "elementary", "illuminating", etc.? And which is more important when these goals conflict? If the best proofs of some number theoretic facts are essentially complex analytic, does complex analysis become a part of number theory? Or, conversely, is this exactly what we need to establish complex analysis as an independently interesting field of math?

Set theory as a foundation for mathematics (∗∗∗∗, Kenny, 2 days)
All of mathematics can be done in set theory. And in fact, we seem to need something like this to make sure that real analysis is consistent and all. But because of Godel, we see that we can't use something simpler to prove that this system is consistent. So we are either stuck with there being "no fact of the matter" about statements like CH, or with using methods other than proof to increase our mathematical knowledge. But this is what we needed to do to get our axioms in the first place! So how and why do we do it?

X-Treme division by zero! (in probability) (∗∗∗∗, Kenny, 2 days)
What happens when we learn something that had probability zero? Didn't probability zero mean impossible anyway? I'll tell you why it doesn't, and discuss a few of the approaches that people have suggested for dealing with this, so that we can avoid dividing by zero. (Rule 4 applies to everyone...) This is also my thesis research, so I'm glad to hear objections and counterexamples! But it may also be somewhat rambling...

Marisa's classes

Field extensions and Greek geometry (∗∗∗∗, Marisa, 4 days)
The goal of this week will be to explore doubling the cube, trisecting an angle and squaring the circle, all of which we can prove to be impossible using field extensions. The math will be somewhat heavy-duty: we'll discuss fields, extensions, degree, homomorphisms, irreducible polynomials, minimal polynomials, closure, and splitting fields.
Prerequisites: Basic groups, rings, fields.
Homework: recommended.

More topological graph theory (∗∗∗, Marisa, 4 days)
This is a continuation of Weeks 3 and 4. We will prove some neat theorems on the projective plane and the torus. See me if you want to join.
Prerequisites: The first two weeks of "Topological graph theory".

Mark's classes

Determinants (∗∗, Mark, 2–3 days)
Determinants come up all the time! We'll see what they mean geometrically, and how to define them algebraically and prove that they have the properties we want (such as $\det(AB) = \det A \det B$).
Prerequisites: A bit of linear algebra; at a minimum, matrix multiplication.

Green's theorem (∗∗∗, Mark, 1 day)
What got cut at the end of the multivariable calculus class: a beautiful theorem which relates double integrals to line integrals, and which can be used to measure the area of a region by traveling around the perimeter with a suitable device.
Prerequisites: Some multivariable integration. I would probably review line integrals, though.

Integration by parts and the Wallis product (∗∗, Mark, 1–2 days)
Come learn one of the two most important techniques of integration and, as a bonus, see a proof of the famous identity

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots$$

stated by John Wallis in the 17^{th} century.
Prerequisites: Basic knowledge of integration.

More commutative algebra and algebraic geometry (∗∗∗–∗∗∗∗, Mark, 2–4 days)
A continuation of the class from Week 4.
Prerequisites: Depends on how far we get in Week 4.

Multiplicative functions (∗∗–∗∗∗, Mark, 2–3 days)
Many number-theoretic functions, including the Euler ϕ-function and the number of divisors, share the property that

$$f(mn) = f(m)f(n) \qquad \text{whenever } \gcd(m,n) = 1.$$

There is an interesting operation (motivated by L-series) on the set of such *multiplicative* functions, which gives that set a nice structure. If this sounds attractive and/or if you'd like to be able to find the sum of the 10th powers of all of the divisors of $686,000,000,000$ in about one minute, do consider taking this class.

Prerequisites: No fear of summation notation, and a bit of number theory experience (not necessarily as much as the "Intro to number theory" class).

Primitive roots ($**$–$***$, Mark, 2 days)

X-treme number wheels! Seriously, the question is for which n there exist numbers b such that *all* numbers modulo n that are relatively prime to n can be written as powers of b.

Prerequisites: Basic number theory (or at least Euler's theorem).

The quadratic reciprocity law ($**$–$***$, Mark and Noah, 2–3 days)

Let p and q be distinct primes. What is the relation between the following questions?

"Is q a square modulo p?"
"Is p a square modulo q?"

Come learn the answer, an important and surprising result in number theory for which Gauss gave 6 different proofs.

Prerequisites: Fermat's Little Theorem.

Taylor and Laurent series ($***$, Mark, 2–3 days)

Come and find out why analytic (complex differentiable) functions are always given by a power series, and how to express a function in a series on a "ring" around z_0 even if the function "blows up" at z_0.

Prerequisites: Complex analysis or basic familiarity with complex numbers and willingness to accept one result (the Cauchy Integral Formula) from complex analysis.

M@'s classes

Algebraic topology ($****$, M@ and Sam, 2 hours a day for 4 days)

This class will be both a continuation of the Moore Method Topology class and a new entry point for people who have had a little point set topology or metric space theory before. We will explore the most fundamental algebraic invariants of a topological space X - the fundamental group $\pi_1(X)$ and homology groups $H_n(X)$. The fundamental group measures algebraically how much one can deform loops in X (and is the subject of the famous Poincaré conjecture), and the homology groups basically allow you to detect higher dimensional "holes" in a space, using only linear algebra.

As applications of this machinery, we will cover some of the following topics, depending on interest of students and instructors: classification of surfaces (compact 2-manifolds, e.g. sphere, torus, Klein bottle, the projective plane, ...), distinguishing between topological m-dimensional and n-dimensional Euclidean space when $m \neq n$, Brouwer and Lefshetz fixed point theorems, ...

To really get a feel for this material in such a short time, we'd like to run this class for 2 hours a day; probably the format will be something like one hour a day of lecture, and one hour of recitation where students work on problems and present their work on the board, as in Moore Method.

Related to: "MM Topology", "Groups and group actions", linear algebra, "Quantum topology", "7MP: The Poincaré conjecture".

Homework: Possibly some, but a willingness to work on problem sets in class daily and to present solutions on the board is strongly recommended.

Tilings and wallpaper groups ($*$–$**$, M@, 2 days)

We will briefly survey the vast and beautiful field of of mathematical tilings. We may talk about, among other things: polyominos, tangrams, MC Escher and the Alhambra, questions of decidability, tilings in the hyperbolic plane, symmetries of tilings, and connections between aperiodicity and chemistry. This is one of the few areas of modern mathematics where amateurs have made notable progress in the past 30 years, yet many fundamental problems remain unresolved.

Among the many open questions: Is there an aperiodic tile? Which convex pentagons tile the plane? Is there a polyomino of odd order? Is there an algorithm to decide if a given convex shape can actually tile the plane? Is there an all-purpose tile? To learn what these questions mean, come to the class.

In classifying the tilings of the plane, it's useful to consider the various types of symmetry that can arise. It turns out there are exactly 17 wallpaper groups, and one goal of this class is to sketch a proof of this.

Related to: "Groups and group actions", "Hyperbolic geometry", "Combinatorics".

Miljan's classes

The Kronecker approximation theorem ($*$–$**$, Miljan, 1–2 days)

The Kronecker Approximation Theorem is one of those mathematical theorems which assert, roughly, that "what is not impossible will happen sometimes, however improbable it may be". It states that if θ is irrational, then the set of points $\{n\theta\}$ is dense in the interval $(0, 1)$, where $\{x\} = x - \lfloor x \rfloor$ is the decimal part of x. We shall prove the theorem and apply it to simple but entertaining problems.

Prerequisites: none.

Related to: "Intro to number theory", "Rational numbers: in space!"

The Minkowski convex body theorem ($*$–$**$, Miljan, 1–2 days)

Around 1891, Hermann Minkowski discovered his Convex Body Theorem, opening up a new field of study which he called geometry of numbers. The theorem states that any convex region symmetrical about the origin and of area greater than 4 contains an integral lattice point other than the origin. We are going to prove the theorem and give some cool applications such as proving that every natural number can be represented as a sum of four squares of integers!

Prerequisites: modular arithmetic.

Related to: "Intro to number theory", "Rational numbers: in space!"

Mira's classes

Bayesian statistics; or, Don't listen to anything they teach you in school! ($**$, Mira, 4 days)

Statistics is the science of analyzing data in the presence of uncertainty or with incomplete information. Since there is little in the world that is certain, and information is always scarce, we humans can't go a day without doing some kind of statistics – in our routine cognitive functions, in science, in the political arena, etc.

When you start studying it in school, statistics at first looks a lot like math. Yet if you've ever taken a statistics class, you might have felt your inner mathematician getting increasingly disgruntled and annoyed – and with good reason! Many mathematicians shun statistics as not being a legitimate branch of mathematics. Among the general public too, statistics has a bad rep. Try to imagine the famous quote by Mark Twain ("There are three kinds of lies: lies, damned lies, and statistics") applied to mathematics. Impossible! So what's the difference?

The difference is that, unlike math, statistics (as it is usually done) appears to be just a scrapbook of arbitrary tests and procedures, with no basic underlying principle. This attitude makes it easy to come up with a variety of "lies", through either negligence or malice. But there are serious problems with statistics even when it is done carefully and honestly. For instance, if you look at what some of the standard statistical tests are actually measuring, you will find that most of them are not asking the questions that they're supposed to be answering. Instead they're measuring something related, but different and much more convoluted.

Why is statistics so screwed up? There are interesting historical and philosophical reasons, and we'll discuss them. The good news is: there is an alternative. There is a way of doing statistics which is really math, which doesn't substitute artificial questions for the questions you actually want answered, and which makes perfect sense every step of the way. We only have four days, so we may not get to a lot of the technical stuff. Still I hope to give you a sense of how Bayesian statistics works and to convince you that it's the way to go.

Prerequisites: Basic probability theory, calculus. If you don't have calculus, you should still be able to get something out of the course, as long you're willing to tune out for a few difficult stretches.

Homework: Optional.

Related to: "Information theory", Josh Tenenbaum's "Probability and the mind" (but this course will be less mind and more probability.)

Hidden Markov models (∗∗∗, Mira, 4 days)

Say you're a computer scientist, trying to teach a machine to recognize human handwriting. Or maybe you're a climatologist, trying to make sense of weather patterns. Or a geneticist, trying to gather meaningful information from a strand of DNA.

In all cases, you observe a sequence of chaotic-looking data. You believe the sequence follows some relatively simple laws (English spelling; global climate trends; the genetic code), and you either want to learn more about these laws or to use what you already know about them (as in handwriting recognition). The problem is that these laws often apply not to the phenomena that you can actually observe, but to some "hidden states" of the system to which you have no direct access. For instance, the rules of English spelling are formulated in terms of letters of the alphabet – but what you (or the machine) actually observe are just squiggles on a page. If you can identify these squiggles as letters, you're done; if not, how are the rules of spelling going to help you?

Hidden Markov Models (HMMs) provide a powerful mathematical framework for uncovering the "hidden" structure of observations. In addition to the examples above, HMMs are used in economics, computer speech recognition, animal behavior, computer vision, and psychology. In this class, we'll develop the three most important HMM algorithms and apply them to some simple examples (hopefully using a computer, if I get my act together in time). Information theorists: Gibbs inequality turns out to be a key ingredient here as well!

Homework: Optional

Math and genetics (∗∗, Mira, 1 day (Staff Research Colloquium Series))

This is a colloquium on Mira's research!

More information theory (∗∗∗, Mira, 4 days)

You want more? I got more! I'm not sure exactly what though . . . I plan to spend Week 4 (when I'm away from Mathcamp) thinking about what's best to do next.

We'll definitely spend the first day *actually* proving the Noisy Channel Coding Theorem (as opposed to my unsuccessful attempt to go through the proof in 15 minutes on the last day of the class in Week 2). After that, we'll probably look at error-correcting codes: the way people try to get around noisy channels in practice.

Homework: Optional (since it's Week 5), but highly recommended.

Prerequisites: The first two weeks of "Information theory".

Noah's classes

Building knot complements (∗∗, Noah, 1 day)

One of the most important breakthroughs in knot theory in the '80s was that if you take the complement of a knot (that is, take the part of space that isn't the knot), then typically it has an interesting geometric structure. In this course we'll state and discuss the main result about knot complements, but we'll spend most of the time figuring out how to make paper models of these strange spaces.

Prerequisites: Knot theory.

Continued fractions... in space! (∗∗–∗∗∗, Noah, 2–4 days)
Or, "Fractions... in space!, continued". More fun proving results about fractions by using plane geometry. This time we'll show how to actually compute the good approximations to a number, and prove some nifty results about continued fractions.
Prerequisites: "Rational numbers... in space!"

Introduction to Hopf algebras (∗∗∗∗, Noah, 2–3 days).
Hopf algebras are like groups, except not commutative. If you thought groups were already non-commutative, well, these ones are even more non-commutative. The goal of the class will be to work up to the definition of quantum 2×2 matrices.
Prerequisites: Group representation theory, tensor products.

The quadratic reciprocity law
(See Mark's classes).

Quantum computing (∗∗∗, Noah, 2–3 days)
Alfonso has finally finished the perfect conflict-free schedule and has encrypted it and stored it on his computer. In order to break the code, Dan's minions would need to be able to find the factors of a large number with no small factors. Dan's minions have all been taking his TCS class and think they should try to use a computer to do the factoring. However, these ordinary minions would be useless, since no one knows an efficient way to factor using an ordinary computer. But if Dan had *quantum* minions, then this would be no problem at all! We'll discuss what a quantum computer is, what makes them different from ordinary computers, whether all quantum computers are the same, and (if we have 3 hours) how to factor large numbers.
Prerequisites: "Quantum mechanics" OR "Vectors and matrices, tensors and spinors".
Related to: "Theoretical computer science".

Sam's classes

Algebraic topology (∗∗∗∗, M@ and Sam, 2 hours a day for 4 days)
(See M@'s classes).

Visitor classes

Arts and crafts in spherical, euclidean, and hyperbolic space (∗, Yvonne, 1 day)
Do you own a model of hyperbolic space yet? Why not? Make and bring home your personal own copy of hyperbolic space. Learn about geometry, the shape of space, donuts, and double donuts in the process. All (Euclidean 3-space embeddable) construction materials will be provided.
Prerequisites: A desire to visualize funky shapes

Calculus without calculus (∗∗, Brenda, 2 days)
If you've ever taken a calculus class, you've almost certainly seen certain types of problems. Without a doubt, you've learned how to find the equations of tangents to curves. In all likelihood, you've learned how to maximize an area with a given shape and perimeter, and minimize the perimeter of a region of given shape and area. You've probably also seen the ol' qqswim–and–run" problem of finding the route that minimizes the amount of time it takes to swim to the shore, and then run to a certain place on land. As it turns out, all of these problems — and more — can be solved without evaluating a single limit or derivative. In this brief course, we'll exploit the geometric properties of diagrams, and we'll explore some powerful inequalities that let us solve optimization problems swiftly and elegantly. Come to "Calculus without calculus", and learn the math your calculus teacher doesn't want you to know. Previous knowledge of calculus isn't necessary,

and in fact, those who have taken a calculus class before may find themselves distressed to learn just how much calculus they've used unnecessarily in the past.

Prerequisites: none.

Coarse geometry of groups (∗∗, Yvonne, 2 days)

"Coarse" might be the opposite of "refined" in English, but not in math, and especially not in group theory. We often think of curvature as a property of objects like surfaces with uncountably many points. But groups – countable ones and even finite ones! – have curvature too. To get there, we'll develop the background of and discuss so–called "coarse" geometric structures on groups. Our tools: triangles, graphs, homophones, and the groups themselves.

Prerequisites: group theory.

Inversive geometry (∗∗, Brenda, 3 days)

Here's a problem that's simple to state, but difficult to attack using the standard tools of Euclidean geometry: given three circles, two of which are tangent to one another, construct all of the circles that are tangent to all three. In this class, we'll look at a simple but powerful transformation called inversion, which effectively amounts to "turning a circle inside out" by sending the points inside the circle outside, and vice versa. Using this method, we can turn the above question about three circles into an easier one involving a circle and two lines. Over three classes, we'll look at a handful of surprising and elegant results in Euclidean geometry that can be proven using this tool.

Kakeya needle problem (∗, Yvonne, 1 day)

In 1917, the Japanese mathematician S. Kakeya proposed a problem: What is the smallest area through which a needle of length one can be rotated 360 degrees? Clearly a circle of diameter length one — area pi/4 — would do the trick: just pivot the needle about its centre. But we can do better than that: if we take an equilateral triangle of altitude one (and hence side length $2/\sqrt{3}$), we can slide the line segment up one side of the triangle, rotate it at the vertex, slide it down to the next vertex, and so forth, until the needle is fully rotated. This triangle has area less than $\pi/4$. In fact, a shape called a deltoid, which looks like a triangle whose edges are curved inward, does the trick - and its area is just $\pi/8$ – just half the area of the circle. Can we do better? In 1928 the mathematician A.S. Besicovich came up with the unexpected answer to Kakeya's problem: no matter how small an area you choose, it's possible to rotate a needle of length one through a shape with that area. In this class we'll explore this result, and look at some related questions.

Prerequisites: none.

Proof and purity (∗, Kenny and Moon, 1 day)

(See Kenny's classes.)

Set! (∗, Yvonne, 1–2 days)

The sets of SET relate to the surface of a donut. We will discover why and then generalize the game of SET.

Prerequisites: none

Visualizing groups (∗∗, Yvonne, 2 days)

In the 19th century, mathematician Arthur Cayley discovered a way to capture all the information in a group by using a graph now called the Cayley graph. These graphs turn out to reveal the intrinsic geometry of a group. Using homophones(!) and symmetries as a starting point, we will explore the geometry of groups and see fractals along the way.

Prerequisites: definition of a group.

Related to: group theory, graph theory.

Brief visitor bios

Brenda Fine

Brenda is a former geometry student at the University of British Columbia and former Mathcamp mentor, currently working an education gig in Vancouver, BC. An evangelical mathematician who strives to spread the gospel of the projective plane far and wide, Brenda divides her free time among her pottery studio, the ski hills, and the bike routes of Vancouver.

Yvonne Lai (UC Davis)

Ever since lectures at Mathcamp 96, Yvonne has been convinced of the beauty of geometry and topology. Now she works on geometric group theory and hyperbolic geometry as a grad student at UC Davis and is the founder and co-director of the Davis Math Circle.

She was a JC from 2000-2002 and a mentor in 2004. She likes action movies, her Mathcamp sarong, and the idea that the positively-curved Poincaré Dodecahedral Space might be the shape of our universe (even though her research area would have her believe that negative curvature is the best thing since sliced bread).