Class descriptions—Week 4, Mathcamp 2006

Contents

Attention! 9am classes will happen at 1:10pm on Tuesday this week! 1:10pm classes do not meet on Tuesday this week.

Classes beginning in Week 4

Domino tilings of Aztec diamonds (∗∗, D.A., 1:10pm, We–Th)

Matchings of graphs – and tilings of related regions – have long interested chemists, since they can be used to model patterns of bonding in molecules. The basic question here is, "How many ways can you tile a given finite region?" In the 1980s, it was discovered that for one such region, called the Aztec diamond, this question has an amazingly beautiful answer. The enumeration was proved around 1990 in at least four independent ways; several new proofs have appeared since then, drawing on techniques as diverse as elementary combinatorics, representation theory, and statistical mechanics. In this two-day course, I'll show you the answer to the question, one way to prove it, and perhaps some techniques for attacking similar problems.

Prerequisites: none.

Homework: none.

Related to: graph theory (weeks 2–4), "Combinatorics" (Week 1).

Generating functions and Feynman diagrams (∗∗∗, Anti and Holly, 11am, week 1 of 1)

Feynman diagrams are funny pictures coming from Quantum Field Theory that describe how subatomic particles interact, in which particle paths are represented by straight lines or squiggly lines. For example, we have the following two pictures.

The diagram on the left describes an electron and positron annihilating to make a photon, which then decays into an electron-positron pair, while the diagram on the right describes an electron and positron passing each other and exchanging a photon. These are just two of an infinite collection of diagrams which describe all the ways in which an electron and positron can interact. In QFT, each diagram has an 'amplitude' to happen, and what really happens is predicted by summing the infinite series of amplitudes with one term for each picture.

This series is what mathematicians call the 'generating function' for the diagrams, but the physicists are using it 'backwards' from the way mathematicians generally do. Generating functions are more often used to encode information about a sequence of numbers within one function, rather than to calculate a function by expanding it in a series. Studying a generating function is often much more efficient than working with a recursive formula or an nth term formula for a sequence. In this class, we'll get comfortable with generating functions first, and see how, by manipulating them, we can learn about the corresponding sequences. Then we'll move on to the case of Feynman diagrams.

Prerequisites: some calculus.

Homework: none.

Related to: "Generating functions, partitions, and Catalan numbers" (Week 4), quantum field theory.

Generating functions, Catalan numbers, and partitions (∗∗, Julian and Mark, 9am (except 1:10pm on Tu), week 1 of 1)

Generating functions provide a powerful technique, used by Euler and many later mathematicians, to analyze sequences of numbers; often, they also provide the pleasure of working with infinite series without having to worry about convergence.

The sequence of Catalan numbers starts off 1, 2, 5, 14, 42, . . . These integers come up in the solution of many counting problems, as you'll see in Jeremy's colloquium on Saturday. We'll use generating functions to come up with an explicit formula for them. A *partition* of a positive integer n is a way to write n as a sum of one or more positive integers, say in nonincreasing order; for example, the seven partitions of 5 are

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

The number of such partitions is given by the partition function $p(n)$; for example, p(5) = 7. Although an "explicit" formula for p(n) is known and we may even show it, it's quite complicated. In our class, we'll combine generating functions and a famous combinatorial argument due to Franklin to find a beautiful recurrence relation for the (rapidly growing) partition function. This formula was used by MacMahon to make a table of values for $p(n)$ through $p(200) = 3972999029388$, well before the advent of computers!

Prerequisites: Summation notation; geometric series. Calculus is useful. Some experience with more general power series may help, but is not really needed.

Homework: recommended.

Related to: "Generating functions and Feynman diagrams" (Week 4), "The top 10 (or so) combinatorial interpretations of the Catalan numbers" (Week 4), "Combinatorially thinking" (Week 4), "Combinatorics" (Week 1).

Knots and links (∗∗, Ari, 10am, week 1 of 1)

People have been studying knots and links seriously for a bit more than a century. We'll learn a bit about the curious origins of the field, then investigate basic properties of these topological objects. Among the questions we'll answer are: What's the difference between a knot in 3-d space and a drawing of a knot on a blackboard, and how can we formulate this mathematically? Can you make any knot by gluing the ends of a braid together? Can you tell by staring at a tangled loop of string whether it can be unknotted? What about detecting whether two knots are the same? This last one will lead us the wild world of knot invariants, which are combinatorial, algebraic, or topological gadgets used to tell knots apart.

Prerequisites: none.

Homework: Possibly a few exercises mentioned in class.

Related to: "Topology" (Weeks 1–4), "Intro to graph theory" (Week 2), "Topological graph theory" (Weeks 3–4), "Singularities and knots" (Week 4).

The Prime Number Theorem for polynomials (∗∗∗, Noah, 11am, week 1 of 1)

The Prime Number Theorem (PNT) for the integers states that the probabilty that an integer of size x is prime is approximately $\frac{1}{\log x}$. This allows one to approximate things like the number of primes up to a certain number, or the size of the nth prime. Proving the PNT for the integers requires complex analysis and other advanced tools. In this class we will instead consider the analogue of the PNT for polynomials over a finite field. Here we can get an explicit formula for the number of prime polynomials of a given size. Along the way we'll discuss the analogue of the zeta function and some other nifty stuff.

Prerequisites: modular arithmetic, ("FLT for polynomials (Week 2)" OR "Pathway to analytic number theory" (Week 3)).

Homework: optional.

Related to: number theory, analysis, "The Riemann Hypothesis" (Week 4).

The Reeb foliation of the 3–sphere ($*$, Dan, 11am, Tu)

The sphere you know is the one that satisfies the equation $x^2 + y^2 + z^2 = 1$. But take it all up a dimension — $x^2 + y^2 + z^2 + w^2 = 1$ — and you get the **3-sphere**, an object that only lives inside four dimensions. Inside, however, it's just like the space we live in (i.e. it is locally three-dimensional), so hopefully we can tell what's going on in there!

So how do you study an object that you can't really see? That will be our motivating question, and from there we'll find all kinds of crazy things. In one hour, I'll take you on a wild tour of this fascinating space. Thanks to lots of careful pictures, you should get a good intuition for generalizing to the strange world of "3-manifolds."

Prerequisites: none.

Homework: none.

Related to: "The Poincaré conjecture" (Fr).

Semidefinite programming ($****$, Ellen, 10am, week 1 of 1)

Everyone has problems. How do we separate campers into teams for rock, paper, scissors tag so as to maximize animosity between the teams? What is the minimum number of classtime slots that Alfonso should schedule so that each mentor is teaching a class that she requested, no mentor is teaching two classes in one time slot, and there are no two classes of the same star level in a time slot? Happily for us, these seemingly difficult problems can be approximately solved in polynomial time with the wonders of semidefinite programming! In this class, we will talk about what semidefinite programming is, and how it works in the cases of the two problems above. In particular, using semidefinite programs we can find a number that lies between the clique number and chromatic number of a graph. We can also approximate a solution to MAX CUT (a question about cutting edges in a graph) within a factor of .878 (if we could approximate it within a factor of .94117, then P=NP !!!).

Prerequisites: linear algebra, knowledge of what a graph is.

Homework: recommended.

Related to: "Theoretical computer science" (Weeks 1–3), "Linear programming" (Week 2), "Convex cones" (Week 3), graph theory (Weeks 2–4), "The fifteen theorem" (Week 2).

Singularities and knots ($****$, D.A., 11am, week 1 of 1)

Mathematicians often say singularities are bad, but really they're just non-conformists – they stand out from the crowd of smooth points on a surface. In this course, we'll see how singularities on complex curves (which are real 2-dimensional surfaces) come with unusual piercings in the form of knots and links, and also how you can (to some extent) recognize a curve by the knot it's wearing.

Prerequisites: complex numbers, basic calculus, comfort with algebraic manipulation of polynomials.

Homework: optional.

Related to: "Constructive geometry" (Week 1), "Topology" (Weeks 1–4), "Knots and links" (Week 4), "Tropical curves" (Week 3). And math.

Some commutative algebra, and a bit beyond (∗∗∗-∗∗∗∗, Mark, 1:10pm, week 1 of 1 (no class on Tu))

Modern algebraic geometry relies heavily on commutative algebra. In this class, which may spill over into Week 5 if there is enough interest, we'll start by introducing some basic algebraic ideas (such as prime and maximal ideals and quotient rings) and then see, although not always with proof, how some geometric ideas, such as dimension, can be defined and/or studied using the language of algebra. If time permits, a bit of point-set topology (on a certain set of ideals!) will show up also, but this will be developed as needed.

Prerequisites: Nothing specific that you couldn't catch up on as needed, but if you have never seen any abstract algebra (groups, rings, or fields) at all, the class will likely move too fast.

Homework: Optional, probably.

Related to: "Fermat's dream I and II" (Weeks 1–4), "Finite fields" (Week 3), "Intro to groups and actions" (Weeks 1–2), "Singularities and knots" (Week 4).

Voting theory (∗∗, Rob, 1:10pm, Fr–Sa)

You and your friends are trying to decide what topping to put on a pizza. You decide to vote. The vote gets muddled when someone asks if you're allowed to vote on multiple toppings, and someone else says that everyone should vote for two toppings because that's how many you'll put on the pizza. Then someone else asks if she can just vote for mushrooms twice, because that's all she really wants. In the end, somehow you end up with two toppings that the majority don't even like.

Knowing the mathematical properties of voting can help you choose a method that's actually more likely to give people what they want. Or, if you're evil, what *you* want. Or it might just make you realize what's going wrong in both petty votes about pizza and important political elections.

Prerequisites: none.

Homework: none.

Related to: "Matrix game theory" (Week 3), "The stable marriage algorithm" (Week 2). And real life!

Marathon classes

A marathon class is a chance to learn material in greater depth and continue to keep up a good work ethic as it gets later in camp. Marathon classes typically run for up to six hours per day (including time for group work or homework), taking up most of the class periods (so you do have to miss normal classes). On the other hand, you get closer interaction with a mentor and learn a great deal about what you're studying. You also have a chance to focus on one topic at a time, which will let you improve your learning of that topic. Marathon classes have in the past been extremely well received by the students that took them. When the marathon class is done, you can return to a normal class schedule, or try another marathon class.

Ordinals and Cardinals (Kenny, ∗∗∗, marathon on Week 4)

You've probably heard by now that there's more than one kind of infinity. In fact, there's even at least two different ways to measure the sizes of infinity, depending on whether you're counting how long a sequence is (ordinal) or how big a set is (cardinal). In this class, I'll cover enough set theory to help you calculate with these kinds of infinity, when you combine sets or sequences of the appropriate size. We'll figure out what all these $\aleph$'s and ω's that set theorists talk about mean, and come up with more types of infinity than you could have possibly imagined before, and see where familiar sets like $\mathbb{N}$ and $\mathbb{R}$ fit in among them.

Prerequisites: A fair amount of rigor and some experience with very abstract mathematics. A willingness not to flinch when we discuss the details of how the Axiom of Choice works.

I will hold an informational session on Monday 24 (the day after the Puzzle Hunt) for everyone who is interested in the class. If you can't make it to this, then you need to talk to me at some point during the week. Either way, if you are interested in the class, do not wait till Monday to talk to me!

Right now I'm thinking that the class will be organized with alternating hours of lecture and problem time. This may change depending on interest and how things progress. We can discuss this at the informational session as well.

Visitor classes

Combinatorially thinking (∗∗, Arthur Benjamin and Jennifer Quinn, 10am, week 1 of 1)

Combinatorial proof is an art that requires cleverness, creativity, and the ability to count. Arguments primarily take one of two forms:

- A counting question is posed and answered in two different ways. Since both answers solve the same questions they must be equal.
- Two different sets are described, counted, and a correspondence found between them. One-to-one correspondences guarantee sets of the same size. Almost one-to-one correspondences take error terms into account. Even many-to-one correspondences may be utilized.

Faced with an identity, how do you create a combinatorial proof? This course will provide you with some useful combinatorial interpretations, well–selected examples, and the challenge of finding your own combinatorial proofs. Each day will focus on a different mathematical substrate — sometimes we explore identities involving numbers that are defined through counting (like binomial coefficients, Stirling numbers, and Catalan numbers) and other times you will need to acquire a combinatorial appreciation for quantities involved (like harmonic numbers, continued fractions, determinants, Fibonacci numbers, and the golden ratio). An extensive list of identities — some with known interpretations and others without — will serve as the basis for your exploration.

Prerequisites: A familiarity with basic counting techniques is helpful but not necessary.

Homework: Only recommended. We will ask you to share overnight successes.

Related to: "Combinatorics" (Week 1), "Generating functions, Catalan numbers, and partitions" (Week 4), "Polytopes" (Week 4), "Generating functions and Feynman diagrams" (Week 4), "Domino tilings of Aztec diamonds" (Week 4).

The geometry of continuous fractions (∗∗∗, Moon, 9am (except 1:10pm on Tu), week 1 of 1)

Would you like some number theory with that hyperbolic geometry?

Come to this class for a brief introduction to the hyperbolic plane an meet a triangulation with a special property, it encodes number theoretic information about the real numbers. Some key foundational objects of algebraic topology will enter the mix and in the end the magic lands on everyone's favourite Lie group lattice $\mathrm{SL}_2(\mathbb{C})$.

Related to: "Hyperbolic geometry" (Week 1), number theory.

How to count, or Finite calculus (∗, Julian, 11am, Th–Fr)

How many ways can you skin a cat? Or even sum a series? On the first day, we'll explore a number of ways, including building repertoires and expansion and contraction. On the second day, we'll develop finite calculus (no regular calculus needed!), and use that to discover a really awesome method of adding up.

Prerequisites: none.

Homework: none.

Related to: "Combinatorics" (Week 1), "Combinatorially thinking" (Week 4), "Generating functions, partitions, and Catalan numbers" (Week 4), "The top 10 combinatorial interpretations of the Catalan numbers" (Week 4).

Perfect numbers and Mersenne primes (∗∗, Holly, 11am, Sa)

In honor of my 28th birthday, this one-off is dedicated to the subject of perfect numbers! A perfect number is a number for which all its divisors (not equal to itself) add up to the original number.

So, 28 is perfect. There are some interesting things known and not known about perfect numbers, including their connection to Mersenne primes. Come find out more!

Prerequisites: none.

Homework: none.

Related to: number theory.

Polytopes (∗∗–∗∗∗, Jeremy Martin, 11am, week 1 of 1 (no class on Tu))

We'll study some of the geometry and combinatorics of polytopes, which can be defined roughly as "thingamajigs that you could build out of Zometool if Zometool came in seven or eight (or more) dimensions". Since my Zometool set only goes up to three dimensions, we'll be forced to find other, non-visual, ways of studying polytopes. For example, it's hard to draw a five-dimensional orange, but it *is* possible to figure out what happens when you peel it or put it in a blender. Aside from the inherent coolness of polytopes, maybe we'll look at their importance in optimization problems, the problem of counting the points inside a polytope (no, not *all* the points, silly!). I might throw in some turbo-charged abstract algebra if we all feel suitably ambitious.

Homework: none.

Prerequisites: combinatorics.

Related to: "Convex cones" (Week 3), "Combinatorially thinking" (Week 4), "Generating functions, partitions, and Catalan numbers" (Week 4), "Generating functions, groupoids, and Feynman diagrams" (Week 4).

Rational trigonometry and universal geometry (∗∗, Julian, 11am, Tu–We)

The intersecting chords theorem(*). Nice result. At least on the Euclidean plane ($\mathbb{R}^2$). But what would happen if we were to use to a different field, working on say $\mathbb{C}^2$ or $\mathbb{F}_7^2$ (here, $\mathbb{F}_7$ is the integers mod 7)? What could the statement of the theorem mean? And could it still, in some sense, be true?

And is there a way of doing some trigonometry (triangle measuring questions), getting exact answers without needing a calculator or tables?

Come and learn about some very recent developments in the fields of trigonometry and (elementary) geometry in these exotic settings.

(*) The intersecting chords theorem: if AB and CD are two chords of a circle which intersect at P, then AP.PB = CP.PD.

Prerequisites: none.

Homework: none.

Colloquia

Colloquia do not have star ratings: speakers try to make colloquia interesting and accessible to all Mathcampers.

Billiards (Moon Duchin, 4pm, We)

Consider a point bouncing around in a Euclidean polygon and study the trajectories: it turns out that this problem is surprisingly hard, it ties in a lot of cool math, and it won't help your pool one bit. We will see in this talk that the dynamical system of a billiard is described by families of curves on an asociated topological surface. Cameo appearances will be made by complex analysis, classical number theory, and Galois theory.

Proofs that really count, Part II (Arthur Benjamin, 4pm, Tu)

Counting leads to beautiful, often elementary, and very concrete proofs. As human beings we learn to count from a very early age. A typical 2 year old will proudly count to ten for the coos and applause of adoring parents. Although many adults readily claim ineptitude in mathematics no one ever owns up to an inability to count. Counting is one of our first tools, and it is time to appreciate its full mathematical power. Every proof in this talk reduces to a counting problem – typically enumerated in two different ways. While not necessarily the simplest approach, it offers another method to gain understanding of mathematical truths. To a combinatorialist, this kind of proof is the only right one. Hopefully when you encounter identities in the future, the first question to pop into your mind will not be "Why is this true?" but "What does this count?"

The Seven Millenium Problem Series

On May 2000, in order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts selected the Seven Millenium Prize Problems. They focused on important questions that had resisted solution over the years, and allocated an award of $1M for the solution of each one of them. In this series of colloquia we will present these problems to you, in the hope that you will some day solve them and endow Mathcamp with the award!

The Birch and Swinnerton–Dyer conjecture (∗∗∗, Dave S, 10am, week 1 of 1)

This remarkable conjecture relates the value of a function at a point where it is not at present known to be defined, to the order of a group which is not known to be finite. – John Tate

One of the core goals of number theory is to study the rational solutions to polynomial equations with integer coefficients. We've got some very good techniques for proving that equations have no rational solutions: for instance, the equation $x^2 + y^2 = 3$ has no solutions with (x, y) both rational, because a solution would yield a contradiction modulo 4. On the other hand, when an equation actually does have rational solutions, we are much less good at finding them.

The simplest equations for which we do not have a proven method for finding all rational solutions are the *elliptic curves*: equations of the form $y^2 = x^3+ax+b$. The conjecture of Birch and Swinnerton-Dyer, among its many consequences, would provide an algorithm for finding all rational solutions to elliptic curves, though you definitely wouldn't be able to tell this just by looking at the statement of the problem. The BSD conjecture proposes a deep analogy beetween elliptic curves and (generalizations of) the integers, and is an extraordinary confluence of three major themes of modern number theory: the local-to-global principle, special values of L-functions, and modularity. I'll describe these themes as best I can in a week–long class.

Prerequisites: modular arithmetic.

Homework: none.

Related to: "Finite fields" (Week 3), "Knots and singularities" (Week 4), "Fermat's dream II" (Weeks 3–4).

The Poincaré conjecture (Noah, 4pm, Fr)

Unlike the other 7MP talks, in this talk I'll give you an outline of how to win a million dollars. Very roughly, the Poincare conjecture gives a simple criterion for distinguishing the 3-dimensional sphere ($x^2 + y^2 + z^2 + w^2 = 1$ in 4-dimensional space) from all similar spaces. I'll be giving a more precise statement of this conjecture, and giving a brief outline of how to prove it.

Related to: "The Reeb foliation of the 3-sphere" (Week 4).

The Riemann hypothesis (Miljan, 4pm, Th)

The distribution of prime numbers among all natural numbers does not follow any regular pattern. However the German mathematician G.F.B. Riemann (1826 – 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function

$$\zeta(s) \;=\; 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

called the Riemann Zeta function. It is defined for all complex numbers $s \neq 1$. It has zeros at the negative even integers (i.e. at $s = -2$, $s = -4$, $s = -6$, ...) which are called the *trivial zeroes.* The Riemann hypothesis is concerned with the *non–trivial zeroes*, and states that:

The real part of any non–trivial zero of the Riemann zeta function is 1/2.

Thus the non–trivial zeros should lie on the so–called *critical line* $1/2 + it$, with t a real number and i the imaginary unit.

This hypothesis has been checked for the first ten billions zeroes. The Riemann Hypothesis is now unquestionably the most important problem in mathematics and it continues to attract the attention of the best mathematicians, not only because it has gone unsolved for so long but also because it appears tantalizingly vulnerable and because its solution would probably bring to light new techniques of far reaching importance.

Prerequisites: complex numbers.

Related to: "Pathway to analytic number theory" (Week 3).

Digestif

A *digestif* is a crispy, exciting thirty–minute minicolloquium to keep you from napping on Saturday after lunch!

The top ten (or so) combinatorial interpretations of the Catalan numbers (Jeremy, 2–2:30, Sa)

The Catalan numbers 1, 2, 5, 14, 42, ... are ubiquitous in combinatorics; they count binary trees, triangulations of a polygon, noncrossing matchings, lattice paths, pattern-avoiding permutations, and a whole slew of other cool combinatorial doodads. Many of these seemingly different doodads have a lot more in common than appears at first sight. I'll show off my favorite N kinds of Catalan doodads and explain some of the subtle (or sometimes not-so-subtle) connections between them.

Prerequisites: none.

Brief visitor and new staff bios

Arthur Benjamin (Harvey Mudd College – math)

Arthur Benjamin enjoys game theory and combinatorics, with a special fondness for Fibonacci numbers. Many of these ideas appear in the book (co-authored with Jennifer Quinn), *Proofs That Really Count: The Art of Combinatorial Proof*, published by the Mathematical Association of America (MAA). Art and Jenny are the co-editors of the Math Horizons magazine. Art has received the national teaching award from the MAA. He is also a magician who performs his mixture of math, magic and rapid mental calculations to audiences all over the world.

Moon Duchin (UC Davis – math)

Moon Duchin is interested in geometry, topology, and dynamics, in lots of different combinations. Lately she's got geometric group theory on her mind. She also thinks about philosophy, cultural studies, gender theory, what they have to say about math, and what math has to say back!

Julian Gilbey

Julian is a Mathcamp veteran, having been a mentor at MC00, MC01, MC02 and a visitor at MC04. He has many stories about the delights of teaching 6th–12th grades in an inner-city-type public school in London, which is what he has been doing this year. His main focus recently has been on staying sane, which he has somewhat achieved by reading math in those ever-so-precious moments of free time he has had this year. He is also the proud owner of two sarongs, acquired at MC00.

Jeremy Martin (Kansas University – math)

Jeremy was a Mentor and member of the *Junta* at Mathcamp in 1997 and 1998, and he comes back to visit every chance he gets. Jeremy's research is in geometric combinatorics, which means that he spends a lot of time drawing pictures and counting things. His interests outside mathematics include bridge, chess, cooking, and choral music.

Jennifer Quinn (Association for Women in Mathematics)

Jennifer Quinn thinks that beautiful proofs are as much art as science. Simplicity, elegance, and transparency should be the driving principles. Simply understanding mathematical truth is not sufficient. Instead, strive to put mathematics into a concrete framework where truth becomes apparent and ideas quickly generalize. Together Jenny and her co-author, co-editor, and friend Arthur Benjamin will guide mathcampers in an exploration of mathematical truths by the basic combinatorial techniques of counting and matching.

Holly Swisher (Oregon State University – math)

A former Mathcamp mentor, Holly got her Ph.D. in analytic number theory at University of Wisconsin in 2005. She did a one year postdoc at Ohio State University, and is starting a tenure track position at Oregon State this fall. When she's not contemplating Ramanujan's congruence formulas for partitions and modular forms, Holly likes Eastern European folk dancing, hockey, and roller derby.